ON A RESULT OF ZHANG AND XU CONCERNING THEIR OPEN PROBLEM

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Abstract. The motivation of this paper is to study the uniqueness of meromorphic functions sharing a nonzero polynomial with the help of the idea of normal family. The result of the paper improves and generalizes the recent result due to Zhang and Xu [24]. Our another remarkable aim is to solve an open problem as posed in the last section of [24].

1. Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Suppose $f$ and $g$ are two non-constant meromorphic functions and $a \in \mathbb{C}$. We say that $f$ and $g$ share the value $a$ with counting multiplicities (CM), provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share the value $a$ with ignoring multiplicities (IM), provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. Moreover we say that $f$ and $g$ share $\infty$ CM, if $1/f$ and $1/g$ share $0$ CM, and we say that $f$ and $g$ share $\infty$ IM, if $1/f$ and $1/g$ share $0$ IM.

In this paper we take up the standard notations and definitions of the value distribution theory (see [7]). For a non-constant meromorphic function $f$ we denote by $S(r, f)$ any quantity satisfying the relation $S(r, f) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure.

We define $T(r) = \max\{T(r, f), T(r, g)\}$ and we use the notation $S(r)$ to denote any quantity satisfying the relation $S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

A meromorphic function $a$ is said to be a small function of $f$ if $T(r, a) = S(r, f)$, i.e., if $T(r, a) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure.

If $f(z_0) = z_0$, where $z_0 \in \mathbb{C}$, then $z_0$ is called a fixed point of $f(z)$. We use the following definition throughout this paper

$$\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)},$$

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where \( a \in \mathbb{C} \cup \{ \infty \} \).

First we recall the following result due to W.K. Hayman.

**Theorem A** ([6]). Let \( f \) be a transcendental meromorphic function and \( n \in \mathbb{N} \setminus \{1, 2\} \). Then \( f^n f' = 1 \) has infinitely many solutions.

Corresponding to Theorem A, C.C. Yang and X.H. Hua obtained the following result.

**Theorem B** ([19]). Let \( f \) and \( g \) be two non-constant meromorphic functions, \( n \in \mathbb{N} \) with \( n \geq 11 \). If \( f^n f' \) and \( g^n g' \) share 1 CM, then either \( f(z) = c_1 e^{cz^2} \), \( g(z) = c_2 e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \( (c_1 c_2)^{n+1} c^2 = -1 \) or \( f \equiv tg \) for a constant \( t \) such that \( t^{n+1} = 1 \).

In 2002, using the idea of sharing fixed points, M.L. Fang and H.L. Qiu further generalized and improved Theorem B in the following manner.

**Theorem C** ([11]). Let \( f \) and \( g \) be two non-constant meromorphic functions, and let \( n \in \mathbb{N} \) with \( n \geq 11 \). If \( f^n f' - z \) and \( g^n g' - z \) share 0 CM, then either \( f(z) = c_1 e^{cz^2} \), \( g(z) = c_2 e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are three non-zero complex numbers satisfying \( 4(c_1 c_2)^{n+1} c^2 = -1 \) or \( f = tg \) for a complex number \( t \) such that \( t^{n+1} = 1 \).

For the last couple of years a handful numbers of astonishing results have been obtained regarding the value sharing of non-linear differential polynomials which are mainly the \( k \)-th derivative of some linear expression of \( f \) and \( g \).

In 2010, J.F. Xu, F. Lü and H.X. Yi studied the analogous problem corresponding to Theorem C where in addition to the fixed point sharing problem, sharing of poles are also taken under supposition. Thus the research has somehow been shifted to wards the following direction.

**Theorem D** ([16]). Let \( f \) and \( g \) be two non-constant meromorphic functions, and let \( n, k \in \mathbb{N} \) with \( n > 3k + 10 \). If \( (f^n)^{(k)} \) and \( (g^n)^{(k)} \) share \( z \) CM, \( f \) and \( g \) share \( \infty \) IM, then either \( f(z) = c_1 e^{cz^2} \), \( g(z) = c_2 e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \( 4n^2 (c_1 c_2)^{n+1} c^2 = -1 \) or \( f \equiv tg \) for a constant \( t \) such that \( t^n = 1 \).

**Theorem E** ([16]). Let \( f \) and \( g \) be two non-constant meromorphic functions satisfying \( \Theta(\infty, f) > \frac{3}{5} \), and let \( n, k \in \mathbb{N} \) with \( n \geq 3k + 12 \). If \( (f^n(f - 1))^{(k)} \) and \( (g^n(g - 1))^{(k)} \) share \( z \) CM, \( f \) and \( g \) share \( \infty \) IM, then \( f \equiv g \).

Recently, X.B. Zhang and J.F. Xu further generalized and improved the results obtained in [16] in the following manner.

**Theorem F** ([24]). Let \( f \) and \( g \) be two transcendental meromorphic functions, let \( p(z) \) be a non-zero polynomial with \( \deg(p) = l \leq 5 \), \( n, k, m \in \mathbb{N} \) with \( n > 3k + m + 7 \). Let \( P^*(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0 \) be a non-zero polynomial. If \( [f^n P^*(f)]^{(k)} \) and \( [g^n P^*(g)]^{(k)} \) share \( p \) CM, \( f \) and \( g \) share \( \infty \) IM then one of the following three cases hold:

1. \( f(z) \equiv t g(z) \) for a constant \( t \) such that \( t^d = 1 \), where \( d = \gcd(n + m, \ldots, n + m - i, \ldots, n) \), \( a_{m-i} \neq 0 \) for some \( i = 1, 2, \ldots, m \),
(2) \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0 \), where \( R(\omega_1, \omega_2) = \omega_1^m(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0); \)

(3) \( P^*(z) \) reduces to a non-zero monomial, namely \( P^*(z) = a_i z^i \neq 0 \) for some \( i \in \{0, 1, \ldots, m\}; \) if \( p(z) \) is not a constant, then \( f(z) = c_1 e^{cQ(z)}, g(z) = c_2 e^{-cQ(z)}, \) where \( Q(z) = \int_0^z p(t) dt \), \( c_1, c_2 \) and \( c \) are constants such that \( a_i^2(c_1c_2)^{n+i}|(n+i)c|^2 = -1, \) if \( p(z) \) is a non-zero constant \( b, \) then \( f(z) = c_2 e^{cz}, g(z) = c_4 e^{-cz}, \) where \( c_3, c_4 \) and \( c \) are constants such that \( (-1)^k a_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2. \)

Zhang and Xu made the following commen in Remark 1.2 [24]:

"From the proof of Theorem 1.3, when \( \deg(p) \) becomes large we can see that the computation will be very complicated and so we are not sure whether Theorem 1.3 holds for the general polynomial \( p(z) \)."

In addition they [24] posed the following open problem at the end of their paper.

**Open problem.** What happens to Theorem 1.3 [24] if the condition \(" l \leq 5 \)" is removed?

Regarding the above result, the first author [13] asked the following question in 2016.

**Question 1.** Can the lower bound of \( n \) be further reduced in Theorem F?

Keeping in mind the above question, the first author obtained the following result.

**Theorem G** ([13]). Let \( f \) and \( g \) be two transcendental meromorphic functions, let \( p(z) \) be a nonzero polynomial with \( \deg(p) \leq n - 1 \), \( n \geq 1 \), \( k \geq 1 \) and \( m \geq 0 \) be three integers such that \( n > 3k + m + 6 \) and \( P^*(z) \) be defined as in Theorem F. If \( [f^n P^*(f)]^{(k)}, [g^n P^*(g)]^{(k)} \) share \( p \, \text{CM} \) and \( f, g \) share \( \infty \, \text{IM} \) then the conclusion of Theorem F holds.

This paper is motivated by the following questions

**Question 2.** Can one remove the conditions \(" l \leq 5 \)" and \(" \deg(p) \leq n - 1 \)" respectively in Theorems F and G?

**Question 3.** Can one deduce a generalized result in which Theorems F and G will be included?

**Question 4.** Can the lower bound of \( n \) be further reduced in Theorem G?

Our main objective to write this paper is to solve the above questions.

## 2. Main result and definitions

Throughout this paper, we always use \( P(z) \) to denote an arbitrary non-constant polynomial of degree \( n \) as follows

\[
P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0
\]

\[
(2.1) = a_n (z - e_1)^{d_1} (z - e_2)^{d_2} \cdots (z - e_s)^{d_s},
\]
where \( a_i \in \mathbb{C} \) (\( i = 0, 1, \ldots, n \)) with \( a_n \neq 0 \), \( e_j (j = 1, 2, \ldots, s) \) are distinct numbers in \( \mathbb{C} \) and \( d_1, d_2, \ldots, d_s \in \mathbb{N} \cup \{0\} \), \( n, s \in \mathbb{N} \) with

\[
\sum_{i=1}^{s} d_i = n. \]

Let \( d = \max\{d_1, d_2, \ldots, d_s\} \) and \( e \) be the corresponding zero of \( P(z) \) of multiplicity \( d \). We set an arbitrary non-zero polynomial \( P_1(z) \) by

\[
P_1(z) = a_n \prod_{\substack{i=1, d_i \neq d \leq k+1 \atop d_i \leq d}} (z - e_i)^{d_i} = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,
\]

where \( a_n = b_m \) and \( m = n - d \). Let \( z_1 = z - e \). Then

\[
P_1(z) = P_1(z_1 + e) = P_2(z_1) = c_m z_1^m + c_{m-1} z_1^{m-1} + \cdots + c_1 z_1 + c_0,
\]

where \( c_m = b_m = a_n \). Obviously

\[
P(z) = (z - e)^d P_1(z) = z_1^d P_2(z_1).
\]

Let

\[
m_1 = \sum_{\substack{i=1, d_i \neq d \leq k+1 \atop d_i \leq d}} d_i,
\]

where \( k \in \mathbb{N} \). Suppose \( \Gamma = m_1 + (k + 2)m_2 \), where \( m_2 \) is the number of zeros of \( P_1(z) \) with multiplicities \( \geq k + 2 \). Clearly \( \Gamma \leq \deg(P_1) = m \).

Before going to our main result we now explain the following useful definition and notation.

**Definition 1** ([10][11]). Let \( k \in \mathbb{N} \cup \{\infty\} \). For \( a \in \mathbb{C} \cup \{\infty\} \) we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k+1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f, g \) share the value \( a \) with weight \( k \).

We write \( f, g \) share \((a, k)\) to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly if \( f, g \) share \((a, k)\) then \( f, g \) share \((a, p)\) for any integer \( p, 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \((a, 0)\) or \((a, \infty)\) respectively. If \( a \) is a small function, we define that \( f \) and \( g \) share \((a, k)\) if \( f - a \) and \( g - a \) share \((0, k)\).

In this paper, taking the possible answers of the above questions into background we obtain the following result.

**Theorem 1.** Let \( f \) and \( g \) be two transcendental meromorphic functions and let \( d, n, k \in \mathbb{N} \) and \( m, \Gamma \in \mathbb{N} \cup \{0\} \) such that \( n > 2\Gamma + 3k + 6 \) and \( d > k \). Let \( P(z) \) be a nonzero polynomial and \( P(z) \) be defined as in [2][4]. If \([P(f)]^{(k)}, [P(g)]^{(k)}\) share \((p, k_1)\) where \( k_1 = \left\lceil \frac{3+k}{n-k-1} \right\rceil + 3 \) and \( f, g \) share \((\infty, 0)\) then one of the following three cases holds
1. \( f(z) - e \equiv t(g(z) - e) \) for a constant \( t \) such that \( t^{d_0} = 1 \), where \( d_0 = GCD(d + m, \ldots, d + m - i, \ldots, d) \), \( c_{m-i} \neq 0 \) for some \( i = 1,2, \ldots, m \),

2. \( f_1 \) and \( g_1 \) satisfy the algebraic equation \( R(f_1, g_1) = 0 \), where \( R(\omega_1, \omega_2) = \omega_1^d(c_m\omega^m + c_{m-1}\omega^{m-1} + \cdots + c_0) - \omega_2^d(c_m\omega_2^m + c_{m-1}\omega_2^{m-1} + \cdots + c_0) \), where \( f_1 = f - e \) and \( g_1 = g - e \);

3. \( P(z) \) takes the form \( P(z) = c_i(z-e)^{d+i} \neq 0 \) for some \( i \in \{0,1, \ldots, m\} \). Also if \( p(z) \) is not a constant, then \( f(z) = d_1e^{e^{Q(z)} + e} \), \( g(z) = d_2e^{-c^eQ(z)} + e \), where \( Q(z) = \int_0^z p(t)dt, d_1, d_2 \) and \( e^e \) are constants such that \( c_2^2(d_1d_2)^{d+i}[(d+i)e^e]^2 = -1 \), if \( p(z) \) is a non-zero constant, say \( b \), then \( f(z) = d_2e^{-c^e} + e, g(z) = d_4e^{-c^e} + e \), where \( d_3, d_4 \) and \( e^e \) are constants such that \( (1)^k c_2^2(d_3d_4)^{d+i}[(d+i)e^e]^2k = b^2 \).

Remark 1. In this paper we can able to remove the conditions “\( l \leq 5 \)” and “\( \deg(p) \leq n - 1 \)” respectively in Theorems F and G without imposing any other conditions and keeping all the conclusions intact. As a result both Theorems F and G hold for a general non-zero polynomial \( p(z) \).

Remark 2. Let us take \( d = n, e = 0 \) and \( P_1(z) = a_mz^n + a_{m-1}z^{m-1} + \cdots + a_1z + a_0 \) in \([2,3]\), where \( a_0, a_1, \ldots, a_{m-1}, a_m \) are complex constants. Then by replacing \( n \) by \( d + m \) in Theorem [1] we can easily get a theorem which is the improvement of Theorems F and G.

We give the following definitions and notations which are used in the paper.

**Definition 2** ([9]). Let \( a \in \mathbb{C} \cup \{\infty\} \). For \( p \in \mathbb{N} \) we denote by \( N(r, a; f \mid \leq p) \) the counting function of those \( a \)-points of \( f \) (counted with multiplicities) whose multiplicities are not greater than \( p \). By \( \overline{N}(r, a; f \mid \leq p) \) we denote the corresponding reduced counting function.

In an analogous manner we can define \( N(r, a; f \mid \geq p) \) and \( \overline{N}(r, a; f \mid \geq p) \).

**Definition 3** ([11]). Let \( k \in \mathbb{N} \cup \{\infty\} \). We denote by \( N_k(r, a; f) \) the counting function of \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k \) times if \( m > k \). Then \( N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \cdots + \overline{N}(r, a; f \mid \geq k) \). Clearly \( N_1(r, a; f) = \overline{N}(r, a; f) \).

**Definition 4** ([2]). Let \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share \((a,0)\) for \( a \in \mathbb{C} \cup \{\infty\} \). Let \( z_0 \) be an \( a \)-point of \( f \) with multiplicity \( p \) and also an \( a \)-point of \( g \) with multiplicity \( q \). We denote by \( \overline{N}_L(r, a; f / (\overline{N}_L(r, a; g)) \) the reduced counting function of those \( a \)-points of \( f \) and \( g \), where \( p > q \geq 1 \). Also we denote by \( \overline{N}_E(r, a; f) \) the reduced counting function of those \( a \)-points of \( f \) and \( g \), where \( p = q \geq 1 \).

**Definition 5** ([10] [11]). Let \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share \((a,0)\). We denote by \( \overline{N}_s(r, a; f, g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \). Clearly \( \overline{N}_s(r, a; f, g) = \overline{N}_s(r, a; g, f) \) and \( \overline{N}_s(r, a; f, g) = \overline{N}_L(r, a; f) + L(r, a; g) \).
Then we say that which converges spherically and uniformly on the compact subsets of

Let \( f \) be a non-constant meromorphic function in \( \mathbb{C} \). Then \( h \) is called a normal function if there exists a positive real number \( M \) such that \( h^\#(z) \leq M \) \( \forall z \in \mathbb{C} \), where

\[
h^\#(z) = \frac{|h'(z)|}{1 + |h(z)|^2}
\]
denotes the spherical derivative of \( h \).

**Definition 7.** Let \( h \) be a meromorphic function in \( \mathbb{C} \). Then \( h \) is called a normal function if there exists a positive real number \( M \) such that \( h^\#(z) \leq M \) \( \forall z \in \mathbb{C} \), where

\[
h^\#(z) = \frac{|h'(z)|}{1 + |h(z)|^2}
\]
denotes the spherical derivative of \( h \).

**Definition 8.** Let \( F \) be a family of meromorphic functions in a domain \( D \subset \mathbb{C} \). We say that \( F \) is normal in \( D \) if every sequence \( \{f_n\}_n \subseteq F \) contains a subsequence which converges spherically and uniformly on the compact subsets of \( D \) (see [12]).

3. Lemmas

Let \( F \) and \( G \) be two non-constant meromorphic functions defined in \( \mathbb{C} \). We define the meromorphic functions \( H \) and \( V \) in the following manner

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right)
\]
and

\[
V = \left( \frac{F'}{F-1} - \frac{F'}{F} \right) - \left( \frac{G'}{G-1} - \frac{G'}{G} \right).
\]

**Lemma 1 ([18]).** Let \( f \) be a non-constant meromorphic function and let \( a_n(z)(\neq 0), a_{n-1}(z), \ldots, a_0(z) \) be meromorphic functions such that \( T(r,a_i(z)) = S(r,f) \) for \( i = 0,1,2,\ldots,n \). Then

\[
T(r,a_nf^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0) = nT(r,f) + S(r,f).
\]

**Lemma 2 ([23]).** Let \( f \) be a non-constant meromorphic function and \( p, k \in \mathbb{N} \). Then

\[
N_p(r,0; f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0; f) + S(r,f),
\]

\[
N_p(r,0; f^{(k)}) \leq kN(r,\infty; f) + N_{p+k}(r,0; f) + S(r,f).
\]

**Lemma 3 ([12]).** If \( N(r,0; f^{(k)} | f \neq 0) \) denotes the counting function of those zeros of \( f^{(k)} \) which are not the zeros of \( f \), where a zero of \( f^{(k)} \) is counted according to its multiplicity, then

\[
N(r,0; f^{(k)} | f \neq 0) \leq kN(r,\infty; f) + N(r,0; f |< k) + kN(r,0; f |\geq k) + S(r,f).
\]

**Lemma 4 ([7, Theorem 3.10]).** Suppose that \( f \) is a non-constant meromorphic function, \( k \in \mathbb{N} \setminus \{1\} \). If

\[
N(r,\infty, f) + N(r,0; f) + N(r,0; f^{(k)}) = S\left(r, \frac{f'}{f}\right),
\]
then \( f(z) = e^{az+b} \), where \( a \neq 0 \), \( b \) are constants.
Lemma 5 ([3]). Let \( f(z) \) be a non-constant entire function and let \( k \in \mathbb{N} \setminus \{1\} \). If \( f(z)f^{(k)}(z) \neq 0 \), then \( f(z) = e^{az+b} \), where \( a \neq 0 \) and \( b \) are constant.

Lemma 6 ([20], Theorem 1.24]). Let \( f \) be a non-constant meromorphic function and let \( k \in \mathbb{N} \). Suppose that \( f^{(k)} \neq 0 \), then
\[
N(r,0; f^{(k)}) \leq N(r,0; f) + kN(r,\infty; f) + S(r,f).
\]

Lemma 7. Let \( f, g \) be non-constant meromorphic functions and let \( n, k, \Gamma \in \mathbb{N} \) with \( n > k + \Gamma + 2 \). Let \( P(z) \) be defined as in (2.1) and \( a(z)(\neq 0,\infty) \) be a small function of \( f \). If \( [P(f)]^{(k)} \) and \( [P(g)]^{(k)} \) share \( (a,0) \), then \( T(r,f) = O(T(r,g)) \), \( T(r,g) = O(T(r,f)) \).

Proof. Let \( f_1 = f - c. \) Clearly \( F = f_1^d P_1(f) \). By the second fundamental theorem for small functions (see [17]), we have
\[
(3.5) \quad T(r,F^{(k)}) \leq N(r,f) + \overline{N}(r,0; F^{(k)}) + \overline{N}(r,a; F^{(k)}) + (\varepsilon + o(1))T(r,f)
\]
for all \( \varepsilon > 0 \). From (3.5) and Lemmas 1 2 with \( p = 1 \) we have
\[
n T(r,f) \leq \overline{N}(r,f) + N_{k+1}(r,0; F) + \overline{N}(r,a; F^{(k)}) + (\varepsilon + o(1))T(r,f)
\]
\[
\leq \overline{N}(r,f) + (k+1)\overline{N}(r,0; f_1) + N_{k+1}(r,0; P_1(f))
+ \overline{N}(r,a; [P(f)]^{(k)}) + (\varepsilon + o(1))T(r,f)
\]
\[
\leq \overline{N}(r,f) + (k+1)\overline{N}(r,0; f_1) + N_{k+2}(r,0; P_1(f))
+ \overline{N}(r,a; [P(f)]^{(k)}) + (\varepsilon + o(1))T(r,f)
\]
\[
\leq (k+\Gamma+2)T(r,f) + \overline{N}(r,a; [P(g)]^{(k)}) + (\varepsilon + o(1))T(r,f),
\]
i.e.,
\[
(n - k - \Gamma - 2)T(r,f) \leq \overline{N}(r,a; [P(g)]^{(k)}) + (\varepsilon + o(1))T(r,f).
\]
Since \( n > k + \Gamma + 2 \), take \( \varepsilon < 1 \) and we have \( T(r,f) = O(T(r,g)) \). Similarly we have \( T(r,g) = O(T(r,f)) \). This completes the proof. \( \square \)

Lemma 8. Let \( f \) and \( g \) be two non-constant meromorphic functions. Let \( P(z) \) be defined as in (2.1) and \( k, \Gamma, n \in \mathbb{N} \) with \( n > 3k + 2 \Gamma \). If \( [P(f)]^{(k)} \equiv [P(g)]^{(k)}, \) then \( P(f) \equiv P(g). \)

Proof. We have \( [P(f)]^{(k)} \equiv [P(g)]^{(k)}. \) Integrating we get
\[
[P(f)]^{(k-1)} \equiv [P(g)]^{(k-1)} + c_{k-1}.
\]
If possible suppose \( c_{k-1} \neq 0 \). Now in view of Lemma 2 for \( p = 1 \) and using the second fundamental theorem we get
\[
n T(r,f) = T(r,P(f)) + O(1) \leq T(r,[P(f)]^{(k-1)}) - \overline{N}(r,0; [P(f)]^{(k-1)})
+ N_k(r,0; P(f)) + S(r,f)
\]
\[
\leq \overline{N}(r,0; [P(f)]^{(k-1)}) + \overline{N}(r,\infty; f) + \overline{N}(r,c_{k-1}; [P(f)]^{(k-1)})
- \overline{N}(r,0; [P(f)]^{(k-1)}) + N_k(r,0; P(f)) + S(r,f)
\]
\[
= O(1).
\]
\[ \leq N(r, \infty; f) + N(r, 0; [P(g)]^{(k-1)}) + N_k(r, 0; P(f)) + S(r, f) \]
\[ \leq N(r, \infty; f) + (k-1)N(r, \infty; g) + N_k(r, 0; P(g)) + N_k(r, 0; P(f)) + S(r, f) \]
\[ \leq N(r, \infty; f) + (k-1)N(r, \infty; g) + k \ N(r, 0; g_1) + N_k(r, 0; P_1(g)) \]
\[ + k \ N(r, 0; f_1) + N_k(r, 0; P_1(f)) + S(r, f) \]
\[ \leq (k + \Gamma + 1)T(r, f) + (2k + \Gamma - 1)T(r, g) + S(r, f) + S(r, g) \]
\[ \leq (3k + 2 \Gamma)T(r) + S(r). \]

Similarly we get
\[ n \ T(r, g) \leq (3k + 2 \Gamma)T(r) + S(r). \]

Combining we get
\[ n \ T(r) \leq (3k + 2 \Gamma)T(r) + S(r), \]
which is a contradiction since \( n > 3k+2\Gamma \). Therefore \( c_{k-1} = 0 \) and so \( [P(f)]^{(k-1)} \equiv [P(g)]^{(k-1)}. \) Proceeding in this way after \((k-1)\)-th step, we obtain \([P(f)]' \equiv [P(g)]'\).

Integrating we get \( P(f) \equiv P(g) + c_0. \) If possible suppose \( c_0 \neq 0. \) Now using the second fundamental theorem we get
\[ n \ T(r, f) = T(r, P(f)) + O(1) \]
\[ \leq N(r, 0; P(f)) + N(r, \infty; P(f)) + N(r, c_0; P(f)) + S(r, f) \]
\[ \leq N(r, 0; P(f)) + N(r, \infty; f) + N(r, 0; P(g)) + S(r, f) \]
\[ \leq N(r, 0; f_1) + N(r, 0; P_1(f)) + N(r, \infty; f) + N(r, 0; g_1) \]
\[ + N(r, 0; P_1(g)) + S(r, f) \]
\[ \leq (\Gamma + 2)T(r, f) + (\Gamma + 1)T(r, g) + S(r, f) + S(r, g) \]
\[ \leq (2\Gamma + 3)T(r) + S(r). \]

Similarly we get
\[ n \ T(r, g) \leq (2\Gamma + 3)T(r) + S(r). \]

Combining these we get
\[ (n - 2\Gamma - 3)T(r) \leq S(r), \]
which is a contradiction since \( n > 2\Gamma + 3. \) Therefore \( c_0 = 0 \) and so \( P(f) \equiv P(g). \)

This proves the lemma. \( \square \)
Lemma 9. Let \( f, g \) be transcendental meromorphic functions and let \( P(z) \) be defined as in (2.1). Let \( d \geq 1, \ m \geq 0 \) and \( k \geq 1 \) be three integers such that \( d > k \). If \( [P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2 \), where \( p(z) \) is a non-zero polynomial and \( f, g \) share \((\infty, 0)\), then \( P_2(z_1) \) is reduced to a non-zero monomial, namely \( P_2(z_1) = c_i z_1^i \neq 0 \) for some \( i \in \{0, 1, \ldots, m\} \) and so \( P(z) \) takes the form \( P(z) = c_i (z - e)^{d+i} \neq 0 \) for some \( i \in \{0, 1, \ldots, m\} \).

**Proof.** Suppose

\[ [P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2, \]

i.e.,

\[ (3.6) \quad [f_1^d P_2(f_1)]^{(k)}[g_1^d P_2(g_1)]^{(k)} \equiv p^2, \]

where \( f_1 = f - e \) and \( g_1 = g - e \). Since \( f \) and \( g \) share \((\infty, 0)\), it follows that \( f \) and \( g \) are transcendental entire functions.

Suppose on the contrary that, \( P_2(z_1) \) does not reduce to a non-zero monomial. Then without loss of generality, we may assume that

\[ P_2(z_1) = c_m z_1^m + c_{m-1} z_1^{m-1} + \cdots + c_1 z_1 + c_0, \]

where \( c_0 \neq 0, c_1, \ldots, c_{m-1}, c_m \neq 0 \) are complex constants.

Since the number of zeros of \( p(z) \) is finite, it follows that both \( f_1 \) and \( g_1 \) have finitely many zeros. Then \( f_1(z) \) takes the form

\[ f_1(z) = h(z)e^{\gamma(z)}, \]

where \( h \) is a non-zero polynomial and \( \gamma \) is a non-constant entire function. Clearly

\[ f_1^{d+i}(z) = h^{d+i}(z)e^{(d+i)\gamma(z)}, \]

where \( i = 0, 1, \ldots, m \). Then by induction we have

\[ (3.7) \quad [c_i f_1^{d+i}(z)]^{(k)} = t_i(\gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)})e^{(d+i)\gamma(z)}, \]

where \( t_i(\gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)})(i = 0, 1, \ldots, m) \) are differential polynomials in \( \gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)} \). Since \( f_1(z) \) is a transcendental entire function, from (3.7) we see that

\[ t_i(\gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)}) \neq 0, \]

\( i = 0, 1, \ldots, m \). Note that

\[ (3.8) \quad [f_1^d P_2(f_1)]^{(k)} = \sum_{i=1}^{m} [c_i f_1^{d+i}]^{(k)} = \sum_{i=0}^{m} t_i e^{(d+i)\gamma} = e^{d\gamma} \sum_{i=0}^{m} t_i e^{i\gamma} \]

and so \([f_1^d P_2(f_1)]^{(k)} \neq 0\). Note that \( f_1 = h e^{\gamma} \). So \( f_1' = h' e^{\gamma} + \alpha' h e^{\gamma} \). Therefore \( \frac{f_1'}{f_1} = \frac{h'}{h} + \gamma' \). Since \( \gamma' \) is an entire function, we have

\[ T(r, \gamma') = m(r, \gamma') = m(r, \frac{f_1'}{f_1} - \frac{h'}{h}) \leq m(r, \frac{f_1'}{f_1}) + m(r, \frac{h'}{h}) \]

\[ = S(r, f) + O(\log r) = S(r, f), \]
We now consider the following cases.

(3.9) and using the second fundamental theorem for small functions (see [17]), we obtain

\[ N(r,0; [f_1^d P_2(f_1)]^{(k)}) \leq N(r,0;p^2) \leq S(r,f) . \]

Therefore

\[ T(r,\gamma^{(i)}) = S(r,f) , \]

where \( i = 1, 2, \ldots, k \). Since \( h \) are non-zero polynomial, it follows that \( T(r,t_i) = S(r,f) \), where \( i = 0, 1, \ldots, m \). Note that

\[ \bar{N}(r,0; t_0 + t_1 e^{\gamma} + \cdots + t_m e^{m \gamma}) \leq S(r,f) . \]

Now from (3.6) we have

\[ (3.9) \quad \bar{N}(r,0; t_0 + t_1 e^{\gamma} + \cdots + t_m e^{m \gamma}) \leq S(r,f) . \]

Since \( t_0 + t_1 e^{\gamma} + \cdots + t_m e^{m \gamma} \) is a transcendental entire function and \( t_0(z) \) is a polynomial, it follows that \( t_0 \) is a small function of \( t_0 + t_1 e^{\gamma} + \cdots + t_m e^{m \gamma} \). So from (3.9) and using the second fundamental theorem for small functions (see [17]), we obtain

\[
m T(r,f_1) = T(r,t_1 e^{\gamma} + \cdots + t_m e^{m \gamma}) + S(r,f_1) \\
\leq \bar{N}(r,0; t_m e^{m \gamma} + t_{m-1} e^{(m-1) \gamma} + \cdots + t_1 e^{\gamma}) \\
+ \bar{N}(r,0; t_m e^{m \gamma} + t_{m-1} e^{(m-1) \gamma} + \cdots + t_1 e^{\gamma} + t_0) + S(r,f_1) \\
\leq \bar{N}(r,0; t_m e^{(m-1) \gamma} + t_{m-1} e^{(m-2) \gamma} + \cdots + t_1) + S(r,f_1) \\
\leq (m-1)T(r,f_1) + S(r,f_1),
\]

which is a contradiction. Hence \( P_2(z_1) \) is reduced to a non-zero monomial, namely \( P_2(z_1) = c_i z_1^m \neq 0 \) for some \( i \in \{0, 1, \ldots, m\} \) and so \( P(z) \) takes the form \( P(z) = c_i(z - e)^{d+i} \neq 0 \) for some \( i \in \{0, 1, \ldots, m\} \). This proves the lemma. \( \square \)

**Lemma 10.** Let \( f, g \) be two transcendental meromorphic functions and let \( P(z) \) be defined as in (2.1). Let \( F = \frac{[P(f)]^{(k)}}{p} \), \( G = \frac{[P(g)]^{(k)}}{p} \), where \( p(z) \) is a non-zero polynomial and \( n, k \in \mathbb{N} \), \( m \in \mathbb{N} \cup \{0\} \) such that \( n > 3k + 2 \Gamma + 3 \). If \( f, g \) share \((\infty,0)\) and \( H \equiv 0 \), then either \([P(f)]^{(k)}[P(f)]^{(k)} \equiv p^2\), where \([P(f)]^{(k)}\) and \([P(g)]^{(k)}\) share \( p \) CM or \( P(f) \equiv P(g) \).

**Proof.** Since \( H \equiv 0 \), on integration we get

\[ \frac{1}{F - 1} = \frac{bG + a - b}{G - 1} , \]

where \( a, b \) are constants and \( a \neq 0 \). From (3.10), we see that \( F \) and \( G \) share 1 CM. We now consider the following cases.

**Case 1.** Let \( b \neq 0 \) and \( a \neq b \). If \( b = -1 \), then from (3.10) we have

\[ F = \frac{-a}{G - a - 1} . \]

Therefore

\[ \bar{N}(r,a + 1;G) = \bar{N}(r,\infty;F) = \bar{N}(r,\infty;f) + S(r,f) . \]
So in view of Lemmas 1 and 2 for \( p = 1 \) and using the second fundamental theorem we get
\[
\begin{align*}
\eta T(r, g) &= T(r, P(g)) + S(r, g) \\
&\leq T\left(r, [P(g)]^{(k)}\right) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; [P(g)]^{(k)}) + S(r, g) \\
&\leq T(r, G) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; G) + S(r, g) \\
&\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, a + 1; G) + N_{k+1}(r, 0; P(g)) \\
&\quad - \overline{N}(r, 0; G) + S(r, g) \\
&\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + N_{k+1}(r, 0; P(g)) + S(r, f) + S(r, g) \\
&\leq 2 \overline{N}(r, \infty; g) + (k + 1) \overline{N}(r, e; f) + \Gamma T(r, g) + S(r, f) + S(r, g) \\
&\leq (k + \Gamma + 3) T(r, g) + S(r, f) + S(r, g),
\end{align*}
\]
which is a contradiction since \( n > k + \Gamma + 3 \). If \( b \neq -1 \), from (3.10) we obtain that
\[
F - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2[G + \frac{a-b}{b}]}. 
\]
So
\[
\overline{N}\left(r, \frac{b-a}{b}; G\right) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f). 
\]
Using Lemmas 1, 2 and the same argument as used in the case when \( b = -1 \) we can get a contradiction.

**Case 2.** Let \( b \neq 0 \) and \( a = b \). If \( b = -1 \), then from (3.10) we have \( FG \equiv 1 \), i.e.,
\[
[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2,
\]
where \([P(f)]^{(k)}\) and \([P(g)]^{(k)}\) share \( p \) CM. If \( b \neq -1 \), from (3.10) we have
\[
\frac{1}{F} = \frac{bG}{(1+b)G-1}. 
\]
Therefore
\[
\overline{N}\left(r, \frac{1}{1+b}; G\right) = \overline{N}(r, 0; F). 
\]
So in view of Lemmas 1 and 2 for \( p = 1 \) and using the second fundamental theorem we get
\[
\begin{align*}
\eta T(r, g) &= T(r, P(g)) + S(r, g) \\
&\leq T\left(r, [P(g)]^{(k)}\right) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; [P(g)]^{(k)}) + S(r, g) \\
&\leq T(r, G) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; G) + S(r, g) \\
&\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+b}; G\right) + N_{k+1}(r, 0; P(g)) \\
&\quad - \overline{N}(r, 0; G) + S(r, g) \\
&\leq (k + 1) \overline{N}(r, e; f) + \Gamma T(r, g) + S(r, f) + S(r, g) \\
&\leq (k + \Gamma + 3) T(r, g) + S(r, f) + S(r, g),
\end{align*}
\]
which is a contradiction since \( n > k + \Gamma + 3 \). If \( b \neq -1 \), from (3.10) we obtain that
\[
F - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2[G + \frac{a-b}{b}]}. 
\]
So
\[
\overline{N}\left(r, \frac{b-a}{b}; G\right) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f). 
\]
Without loss of generality, we suppose that there exists a set \( I \) such that all zeros of functions in \( I \) have multiplicity greater than or equal to \( 3 \), and set \( S(r) \) and set \( N(r, \infty; g) \) for \( r \in I \). So for \( r \in I \) we have

\[
(n - 3k - 2 \Gamma - 3) T(r, g) \leq S(r, g),
\]

which is a contradiction since \( n > 3k + 2 \Gamma + 3 \).

**Case 3.** Let \( b = 0 \). From (3.10) we obtain

\[
F = \frac{G + a - 1}{a}.
\]

If \( a \neq 1 \) then from (3.11) we obtain \( N(r, 1 - a; G) = N(r, 0; F) \). We can similarly deduce a contradiction as in Case 2. Therefore \( a = 1 \) and from (3.11) we obtain \( F \equiv G \), i.e., \( [P(f)]^{(k)} = [P(g)]^{(k)} \). Then by Lemma 8 we have \( P(f) \equiv P(g) \). This completes the proof. \( \square \)

**Lemma 11** ([17, Lemma 3.5]). Suppose that \( F \) is meromorphic in a domain \( D \) and set \( f = \frac{F'}{F} \). Then for \( n \geq 1 \),

\[
\frac{F^{(n)}}{F} = f^n + \frac{n(n - 1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} (f')^2 + P_{n-3}(f),
\]

where \( a_n = \frac{1}{6} n(n - 1)(n - 2) \), \( b_n = \frac{1}{8} n(n - 1)(n - 2)(n - 3) \) and \( P_{n-3}(f) \) is a differential polynomial with constant coefficients, which vanishes identically for \( n \leq 3 \) and has degree \( n - 3 \) when \( n > 3 \).

**Lemma 12** ([3]). Let \( f \) be a meromorphic function on \( \mathbb{C} \) with finitely many poles. If \( f \) has bounded spherical derivative on \( \mathbb{C} \), then \( f \) is of order at most 1.

**Lemma 13** ([20, Theorem 2.11]). Let \( f \) be a transcendental meromorphic function in the complex plane such that \( \rho(f) > 0 \). If \( f \) has two distinct Borel exceptional values in the extended complex plane, then \( \mu(f) = \rho(f) \) and \( \rho(f) \) is a positive integer or \( \infty \).

**Lemma 14** ([22]). Let \( F \) be a family of meromorphic functions in the unit disc \( \Delta \) such that all zeros of functions in \( F \) have multiplicity greater than or equal to \( l \) and all poles of functions in \( F \) have multiplicity greater than or equal to \( j \) and \( \alpha \) be a real number satisfying \(- l < \alpha < j \). Then \( F \) is not normal in any neighborhood of \( z_0 \in \Delta \), if and only if there exist

(i) points \( z_n \in \Delta \), \( z_n \to z_0 \),

(ii) positive numbers \( \rho_n \), \( \rho_n \to 0^+ \) and

(iii) functions \( f_n \in F \),

such that \( \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta) \) spherically locally uniformly in \( \mathbb{C} \), where \( g \) is a non-constant meromorphic function. The function \( g \) may be taken to satisfy the normalisation \( g^\#(\zeta) \leq g^\#(0) = 1(\zeta \in \mathbb{C}) \).
Remark 3. Suppose in Lemma 14 that $F$ is a family of holomorphic functions in the domain $D$ and there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$, whenever $f = 0$. Then the real number $\alpha$ in Lemma 14 can be such that $0 \leq \alpha \leq k$. In that case also $f_n(z_n + \rho_n \zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where $g$ is a non-constant holomorphic function. The function $g$ may be taken to satisfy the normalisation $g^\#(\zeta) \leq g^\#(0) = kA + 1(\zeta \in \mathbb{C})$.

Lemma 15 (20). Let $f_j$ $(j = 1, 2, 3)$ be meromorphic functions, where $f_1$ be non-constant. Suppose that

$$\sum_{j=1}^{3} f_j \equiv 1$$
and

$$\sum_{j=1}^{3} N(r, 0; f_j) + 2 \sum_{j=1}^{3} N(r, \infty; f_j) < (\lambda + o(1))T(r),$$
as $r \to +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1 \leq j \leq 3} T(r, f_j)$. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 16. Let $f$, $g$ be two transcendental entire functions such that $f$ and $g$ have no zeros and $p$ be a non-constant polynomial. Suppose $(f^n)'(g^n)' \equiv p^2$, where $n \in \mathbb{N}$. Now

(i) if $p(z)$ is not a constant, then $f(z) = d_1 e^{Q(z)}$, $g(z) = d_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t) dt$, $d_1$, $d_2$ and $c$ are constants such that $(nc)^2(d_1 d_2)^n = -1$,

(ii) if $p(z)$ is a non-zero constant, say $b$, then $f(z) = d_3 e^{cz}$, $g(z) = d_4 e^{-cz}$, where $d_3$, $d_4$ and $c$ are constants such that $(-1)^k d_3 d_4^n (nc)^{2k} = b^2$.

Proof. Proof of lemma follows from proof of Theorem 1.3 [24].

Lemma 17. Let $f$, $g$ be two transcendental meromorphic functions such that $f$, $g$ share $(\infty, 0)$ and $p$ be a non-zero polynomial. Let $n$, $k \in \mathbb{N}$ such that $n > k$. Suppose $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p(z)$ and $(g^n)^{(k)} - p(z)$ share 0 CM. Now

(i) if $p(z)$ is not a constant, then $f(z) = d_1 e^{Q(z)}$, $g(z) = d_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t) dt$, $d_1$, $d_2$ and $c$ are constants such that $(nc)^2(d_1 d_2)^n = -1$,

(ii) if $p(z)$ is a non-zero constant, say $b$, then $f(z) = d_3 e^{cz}$, $g(z) = d_4 e^{-cz}$, where $d_3$, $d_4$ and $c$ are constants such that $(-1)^k d_3 d_4^n (nc)^{2k} = b^2$.

Proof. Suppose

\begin{equation}
(3.12) \quad (f^n)^{(k)}(g^n)^{(k)} \equiv p^2.
\end{equation}

Since $f$ and $g$ share $(\infty, 0)$, from (3.12) one can easily say that $f$ and $g$ are transcendental entire functions. Let

\begin{equation}
(3.13) \quad F_1 = \frac{(f^n)^{(k)}}{p} \quad \text{and} \quad G_1 = \frac{(g^n)^{(k)}}{p}.
\end{equation}

From (3.12) we get

\begin{equation}
(3.14) \quad F_1 G_1 \equiv 1.
\end{equation}
If $F_1 \equiv c^*_1 G_1$, where $c^*_1 \in \mathbb{C} \setminus \{0\}$, then from (3.14) we have $F_1$ is a constant and so $f$ is a polynomial, which contradicts our assumption. Hence $F_1 \neq c^*_1 G_1$.

Let

$$
(3.15) \quad \Phi = \frac{(f^n)^{(k)} - p}{(g^n)^{(k)} - p}.
$$

We deduce from (3.15) that

$$
(3.16) \quad \Phi \equiv e^{\gamma},
$$

where $\gamma_1$ is an entire function. Let $f_1 = F_1$, $f_2 = -e^{\gamma_1} G_1$ and $f_3 = e^{\gamma_1}$. Here $f_1$ is transcendental. Now from (3.15) and (3.16), we have

$$
f_1 + f_2 + f_3 \equiv 1.
$$

Hence by Lemma [6] we get

$$
\sum_{j=1}^{3} N(r, 0; f_j) + 2 \sum_{j=1}^{3} N(r, \infty; f_j) \leq N(r, 0; F_1) + N(r, 0; e^{\gamma_1} G_1) + O(\log r)
$$

$$
\leq (\lambda + o(1)) T(r),
$$

as $r \to +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1 \leq j \leq 3} T(r, f_j)$.

So by Lemma [15] we get either $e^{\gamma_1} G_1 \equiv -1$ or $e^{\gamma_1} \equiv 1$. But here the only possibility is that $e^{\gamma_1} G_1 \equiv -1$, i.e., $(g^n)^{(k)} \equiv -e^{-\gamma_1} p(z)$ and so from (3.12) we get

$$
(3.17) \quad (f^n)^{(k)} \equiv c^*_2 e^{\gamma_1} p, \quad (g^n)^{(k)} \equiv c^*_2 e^{-\gamma_1} p,
$$

where $c^*_2 = \pm 1$. This shows that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM. Let $z_p$ be a zero of $f(z)$ of multiplicity $p$ and $z_q$ be a zero of $g(z)$ of multiplicity $q$. Since $n > k$, it follows that $z_p$ will be a zero of $(f^n(z))^{(k)}$ of multiplicity $np - k$ and $z_q$ will be a zero of $(g^n(z))^{(k)}$ of multiplicity $nq - k$. Since $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 0 CM, it follows that $z_p = z_q$ and $p = q$. Consequently $f(z)$ and $g(z)$ share 0 CM. Since $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$, so we can take

$$
(3.18) \quad f(z) = h_1(z)e^{\alpha(z)}, \quad g(z) = h_1(z)e^{\beta(z)},
$$

where $h_1$ is a non-zero polynomial and $\alpha$, $\beta$ are two non-constant entire functions. We consider the following cases.

**Case 1.** Suppose 0 is a Picard exceptional value of both $f$ and $g$.

We now consider the following sub-cases.

**Sub-case 1.1.** Let $\deg(p) = l \in \mathbb{N}$.

Since $N(r, 0; f) = 0$ and $N(r, 0; g) = 0$, so we can take

$$
(3.19) \quad f(z) = e^{\alpha(z)}, \quad g(z) = e^{\beta(z)},
$$

where $\alpha$ and $\beta$ are two non-constant entire functions.

We deduce from (3.12) and (3.19) that either both $\alpha$ and $\beta$ are transcendental entire functions or both are polynomials. We consider the following sub-cases.
Sub-case 1.1.1. Let $k \in \mathbb{N} \setminus \{1\}$.
First we suppose both $\alpha$ and $\beta$ are transcendental entire functions. Note that
\[ S(r, n\alpha') = S(r, \frac{(f^n)'}{f^n}), \quad S(r, n\beta') = S(r, \frac{(g^n)'}{g^n}). \]
Moreover we see that
\[ N(r, 0; (f^n)(k)) \leq N(r, 0; p^2) = O(\log r). \]
\[ N(r, 0; (g^n)(k)) \leq N(r, 0; p^2) = O(\log r). \]
From these and using (3.19) we have
\[ (3.20) \quad N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; (f^n)(k)) = S(r, n\alpha') = S(r, \frac{(f^n)'}{f^n}) \]
and
\[ (3.21) \quad N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; (g^n)(k)) = S(r, n\beta') = S(r, \frac{(g^n)'}{g^n}). \]
Then from (3.20), (3.21) and Lemma 4 we must have
\[ (3.22) \quad f(z) = e^{a_3z+b_3}, \quad g(z) = e^{c_3z+d_3}, \]
where $a_3 \neq 0$, $b_3$, $c_3 \neq 0$ and $d_3$ are constants. But these types of $f$ and $g$ do not agree with the relation (3.12).
Next we suppose $\alpha$ and $\beta$ are both non-constant polynomials.
Also from (3.12) we get $\alpha + \beta \equiv C_1$, i.e., $\alpha' \equiv -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$.
If $\deg(\alpha) = \deg(\beta) = 1$, then we again get a contradiction from (3.12). Next we suppose $\deg(\alpha) = \deg(\beta) \geq 2$. Now from (3.19) and Lemma 11 we see that
\[ (f^n)(k) = [n^k(\alpha')k + \frac{k(k - 1)}{2}n^{k-1}(\alpha')^{k-2}\alpha'' + P_{k-1}(\alpha')]e^{n\alpha}. \]
Similarly we have
\[ (g^n)(k) = [n^k(\beta')k + \frac{k(k - 1)}{2}n^{k-1}(\beta')^{k-2}\beta'' + P_{k-1}(\beta')]e^{n\beta} \]
\[ = \left[(-1)^kn^k(\alpha')k - \frac{k(k - 1)}{2}n^{k-1}(-1)^{k-2}(\alpha')^{k-2}\alpha'' + P_{k-1}(-\alpha')\right]e^{n\alpha}. \]
Since $\deg(\alpha) \geq 2$, we observe that $\deg((\alpha')^k) \geq k \deg(\alpha')$ and so $(\alpha')^{k-2}\alpha''$ is either a non-zero constant or $\deg((\alpha')^{k-2}\alpha'') \geq (k - 1) \deg(\alpha') - 1$. Also we see that
\[ \deg ((\alpha')^k) > \deg ((\alpha')^{k-2}\alpha'') > \deg (P_{k-2}(\alpha')) \quad (\text{or} \quad \deg (P_{k-2}(-\alpha'))). \]
Let
\[ [\alpha(z)]' = e_{1\ell}z^\ell + e_{1\ell-1}z^{\ell-1} + \cdots + e_{10}, \]
where $e_{1\ell} \in \mathbb{C} \setminus \{0\}$. Then we have
\[ ([\alpha(z)]')^i = e_{1\ell}^iz^{\ell i} + i e_{1\ell}^{i-1} e_{1\ell-1}z^{\ell i-1} + \cdots, \]
where \( i \in \mathbb{N} \). Therefore we have
\[
(f^n)^{(k)} = [n^k e^{k} e^{kt} + nk^k e^{k-1} e^{1t-1} z^{k-1} + \cdots + (D_1 + D_2) z^{k-t-1} + \cdots] e^{n\alpha}
\]
and
\[
(g^n)^{(k)} = [(-1)^k n^k e^{k} e^{kt} + (-1)^k kn^k e^{k-1} e^{1t-1} z^{k-1} + \cdots + ((-1)^k D_1 + (-1)^{k-1} D_2) z^{k-t-1} + \cdots] e^{n\beta},
\]
where \( D_1, D_2 \in \mathbb{C} \) such that \( D_2 = \frac{k(k-1)}{2} t n^k e^{k-1} e^{1t} \). Since \((f^n)^{(k)}\) and \((g^n)^{(k)}\) share 0 CM, we have
\[
n^k e^{k} e^{kt} + nk^k e^{k-1} e^{1t-1} z^{k-1} + \cdots + (D_1 + D_2) z^{k-t-1} + \cdots
\]
\[
d_1^* \{((-1)^k n^k e^{k} e^{kt} + (-1)^k kn^k e^{k-1} e^{1t-1} z^{k-1} + \cdots + ((-1)^k D_1 + (-1)^{k-1} D_2) z^{k-t-1} + \cdots)\}
\]
where \( d_1^* \in \mathbb{C} \setminus \{0\} \). From (3.23), we get \( D_2 = 0 \), i.e.,
\[
\frac{k(k-1)}{2} t n^k e^{k-1} e^{1t} = 0,
\]
which is impossible for \( k \geq 2 \).

**Sub-case 1.1.2.** Let \( k = 1 \). Remaining part follows from Lemma 16.

**Sub-case 1.2.** Let \( p(z) = b \in \mathbb{C} \setminus \{0\} \). Since \( n > 2k \), we have \( f \neq 0 \) and \( g \neq 0 \). Now using Sub-case 1.1 we can prove that \( f = e^\alpha \) and \( g = e^\beta \), where \( \alpha \) and \( \beta \) are non-constant entire functions. We now consider the following two sub-cases.

**Sub-case 1.2.1.** Let \( k \geq 2 \). We see that \( N(r, 0; (f^n)^{(k)}) = 0 \). Clearly
\[
(f^n)^{(k)}(z) \neq 0, \quad (g^n)^{(k)}(z) \neq 0.
\]
Then from (3.24) and Lemma 5, we must have \( f(z) = e^{az+b} \), \( g(z) = e^{cz+d} \), where \( a \neq 0, b, c \neq 0 \) and \( d \) are constants. From (3.12), it is clear that \( a + c = 0 \). Therefore \( f \) and \( g \) take the forms \( f(z) = d_3 e^{cz}, g(z) = d_4 e^{-cz} \), where \( d_3, d_4, c \in \mathbb{C} \) such that \((-1)^k (d_3 d_4) (nc)^2 k = b^2 \).

**Sub-case 1.2.2.** Let \( k = 1 \). Remaining part follows from Lemma 16.

**Case 2.** Suppose 0 is not a Picard exceptional value of \( f \) and \( g \).

Let \( H = f^n, \hat{H} = g^n, F = \frac{H}{p} \) and \( G = \frac{\hat{H}}{p} \). Let \( \mathcal{F} = \{F_\omega\} \) and \( \mathcal{G} = \{G_\omega\} \), where \( F_\omega(z) = F(z + \omega) = \frac{H(z + \omega)}{p(z + \omega)} \) and \( G_\omega(z) = G(z + \omega) = \frac{\hat{H}(z + \omega)}{p(z + \omega)} \), \( z \in \mathbb{C} \). Clearly \( \mathcal{F} \) and \( \mathcal{G} \) are two families of meromorphic functions defined on \( \mathbb{C} \). We now consider following two sub-cases.

**Sub-case 2.1.** Suppose that one of the families \( \mathcal{F} \) and \( \mathcal{G} \), say \( \mathcal{F} \), is normal on \( \mathbb{C} \). Then by Marty’s theorem \( F_\omega^\#(0) \leq M \) for some \( M > 0 \) and for all \( \omega \in \mathbb{C} \). Hence by Lemma 12 we have \( F \) is of order at most 1. Now from (3.12) we have
\[
\rho(f) = \rho\left(\frac{f^n}{p}\right) = \rho(f^n) = \rho((f^n)^{(k)}) = \rho((g^n)^{(k)})
\]
\[
= \rho(g^n) = \rho\left(\frac{g^n}{p}\right) = \rho(g) \leq 1.
\]
Noting that \( f \) and \( g \) are transcendental entire functions, we observe from (3.25) and Lemma 13 that \( \mu(f) = \rho(f) = 1 \). Now from (3.18) we have
\[
(3.26) \quad f = h_1e^{\alpha}, \quad g = h_1e^{\beta},
\]
where \( \alpha \) and \( \beta \) are non-constant polynomials with degree 1. From (3.12) we see that \( \alpha + \beta \equiv C_1 \) where \( C_1 \) is a constant and so \( \alpha' + \beta' \equiv 0 \). Again from (3.26) we have
\[
\left( f^n(z) \right)^{(k)} = e^{n\alpha} \sum_{i=0}^{k} kC_i(n\alpha')^{k-i} \left( h_1^{n}(z) \right)^{(i)},
\]
where we define \( \left( h_1^{n}(z) \right)^{(0)} = h_1^{n}(z) \). Similarly we have
\[
\left( g^n(z) \right)^{(k)} = e^{n\beta} \sum_{i=0}^{k} kC_i(-1)^{k-i}(n\alpha')^{k-i} \left( h_1^{n}(z) \right)^{(i)}.
\]
Since \( (f^n)^{(k)} \) and \( (g^n)^{(k)} \) share 0 CM, it follows that
\[
(3.27) \quad \sum_{i=0}^{k} kC_i(n\alpha')^{k-i} \left( h_1^{n}(z) \right)^{(i)} \equiv d_2^{n} \sum_{i=0}^{k} kC_i(-1)^{k-i}(n\alpha')^{k-i} \left( h_1^{n}(z) \right)^{(i)},
\]
where \( d_2 \in \mathbb{C} \setminus \{0\} \). But from (3.27) we arrive at a contradiction.

**Sub-case 2.2.** Suppose that one of the families \( \mathcal{F} \) and \( \mathcal{G} \), say \( \mathcal{F} \) is not normal on \( \mathbb{C} \). Then there exists at least one \( z_0 \in \Delta \) such that \( \mathcal{F} \) is not normal \( z_0 \), we assume that \( z_0 = 0 \). Now by Marty’s theorem there exists a sequence of meromorphic functions \( \{F(z + \omega_j)\} \subset \mathcal{F} \), where \( z \in \{z : |z| < 1\} \) and \( \{\omega_j\} \subset \mathbb{C} \) is some sequence of complex numbers such that
\[
F^\#(\omega_j) \to \infty,
\]
as \( |\omega_j| \to \infty \). Note that \( p \) has only finitely many zeros. So there exists a \( r > 0 \) such that \( p(z) \neq 0 \) in \( D = \{z : |z| \geq r\} \). Since \( p(z) \) is a polynomial, for all \( z \in \mathbb{C} \) satisfying \( |z| \geq r \), we have
\[
(3.28) \quad 0 \leftarrow \left| \frac{p'(z)}{p(z)} \right| \leq \frac{M_1}{|z|} < 1, \quad p(z) \neq 0.
\]
Also since \( w_j \to \infty \) as \( j \to \infty \), without loss of generality we may assume that \( |w_j| \geq r + 1 \) for all \( j \). Let \( D_1 = \{z : |z| < 1\} \) and
\[
F(w_j + z) = \frac{H(w_j + z)}{p(w_j + z)}.
\]
Since \( |w_j + z| \geq |w_j| - |z| \), it follows that \( w_j + z \in D \) for all \( z \in D_1 \). Also since \( p(z) \neq 0 \) in \( D \), it follows that \( p(w_j + z) \neq 0 \) in \( D_1 \) for all \( j \). Observing that \( F(z) \) is analytic in \( D \), so \( F(w_j + z) \) is analytic in \( D_1 \). Therefore all \( F(w_j + z) \) are analytic in \( D_1 \). Also from (3.17) we see that every zeros of \( h_1(z) \) must be the zeros of \( p(z) \). Thus we have structured a family \( \{F(\omega_j + z)\} \) of holomorphic functions such that
\[
F(\omega_j + z) \neq 0 \text{ in } D_1 \text{ for all } j.
\]
Then by Lemma 14 there exist
(i) points \( z_j, |z_j| < 1 \).
(ii) positive numbers \( \rho_j, \rho_j \to 0^+ \),

(iii) a subsequence \( \{F(\omega_j + z_j + \rho_j \zeta)\} \) of \( \{F(\omega_j + z)\} \)
such that

\[
h_j(\zeta) = \rho_j^{-k}F(\omega_j + z_j + \rho_j \zeta) \to h(\zeta),
\]
i.e.,

\[
h_j(\zeta) = \rho_j^{-k}H(\omega_j + z_j + \rho_j \zeta) \to h(\zeta)
\]
spherically locally uniformly in \( \mathbb{C} \), where \( h(\zeta) \) is some non-constant holomorphic
function such that \( h^\#(\zeta) \leq h^\#(0) = 1 \). Now from Lemma 12 we see that \( \rho(h) \leq 1 \). By Hurwitz’s theorem we can see that \( h(\zeta) \neq 0 \). In the proof of Zalcman’s lemma (see [14, 21]) we see that

\[
\rho_j = \frac{1}{F^\#(b_j)}
\]
and

\[
F^\#(b_j) \geq F^\#(\omega_j),
\]
where \( b_j = \omega_j + z_j \). Note that

\[
\frac{p'(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to 0,
\]
as \( j \to \infty \). We now prove that

\[
(h_j(\zeta))^{(k)} = \frac{H^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to h^{(k)}(\zeta).
\]
Note that from (3.29)

\[
\rho_j^{-k+1}H'(\omega_j + z_j + \rho_j \zeta) = h'_j(\zeta) + \rho_j^{-k}p'(\omega_j + z_j + \rho_j \zeta)H(\omega_j + z_j + \rho_j \zeta)
\]

\[
= h'_j(\zeta) + \rho_j^{p'(\omega_j + z_j + \rho_j \zeta)}h_j(\zeta).
\]
Now from (3.29), (3.32) and (3.34) we observe that

\[
\rho_j^{-k+1}H'(\omega_j + z_j + \rho_j \zeta) \to h'(\zeta).
\]
Suppose

\[
\rho_j^{-k+l}H^{(l)}(\omega_j + z_j + \rho_j \zeta) \to h^{(l)}(\zeta).
\]
Let

\[
G_j(\zeta) = \rho_j^{-k+l}H^{(l)}(\omega_j + z_j + \rho_j \zeta).
\]
Then

\[ G_j(\zeta) \to h^{(l)}(\zeta). \]

Note that

\[
\rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} = G_j^{(l)}(\zeta) + \rho_j^{-k+l+1} \frac{p'(\omega_j + z_j + \rho_j \zeta)}{p^2(\omega_j + z_j + \rho_j \zeta)} H^{(l)}(\omega_j + z_j + \rho_j \zeta)
\]

(3.35)

\[ = G_j^{(l)}(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} G_j(\zeta). \]

So from (3.32) and (3.35) we see that

\[
\rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to G_j^{(l)}(\zeta),
\]

i.e.,

\[
\rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to h_j^{(l+1)}(\zeta).
\]

Then by mathematical induction we get the desired result (3.33). Let

(3.36)

\[
\left(\hat{h}_j(\zeta)\right)^{(k)} = \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)}.
\]

From (3.12) we have

\[
\frac{H^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} = 1
\]

and so from (3.33) and (3.36) we get

(3.37)

\[
\left(h_j(\zeta)\right)^{(k)} \left(\hat{h}_j(\zeta)\right)^{(k)} = 1.
\]

Now from (3.33), (3.37) and the formula of higher derivatives we can deduce that

\[ \hat{h}_j(\zeta) \to \hat{h}(\zeta) \]

i.e.,

(3.38)

\[ \frac{\hat{H}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to \hat{h}(\zeta), \]

spherically locally uniformly in \( \mathbb{C} \), where \( \hat{h}(\zeta) \) is some non-constant holomorphic function in the complex plane. By Hurwitz’s theorem we can see that \( \hat{h}(\zeta) \neq 0 \). Therefore (3.38) can be rewritten as

(3.39)

\[ \left(\hat{h}_j(\zeta)\right)^{(k)} \to \left(\hat{h}(\zeta)\right)^{(k)} \]

spherically locally uniformly in \( \mathbb{C} \). From (3.33), (3.37) and (3.39) we get

(3.40)

\[ \left(h(\zeta)\right)^{(k)} \left(\hat{h}(\zeta)\right)^{(k)} \equiv 1. \]

Now from (3.40) and \( \rho(h) \leq 1 \) we see that

(3.41)

\[ \rho(h) = \rho(h^{(k)}) = \rho(\hat{h}^{(k)}) = \rho(\hat{h}) \leq 1. \]
Noting that \( h \) and \( \hat{h} \) are transcendental entire functions, we observe from (3.41) and Lemma 13 that \( \mu(h) = \rho(\hat{h}) = 1 \). Therefore we have

\[
(3.42) \quad h(z) = c_1 e^{cz}, \quad \hat{h}(z) = \hat{c}_2 e^{-cz},
\]

where \( c_1, \hat{c}_2 \) and \( c \) are non-zero constants satisfying \((-1)^k(c_1 \hat{c}_2)(c)^{2k} = 1 \). Also from (3.42) we have

\[
(3.43) \quad \frac{h_j'(\zeta)}{h_j(\zeta)} = \rho_j \frac{F'(w_j + z_j + \rho_j \zeta)}{F(w_j + z_j + \rho_j \zeta)} \rightarrow \frac{h'(\zeta)}{h(\zeta)} = c,
\]

spherically locally uniformly in \( \mathbb{C} \). From (3.30) and (3.43) we get

\[
\rho_j \left| \frac{F'(w_j + z_j)}{F(w_j + z_j)} \right| = \frac{1 + |F'(w_j + z_j)|}{|F'(w_j + z_j)|} \rightarrow \lim_{j \to \infty} \frac{|h'(0)|}{|h(0)|} = |c|,
\]

which implies that

\[
(3.44) \quad \lim_{j \to \infty} F(w_j + z_j) \neq 0, \; \infty.
\]

From (3.29) and (3.44) we see that

\[
(3.45) \quad h_j(0) = \rho_j^{-k} F(w_j + z_j) \to \infty.
\]

Again from (3.29) and (3.42) we have

\[
(3.46) \quad h_j(0) \to h(0) = c_1.
\]

Now from (3.45) and (3.46) we arrive at a contradiction. This completes the lemma.

\[\square\]

**Lemma 18.** Let \( f \) and \( g \) be two transcendental meromorphic functions and let \( d(\geq 1), m(\geq 0), k(\geq 1) \) be three integers such that \( d > k \). Let \( P(z) \) be defined as in (2.1) and \( p(z) \) be a non-zero polynomial. Suppose \([P(f)]^{(k)}[P(g)]^{(k)} = p^2\), where \([P(f)]^{(k)}, [P(g)]^{(k)}\) share \( p \) CM and \( f, g \) share \((\infty, 0)\), then \( P_2(z_1) = c_i z_1^k \neq 0 \) for some \( i \in \{0, 1, \ldots, m\} \) and so \( P(z) \) takes the form \( P(z) = c_i (z - e)^{d+i} \neq 0 \) for some \( i \in \{0, 1, \ldots, m\} \); if \( p(z) \) is not a constant, then \( f(z) - e = d_1 e^{cz} Q(z), \; g(z) - e = d_2 e^{-cz} Q(z) \), where \( Q(z) = \int_0^z p(t)dt, \; d_1, d_2 \) and \( c \) are constants such that \( c_i^2 (d_1 d_2)^{d+i}((d+i)e)^2 = -1 \), if \( p(z) \) is a non-zero constant, say \( b \), then \( f(z) - e = d_3 e^{cz}, \; g(z) - e = d_4 e^{-cz} \), where \( d_3, d_4 \) and \( c \) are constants such that \((-1)^k c_i^2(d_3 d_4)^{d+i}((d+i)e)^{2k} = b^2\).

**Proof.** The proof of lemma follows from Lemmas 9 and 17. \[\square\]

**Lemma 19 (1).** Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \((1, k_1)\), where \( 2 \leq k_1 \leq \infty \). Then

\[
\overline{N}(r, 1; f |\equiv 2) + 2 \overline{N}(r, 1; f |= 3) + \cdots + (k_1 - 1) \overline{N}(r, 1; f |\equiv k_1) + k_1 \overline{N}_L(r, 1; f)
\]

\[
+ (k_1 + 1) \overline{N}_L(r, 1; g) + k_1 \overline{N}_E^{(k_1+1)}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g).
\]
Lemma 20. Suppose that $f$ and $g$ be two non-constant meromorphic functions. Let $F = [P(f)]^{(k)}$, $G = [P(g)]^{(k)}$, where $n$, $k \in \mathbb{N}$ and $P(z)$ be defined as in (2.1). Suppose $H \not\equiv 0$. If $f$, $g$ share $(\infty, 0)$ and $F$, $G$ share $(1, k_1)$, where $0 \leq k_1 \leq \infty$ then
\[(n - k - 1) \overline{N}(r, \infty; f) \leq (k + \Gamma + 1) \{T(r, f) + T(r, g)\} + \overline{N}_s(r, 1; F, G) + S(r, f) + S(r, g).\]

Proof. If $\infty$ is a Picard exceptional value of $f$ and $g$, then the result follows immediately.

Next we suppose $\infty$ is not a Picard exceptional value of $f$ and $g$. Since $H \not\equiv 0$, it follows that $F \not\equiv G$. We claim that $V \not\equiv 0$. If possible suppose $V \equiv 0$. Then by integration we obtain
\[1 - \frac{1}{F} = A \left(1 - \frac{1}{G}\right).\]

Note that if $z^*_1$ is a pole of $f$ then it is a pole of $g$. Hence from the definition of $F$ and $G$ we have $1/F(z^*_1) = 0$ and $1/G(z^*_1) = 0$. So $A = 1$ and hence $F \equiv G$, which is a contradiction.

We suppose that $z_0$ is a pole of $f$ with multiplicity $q$ and a pole of $g$ with multiplicity $r$. Clearly $z_0$ is a pole of $F$ with multiplicity $nq + k$ and a pole of $G$ with multiplicity $nr + k$. Clearly $F'(z)/(F(z) - 1) = O((z - z_0)^{nq+k-1})$ and $G'(z)/(G(z) - 1) = O((z - z_0)^{nr+k-1})$. Consequently, $V = O((z - z_0)^{nt+k-1})$, where $t = \min\{q, r\}$. Noting that $f$, $g$ share $(\infty, 0)$, from the definition of $V$ it is clear that $z_0$ is a zero of $V$ with multiplicity at least $n + k - 1$. Now using the Milloux theorem [7, p. 55], and Lemma 1, we obtain from the definition of $V$ that $m(r, V) = S(r, f) + S(r, g)$. Thus using Lemma 1 and (3.4) we get
\[(n + k - 1) \overline{N}(r, \infty; f) \leq N(r, 0; V) \leq T(r, V) + O(1) \leq N(r, \infty; V) + m(r, V) + O(1)\leq N(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_s(r, 1; F, G) + S(r, f) + S(r, g)\leq N_{k+1}(r, 0; P(f)) + N_{k+1}(r, 0; P(g)) + k \overline{N}(r, \infty; f) + k \overline{N}(r, \infty; g) + \overline{N}_s(r, 1; F, G) + S(r, f) + S(r, g)\leq N_{k+1}(r, 0; P(f)) + N_{k+1}(r, 0; P(g)) + 2k \overline{N}(r, \infty; f) + \overline{N}_s(r, 1; F, G) + S(r, f) + S(r, g)\leq (k + \Gamma + 1) T(r, f) + (k + \Gamma + 1) T(r, g) + 2k \overline{N}(r, \infty; f) + \overline{N}_s(r, 1; F, G) + S(r, f) + S(r, g)\]

This gives
\[(n - k - 1) \overline{N}(r, \infty; f) \leq (k + \Gamma + 1) \{T(r, f) + T(r, g)\} + \overline{N}_s(r, 1; F, G) + S(r, f) + S(r, g).\]

This completes the proof. □
4. Proofs of the Theorem

**Proof of Theorem** Let \( F = \frac{[P(f)]^{(k)}}{p} \) and \( G = \frac{[P(g)]^{(k)}}{p} \). Note that since \( f \) and \( g \) are transcendental meromorphic functions, \( p \) is a small function with respect to both \([P(f)]^{(k)}\) and \([P(g)]^{(k)}\). Also \( F, G \) share \((1, k_1)\) except for the zeros of \( p \) and \( f, g \) share \((\infty, 0)\).

**Case 1.** Let \( H \neq 0 \).

From (3.1) it can be easily calculated that the possible poles of \( H \) occur at (i) multiple zeros of \( F \) and \( G \), (ii) those 1 points of \( F \) and \( G \) whose multiplicities are different, (iii) those poles of \( F \) and \( G \) whose multiplicities are different, (iv) zeros of \( F'(G') \) which are not the zeros of \( F(F-1)(G(G-1)) \).

Since \( H \) has only simple poles we get

\[
N(r, \infty; H) \leq N_*(r, \infty; F, G) + N_*(r, 1; F, G) + N(r, 0; F \geq 2)
\]

\[(4.1) + N(r, 0; G \geq 2) + N_0(r, 0; F') + N_0(r, 0; G') + S(r, f) + S(r, g),
\]

where \( N_0(r, 0; F') \) is the reduced counting function of those zeros of \( F' \) which are not the zeros of \( F(F-1) \) and \( N_0(r, 0; G') \) is similarly defined.

Let \( z_0 \) be a simple zero of \( F(z) - 1 \) but \( p(z_0) \neq 0 \). Then \( z_0 \) is a simple zero of \( G - 1 \) and a zero of \( H \). So

\[
N(r, 1; F = 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, f) + S(r, g).
\]

Using (4.1) and (4.2) we get

\[
N(r, 1; F) \leq N(r, 1; F = 1) + N(r, 1; F \geq 2)
\]

\[
\leq N_*(r, \infty; f, g) + N(r, 0; F \geq 2) + N(r, 0; G \geq 2) + N_*(r, 1; F, G)
\]

\[+ N(r, 1; F \geq 2) + N_0(r, 0; F') + N_0(r, 0; G') + S(r, f) + S(r, g)
\]

\[\leq N(r, \infty; f) + N(r, 0; F \geq 2) + N(r, 0; G \geq 2) + N_*(r, 1; F, G)
\]

\[\quad + N_0(r, 0; F') + N_0(r, 0; G') + N(r, 0; G') + S(r, f) + S(r, g).
\]

Now in view of Lemmas 10 and 13 we get

\[
N_0(r, 0; G') + N(r, 1; F \geq 2) + N_*(r, 1; F, G)
\]

\[\leq N_0(r, 0; G') + N(r, 1; F = 2) + N(r, 1; F = 3) + \cdots + N(r, 1; F = k_1)
\]

\[+ N_{E}^{(k_1+1)}(r, 1; F) + N_{L}(r, 1; F) + N_{L}(r, 1; G) + N_*(r, 1; F, G)
\]

\[\leq N_0(r, 0; G') - N(r, 1; F = 3) - \cdots - (k_1 - 2)N(r, 1; F = k_1)
\]

\[- (k_1 - 1)N_{L}(r, 1; F) - k_1N_{L}(r, 1; G) - (k_1 - 1)N_{E}^{(k_1+1)}(r, 1; F)
\]

\[+ N(r, 1; G) - N(r, 1; G) + N_*(r, 1; F, G)
\]

\[\leq N_0(r, 0; G') + N(r, 1; G) - N(r, 1; G) - (k_1 - 2)N_{L}(r, 1; F)
\]

\[- (k_1 - 1)N_{L}(r, 1; G)
\]

\[\leq N(r, 0; G' \mid G \neq 0) - (k_1 - 2)N_{L}(r, 1; F) - (k_1 - 1)N_{L}(r, 1; G)
\]

\[(4.4) \leq N(r, 0; G) + N(r, \infty; g) - (k_1 - 2)N_*(r, 1; F, G) - N_{L}(r, 1; G).
\]
Hence using (4.3), (4.4), Lemmas 2 and 20 we get from the second fundamental theorem that
\[
 n \, T(r, f) \leq T(r, F) + N_{k+2}(r, 0; P(f)) - N_2(r, 0; F) + S(r, f) \\
\leq N(r, 0; F) + N(r, \infty; F) + N(r, 1; F) + N_{k+2}(r, 0; P(f)) \\
- N_2(r, 0; F) - N_0(r, 0; F') + S(r, f) \\
\leq N(r, \infty, f) + N(r, \infty, g) + N(r, 0; F) + N_{k+2}(r, 0; P(f)) \\
+ N(r, 0; F) \geq 2 + N(r, 0; G) \geq 2 + N(r, 1; F) \geq 2 \\
+ N_*(r, 1; F, G) + \tilde{N}_0(r, 0; G') - N_2(r, 0; F) + S(r, f) + S(r, g) \\
\leq 3 \, N(r, \infty, f) + N_{k+2}(r, 0; P(f)) + N_2(r, 0; G) - (k_1 - 2) \, N_*(r, 1; F, G) \\
- N_L(r, 1; G) + S(r, f) + S(r, g) \\
\leq 3 \, N(r, \infty, f) + N_{k+2}(r, 0; P(f)) + k \, N(r, \infty, g) + N_{k+2}(r, 0; P(g)) \\
- (k_1 - 2) \, N_*(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq (3 + k) \, N(r, \infty, f) + (k + \Gamma + 2) \, T(r, f) + (k + \Gamma + 2) \, T(r, g) \\
- (k_1 - 2) \, N_*(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq (k + \Gamma + 2) \, \{T(r, f) + T(r, g)\} + (3 + k) \, N(r, \infty, f) \\
- (k_1 - 2) \, N_*(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq (k + \Gamma + 2) \, \{T(r, f) + T(r, g)\} + \frac{(3 + k)(k + \Gamma + 1)}{n - k - 1} \, \{T(r, f) + T(r, g)\} \\
+ \frac{3 + k}{n - k - 1} \, N_*(r, 1; F, G) - (k_1 - 2) \, N_*(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq \left[ k + \Gamma + 2 + \frac{(3 + k)(k + \Gamma + 1)}{n - k - 1} \right] \, \{T(r, f) + T(r, g)\} \\
+ S(r, f) + S(r, g). \tag{4.5} 
\]
In a similar way we can obtain
\[
 n \, T(r, g) \leq \left[ k + \Gamma + 2 + \frac{(3 + k)(k + \Gamma + 1)}{n - k - 1} \right] \, \{T(r, f) + T(r, g)\} \\
+ S(r, f) + S(r, g). \tag{4.6} 
\]
Adding (4.5) and (4.6) we get
\[
\left[ n - 2\Gamma - 2k - 4 - \frac{(6 + 2k)(k + \Gamma + 1)}{n - k - 1} \right] \, \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g), 
\]
i.e.,
\[
\left[ \frac{n^2 - n(3k + 2\Gamma + 5) - (2k + 2)}{n - k - 1} \right] \, \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g). \tag{4.7} 
\]
Note that
\[
2\Gamma + 3k + 6 > \frac{2\Gamma + 3k + 5 + \sqrt{(2\Gamma + 3k + 5)^2 + 4(2k + 2)}}{2}. 
\]
Consequently when \( n > 2\Gamma + 3k + 6 \), we obtain a contradiction from (4.7).

**Case 2.** Let \( H = 0 \). Then by Lemma 10 we have

\[
(4.8) \quad [P(f)]^{(k)} [P(g)]^{(k)} \equiv p^2
\]
or
\[
(4.9) \quad P(f) \equiv P(g).
\]

From (4.9) we get

\[
(4.10) \quad f_1^d (c_m f_1^m + c_{m-1} f_1^{m-1} + \cdots + c_0) \equiv g_1^d (c_m g_1^m + c_{m-1} g_1^{m-1} + \cdots + c_0).
\]

Let \( h = \frac{f_1}{g_1} \). If \( h \) is a constant, then substituting \( f_1 = g_1 h \) into (4.10) we deduce that

\[
c_m g_1^{d+m} (h^{d+m} - 1) + c_{m-1} g_1^{d+m-1} (h^{d+m-1} - 1) + \cdots + c_0 g_1^d (h^d - 1) \equiv 0,
\]

which implies \( h^{d_0} = 1 \), where \( d_0 = \text{GCD}(d+m, \ldots, d+m-i, \ldots, d) \), \( i \neq 0 \) for some \( i = 0, 1, \ldots, m \). Thus \( f_1 \equiv t g_1 \), i.e., \( f(z) - e \equiv t (g(z) - e) \) for a constant \( t \) such that \( t^{d_0} = 1 \), where \( d_0 = \text{GCD}(d+m, \ldots, d+m-i, \ldots, d) \), \( i \neq 0 \) for some \( i = 0, 1, \ldots, m \).

If \( h \) is not a constant, then from (4.10) we see that \( f_1 \) and \( g_1 \) satisfying the algebraic equation \( R(f_1, g_1) = 0 \), where \( R(\omega_1, \omega_2) = \omega_1^d (c_m \omega_1^m + c_{m-1} \omega_1^{m-1} + \cdots + c_0) - \omega_2^d (c_m \omega_2^m + c_{m-1} \omega_2^{m-1} + \cdots + c_0) \).

Remaining part of the theorem follows from (4.8) and Lemma 18. This completes the proof.

\[ \square \]

**References**


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