SOME FUNCTORIAL PROLONGATIONS OF GENERAL CONNECTIONS

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Abstract. We consider the problem of prolongating general connections on arbitrary fibered manifolds with respect to a product preserving bundle functor. Our main tools are the theory of Weil algebras and the Frölicher-Nijenhuis bracket.

0. Introduction

Our approach to connections on an arbitrary fibered manifold $p: Y \rightarrow M$ is slightly different from the approach by C. Ehresmann, [2], p. 186. Roughly speaking, the fundamental idea in [2] is the development along the individual curves, while the main idea of our approach is the absolute differentiation of the sections of $Y$. This is explained in Chapter 1 of the present paper. But the theory of general connections on $Y$ can be well developed even by using the concept of tangent valued form on $Y$. This was invented by L. Mangiarotti and M. Modugno in [7] and first systematically presented in the book [6]. We repeat the basic ideas in Chapter 2. Chapter 3 is devoted to the case of product preserving bundle functors on the category $Mf$ of smooth manifolds and smooth maps. Our geometrical description of them uses the language of Weil algebras, [5], [6].

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [6].

1. General connections

Let $\pi_Y: TY \rightarrow Y$ denote the tangent bundle of a fibered manifold $p: Y \rightarrow M$. In [6], a general connection of $Y$ is defined as a lifting map

$$\Gamma: Y \times_M TM \rightarrow TY$$

linear in $TM$ and satisfying $\pi_Y \cdot \Gamma = pr_1$, $T \pi \cdot \Gamma = pr_2$, $Y \leftarrow Y \times_M TM \rightarrow TM$. If $x^i$, $y^p$ are some local fiber coordinates on $Y$, then the equations of $\Gamma$ are

$$dy^p = F_i^p(x, y) \, dx^i$$

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with arbitrary smooth functions $F^p_i$. Every vector field $X$ on $M$ defines the $\Gamma$-lift $\Gamma(X): Y \to TY$, $\Gamma(X)(y) = \Gamma(y, X)$. Write $\pi_M: TM \to M$ for the bundle projection.

Equivalently, $\Gamma$ can be interpreted as a section $Y \to J^1Y$ of the first jet prolongation $J^1Y$ of $Y$. It is well known that $J^1Y \to Y$ is an affine bundle with associated vector bundle $VY \otimes T^*M$, where $VY$ is the vertical tangent bundle of $Y$. For a section $s: M \to Y$, its absolute differential $\nabla_{\Gamma}s$ with respect to $\Gamma$ is a section $\nabla_{\Gamma}s: M \to VY \otimes T^*M$ defined by

$$\nabla_{\Gamma}s(x) = j^1_x s - \Gamma(s(x))$$

$x \in M$. Hence the coordinate form of (3) is

$$\frac{\partial s^p}{\partial x^i} - F^p_i(x, s(x)).$$

The curvature $C_{\Gamma}: Y \times_M \Lambda^2 T^*M \to VY$ can be characterized as the obstruction for lifting the bracket

$$C_{\Gamma}(y, X_1, X_2) = [\Gamma(X_1), \Gamma(X_2)](y) - \Gamma([X_1, X_2])(y).$$

By direct evaluation, we find that (5) depends on the values of the vector fields $X_1, X_2$ at $p(y)$ only and the coordinate form of (5) is

$$2\left(\frac{\partial F^p_i}{\partial x^j} + \frac{\partial F^p_j}{\partial y^q} F^q_i\right) \frac{\partial}{\partial y^p} \otimes dx^i \land dx^j.$$

Using the flow prolongation of vector fields, we construct an induced connection $\mathcal{V}_{\Gamma}: VY \times_M TM \to TVY$ on $VY$ as follows, [6]. Consider the flow $Fl^\Gamma_t(X)$ of the vector filed $\Gamma(X)$ and its vertical flow prolongation

$$\mathcal{V}(\Gamma(X)) = \frac{\partial}{\partial t} \bigg|_{t=0} V(Fl^\Gamma_t(X)): VY \to TVY.$$

Write $\eta^p = dy^p$ for the induced coordinates on $VY$. Then the coordinate form of (7) is

$$dy^p = F^p_i(x, y) dx^i,$$

$$d\eta^p = \frac{\partial F^p_i}{\partial y^q} \eta^q dx^i,$$

that determines a general connection $\mathcal{V}_{\Gamma}$ on $VY \to M$. The theoretical meaning of the vertical operator $\mathcal{V}$ is underlined by the following assertion, [6].

**Proposition 1.** $\mathcal{V}$ is the only natural operator transforming general connections on $Y \to M$ into general connections on $VY \to M$.

Consider a section $\varphi: Y \to VY \otimes \Lambda^k T^*M$ with the coordinate expression

$$\eta^p = \varphi^p_{i_1\ldots i_k}(x, y) dx^{i_1} \land \cdots \land dx^{i_k}.$$

According to [6], we construct its absolute exterior differential

$$d_{\mathcal{V}\Gamma}\varphi: Y \to VY \otimes \bigwedge^{k+1} T^*M.$$
as follows. Take (at least locally) an auxiliary linear symmetric connection $\Lambda$ on $M$. Then $V\Gamma \otimes \Lambda^k \Lambda^*$ is a connection on $VY \otimes \Lambda^k T^*M \to Y$ and we can construct the absolute differential

$$\nabla_{V\Gamma \otimes \Lambda^k \Lambda^*} \varphi: Y \to V(VY \otimes \Lambda^k T^*M) \otimes T^*M,$$

[6]. Applying antisymmetrization and natural identifications, we obtain a section $d_{V\Gamma} \varphi: Y \to VY \otimes \Lambda^k T^*M$ independent of $\Lambda$ with the coordinate expression

$$\eta^p = \left(\frac{\partial \varphi^p}{\partial x^i} + \frac{\partial \varphi^p}{\partial y^q} F_i^q \varphi^q_{i_1 \ldots i_k} \right) dx^i \wedge dx^1 \wedge \ldots \wedge dx^k.$$

In [6], we deduced by direct evaluation

**Proposition 2** (Bianchi identity). We have

$$d_{V\Gamma} C\Gamma = 0.$$

2. TANGENT VALUED FORMS

Mangiarotti and Modugno studied systematically the general connections by using the concept of tangent valued forms, [7]. A tangent valued $k$-form $P$ on a manifold $M$ is a section $P: M \to TM \otimes \Lambda^k T^*M$, that can be also interpreted as a map

$$P: TM \times_M \ldots \times_M TM \to TM.$$

If $Q$ is another tangent valued $l$-form on $M$, Mangiarotti and Modugno defined a tangent valued $(k + l)$-form $[P, Q]$ on $M$ by the formula

$$[P, Q](X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma} [P(X_{\sigma_1}, \ldots, X_{\sigma_k}), Q(X_{\sigma_{(k+1)}}, \ldots, X_{\sigma_{(k+l)}})]$$

$$+ \frac{(-1)^{k+l}}{k!(l-1)!l!} \sum_{\sigma} P([Q(X_{\sigma_1}, \ldots, X_{\sigma_l}), X_{\sigma_{(l+1)}}, \ldots], X_{\sigma_{(k+2)}}, \ldots)$$

$$+ \frac{(-1)^{k-1}}{(k-1)!(l-1)!2} \sum_{\sigma} Q([P(X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \ldots, X_{\sigma_{(k+2)}}, \ldots)$$

$$+ \frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!2} \sum_{\sigma} \sigma P(Q([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \ldots, X_{\sigma_{(l+2)}}, \ldots), X_{\sigma_{(k+2)}}, \ldots)$$

(12)

where $X_1, \ldots, X_{k+l}$ are vector fields on $M$, the bracket on the right-hand side are the classical Lie bracket of vector fields, the summations are with respect to all permutations $\sigma$ of $k + l$ letters and $\sigma$ denotes the signum of $\sigma$. The tangent valued
0-forms are the vector fields and (12) reduces to the classical Lie bracket in the case $k = l = 0$.

Later it was clarified, [6], that (12) was introduced in a quite different situation by Frölicher-Nijenhuis, so that this bracket is related with their names today.

The identity of $TM$ is a special tangent valued 1-form on $M$ and we have

$$[\text{id}_{TM}, P] = 0$$

for every tangent valued form $P$. By [6],

$$[P, Q] = -(-1)^{kl}[Q, P]$$

and the graded Jacobi identity holds

$$[P_1, [P_2, P_3]] = [[P_1, P_2], P_3] + (-1)^{k_1k_2} [P_2, [P_1, P_3]]$$

for tangent valued $k_i$-forms $P_i$, $i = 1, 2, 3$.

A general connection $\Gamma : Y \times_M TM \to TY$ defines a tangent valued 1-form $\omega_\Gamma$ on $Y$

$$\omega_\Gamma (Z) = \Gamma (y, Tp(Z)) \ , \ Z \in T_y Y .$$

Even $CT$ can be interpreted as a tangent valued 2-form $C_\Gamma$ on $Y$,

$$C_\Gamma (Z_1, Z_2) = C \Gamma (y, Tp(Z_1), Tp(Z_2)) \ , \ Z_1, Z_2 \in T_y Y .$$

**Proposition 3.** We have $C_\Gamma = \frac{1}{2} [\omega_\Gamma, \omega_\Gamma]$.

**Proof.** This follows directly from Lemma 8.13 in [6]. \qed

Consider an arbitrary tangent valued 1-form $\psi$ of $Y$. Put $P_1 = P_2 = P_3 = \psi$ into (14) and (15) This yields

$$[\psi, [\psi, \psi]] = 0 .$$

If $\psi = \omega_\Gamma$, we obtain

**Proposition 4.** We have $[\omega_\Gamma, [\omega_\Gamma, \omega_\Gamma]] = 0$.

A simple evaluation shows that this relation coincides with the identity from Proposition 2. This gives a simple geometric proof of the Bianchi identity of a general connection $\Gamma$ on $Y$.

3. Weilian prolongations

We recall that Weil algebra is a finite dimensional, commutative, associative and unital algebra of the form $A = \mathbb{R} \times N$, where $N$ is the ideal of all nilpotent elements of $A$. Since $A$ is finite dimensional, there exists an integer $r$ such that $N^{r+1} = 0$. The smallest $r$ with this property is called the order of $A$. On the other hand, the dimension $wA$ of the vector space $N/N^2$ is the width of $A$, [5]. Using systematically our point of view, we say that a Weil algebra of width $k$ and order $r$ is a Weil $(k, r)$-algebra, [5].

The simplest example of a Weil $(k, r)$-algebra is

$$\mathbb{D}_k^r = \mathbb{R} [x_1, \ldots, x_k]/ \langle x_1, \ldots, x_k \rangle^{r+1} = J_0^r (\mathbb{R}^k, \mathbb{R}) .$$

For $k = r = 1$, $\mathbb{D}_1^1 = \mathbb{D}$ is the algebra of Study numbers. In [3] we deduced
Lemma 1. Every Weil \((k, r)\)-algebra is a factor algebra of \(D^r_k\). If \(\varrho, \sigma: D^r_k \to A\) are two algebra epimorphisms, then there exists an algebra isomorphism \(\chi: D^r_k \to D^r_k\) such that \(\varrho = \sigma \circ \chi\).

We are going to present the covariant approach to Weil functors, \([5]\).

Definition 1. Two maps \(\gamma, \delta: R^k \to M\) determine the same \(A\)-velocity \(j^A\gamma = j^A\delta\), if for every smooth function \(\varphi: M \to R\),

\[
\varrho(j^A_0(\varphi \circ \gamma)) = \varrho(j^A_0(\varphi \circ \delta)).
\]

By Lemma 1, this is independent of the choice of \(\varrho\). We say that

\[
\varrho = j^A(\varphi \circ \gamma).
\]

Clearly, \(T^A R = A\).

We say that (19) and (20) represent the covariant approach to Weil functors. The following result is a fundamental assertion, see \([6]\) or \([5]\) for a survey.

Theorem. The product preserving bundle functors on \(Mf\) are in bijection with \(T^A\). The natural transformations \(T^{A_1} \to T^{A_2}\) are in bijection with the algebra homomorphisms \(\mu: A_1 \to A_2\).

We write \(\mu_M: T^{A_1}M \to T^{A_2}M\) for the value of \(\mu: A_1 \to A_2\) on \(M\).

The iteration \(T^{A_2} \circ T^{A_1}\) corresponds to the tensor product of \(A_1\) and \(A_2\). The algebra exchange homomorphism \(ex: A_1 \otimes A_2 \to A_2 \otimes A_1\) defines a natural exchange transformation \(T^{A_2}T^{A_1} \to T^{A_1}T^{A_2}\). We have \(T = T^D\).

The canonical exchange \(\varepsilon^A_M: T^A TM \to TT^A M\) is called flow natural. Indeed, if \(Fl^X_t\) is the flow of a vector field \(X: M \to TM\), then

\[
T^A X = \frac{\partial}{\partial t} \bigg|_{t=0} T^A (Fl^X_t): T^A M \to TT^A M
\]

is the flow prolongation of \(X\). It is related with the functorial prolongation \(T^A X: T^A M \to T^A TM\) by

\[
T^A X = \varepsilon^A_M \circ T^A X.
\]

Consider a tangent valued \(k\)-form \(P\) on a manifold \(M\)

\[
P: TM \times_M \cdots \times_M TM \to TM.
\]

Applying functor \(T^A\), we obtain

\[
T^A P: T^{A_1}TP \times_M \cdots \times_M T^{A_1}TP \to T^{A_1}TP.
\]

Using the flow natural exchange \(\varepsilon^A_M\), we construct

\[
T^A P = \varepsilon^A_M \circ T^A P \circ ((\varepsilon^A_M)^{-1} \times \cdots \times (\varepsilon^A_M)^{-1}).
\]

This is an antisymmetric tensor field of type \((1, k)\), so a tangent valued \(k\)-form on \(T^A M\).

In \([1]\), the following result is deduced.
Proposition 5. The Frölicher-Nijenhuis bracket is preserved under $T^A$, i.e. for every tangent valued $k$-form $P$ and every tangent valued $l$-form $Q$ on the same manifold $M$, we have

$$(23) \quad T^A([P,Q]) = [T^AP, T^AQ].$$

Further, consider a tangent valued $k$-form $P$ on a manifold $M$, a tangent valued $k$-form $Q$ on a manifold $N$ and a smooth map $f: M \to N$. We say that $P$ and $Q$ are $f$-related, if the following diagram commutes

$$\begin{array}{ccc}
\Lambda^kTM & \xrightarrow{P} & TM \\
\Lambda^kTf & \downarrow & Tf \\
\Lambda^kTN & \xrightarrow{Q} & TN
\end{array}$$

In [6], p. 74, one has deduced

Proposition 6. Consider a smooth map $f: M \to N$. Let $P_1$, $Q_1$ or $P_2$, $Q_2$ be two $f$-related pairs of $k$-forms or $l$-forms, respectively. Then the Frölicher-Nijenhuis brackets $[P_1,Q_1]$ and $[P_2,Q_2]$ are also $f$-related.

Consider a general connection $\Gamma$ on $Y$ in the lifting form $\Gamma: Y \times_M TM \to TY$. Applying $T^A$, $x_M^A$ and $x_Y^A$, [1, 5], we can construct the induced connection on $T^A Y \to T^A M$

$$(24) \quad T^A \Gamma: T^A Y \times_{T^A M} TT^A M \to TT^A Y.$$

Consider the connection form $\omega_\Gamma: TY \to TY$ of $\Gamma$. Then Proposition 5 and (24) imply

$$(25) \quad T^A C_\Gamma = \frac{1}{2} [T^A \omega_\Gamma, T^A \omega_\Gamma].$$

Hence the curvature of $T^A \Gamma$ is the $T^A$-prolongation of the curvature of $\Gamma$.

Further, the Bianchi identity of $T^A \Gamma$ is the $T^A$-prolongation of the Bianchi identity of $\Gamma$.

References


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