\section{Introduction}

Let $A$ be a Banach algebra and let $\varphi \in \Delta(A)$, consisting of all nonzero homomorphisms from $A$ into $\mathbb{C}$. The concept of $\varphi$-amenability was first introduced by Kaniuth et al. in \cite{6}. Specifically, $A$ is called $\varphi$-amenable if there exist a $m \in A^{**}$ such that
\begin{enumerate}[(i)]  
  \item $m(\varphi) = 1$;
  \item $m(f \cdot a) = \varphi(a)m(f)$ (for $a \in A$, $f \in A^*$).
\end{enumerate}

Monfared in \cite{10}, introduced and studied the notion of character amenable Banach algebra. $A$ was called character amenable if it has a bounded right approximate identity and it is $\varphi$-amenable for all $\varphi \in \Delta(A)$. Many aspects of $\varphi$-amenability have been investigated in \cite{3, 6, 9}.

Let $A$ be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Following \cite{7}, $A$ is called $\Delta$-weak $\varphi$-amenable if, there exists a $m \in A^{**}$ such that
\begin{enumerate}[(i)]  
  \item $m(\varphi) = 0$;
  \item $m(\psi \cdot a) = \psi(a)$ (for $a \in \ker(\varphi)$, $\psi \in \Delta(A)$).
\end{enumerate}

In this paper we use above definition with a slight difference. In fact we say that $A$ is $\Delta$-weak $\varphi$-amenable if, there exists a $m \in A^{**}$ such that
\begin{enumerate}[(i)]  
  \item $m(\varphi) = 0$;
  \item $m(\psi \cdot a) = \psi(a)$ (for $a \in A$, $\psi \in \Delta(A) \setminus \{\varphi\}$).
\end{enumerate}

The aim of the present work is to study $\Delta$-weak character amenability of certain Banach algebras such as projective tensor product $A \hat{\otimes} B$, Lau product $A \times_{\theta} B$, and $\theta \in \Delta(B)$, abstract Segal algebras and module extension Banach algebras. Indeed, we show that $A \hat{\otimes} B$ (resp. $A \times_{\theta} B$) is $\Delta$-weak character amenable if and
only if both $A$ and $B$ are $\Delta$-weak character amenable. For abstract Segal algebra $B$ with respect to $A$, we investigate relations between $\Delta$-weak character amenability of $A$ and $B$. Finally, for a Banach algebra $A$ and $A$-bimodule $X$ we show that $A \oplus_1 X$ is $\Delta$-weak character amenable if and only if $A$ is $\Delta$-weak character amenable.

2. $\Delta$-WEAK CHARACTER AMENABILITY OF $A \hat{\otimes} B$

We commence this section with the following definition:

Definition 2.1. Let $A$ be a Banach algebra. The net $(a_\alpha)_\alpha$ in $A$ is called a $\Delta$-weak approximate identity if, $|\varphi(aa_\alpha) - \varphi(a)| \longrightarrow 0$, for each $a \in A$ and $\varphi \in \Delta(A)$.

Note that the approximate identity and $\Delta$-weak approximate identity of a Banach algebra can be different. Jones and Lahr proved that if $S = \mathbb{Q}^+$ the semigroup algebra $l^1(S)$ has a bounded $\Delta$-weak approximate identity, but it does not have any bounded or unbounded approximate identity (see [4]).

Definition 2.2. Let $A$ be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. We say that $A$ is $\Delta$-weak $\varphi$-amenable if, there exists a $m \in A^{**}$ such that

(i) $m(\varphi) = 0$;
(ii) $m(\psi \cdot a) = \psi(a)$ ($a \in A, \psi \in \Delta(A) \setminus \{\varphi\}$).

Definition 2.3. Let $A$ be a Banach algebra. We say that $A$ is $\Delta$-weak character amenable if it is $\Delta$-weak $\varphi$-amenable for every $\varphi \in \Delta(A) \cup \{0\}$.

Lemma 2.4. Let $A$ be a Banach algebra such that $0 < |\Delta(A)| \leq 2$. Then $A$ is $\Delta$-weak character amenable.

Proof. If $A$ has only one character, the proof is easy. Let $\Delta(A) = \{\varphi, \psi\}$, where $\varphi \neq \psi$. Hence, by the proof of Theorem 3.3.14 of [5], there exists a $a_0 \in A$ with $\varphi(a_0) = 0$ and $\psi(a_0) = 1$. Put $m = \hat{a}_0$. Then $m(\varphi) = \hat{a}_0(\varphi) = \varphi(a_0) = 0$ and for every $a \in A$, we have

$$m(\psi \cdot a) = \hat{a}_0(\psi \cdot a) = \psi \cdot a(a_0) = \psi(aa_0) = \psi(a).$$

So, $A$ is $\Delta$-weak $\varphi$-amenable. A Similar argument shows that $A$ is $\Delta$-weak $\psi$-amenable. Therefore $A$ is $\Delta$-weak character amenable.

The proof of the following theorem is omitted, since it can be proved in the same direction as Theorem 2.2 of [7].

Theorem 2.5. Let $A$ be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Then $A$ is $\Delta$-weak $\varphi$-amenable if and only if there exists a net $(a_\alpha)_\alpha \subseteq \ker(\varphi)$ such that $|\psi(aa_\alpha) - \psi(a)| \longrightarrow 0$, for each $a \in A$ and $\psi \in \Delta(A) \setminus \{\varphi\}$.

Example 2.6. (i) Let $A$ be a Banach algebra with a bounded approximate identity. By Theorem 2.5, $A$ is $\Delta$-weak 0-amenable.

(ii) Let $S = \mathbb{Q}^+$. Then the semigroup algebra $l^1(S)$ has a bounded $\Delta$-weak approximate identity (see [4]). So, Theorem 2.5 implies that $l^1(S)$ is $\Delta$-weak 0-amenable.
Example 2.7. Let $X$ be a Banach space and let $\varphi \in X^* \setminus \{0\}$ with $\|\varphi\| \leq 1$. Define a product on $X$ by $ab = \varphi(a)b$ for all $a,b \in X$. With this product $X$ is a Banach algebra which is denoted by $A_\varphi(X)$ (see [11]). Clearly, $\Delta(A_\varphi(X)) = \{\varphi\}$. Therefore by Lemma 2.4 $A_\varphi(X)$ is $\Delta$-weak $\varphi$-amenable.

Example 2.8. Let $A$ be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Suppose that $A$ is a $\varphi$-amenable and has a bounded right approximate identity. By Corollary 2.3 of [6], $\ker(\varphi)$ has a bounded right approximate identity. Let $(e_\alpha)$ be a bounded right approximate identity for $\ker(\varphi)$. If there exists $a_0 \in A$ with $\varphi(a_0) = 1$ and $\lim \|\psi(a_0e_\alpha) - \psi(a_0)\| = 0$ for all $\psi \in \Delta(A) \setminus \{\varphi\}$, then $A$ is $\Delta$-weak $\varphi$-amenable. For seeing this suppose that $m$ is $w^*$-$\lim_{\alpha}(\overline{e_\alpha})$. Now, we have

\[ m(\varphi) = \lim_{\alpha} \overline{e_\alpha}(\varphi) = \lim_{\alpha} \varphi(e_\alpha) = 0, \]

and for every $\psi \in \Delta(A) \setminus \{\varphi\}$ and $a \in \ker(\varphi)$,

\[ m(\psi \cdot a) = \lim_{\alpha} \overline{e_\alpha}(\psi \cdot a) = \lim_{\alpha} \psi \cdot a(e_\alpha) = \lim_{\alpha} \psi(ae_\alpha) = \psi(a). \]

Let $a \in A$. Then $a - \varphi(a)a_0 \in \ker(\varphi)$ and for every $\psi \in \Delta(A) \setminus \{\varphi\}$, we have

\[ m(\psi \cdot (a - \varphi(a)a_0)) = \psi(a - \varphi(a)a_0). \]

Therefore $m(\psi \cdot a) = \psi(a)$. So $A$ is $\Delta$-weak $\varphi$-amenable.

For $f \in A^*$ and $g \in B^*$, let $f \otimes g$ denote the element of $(A \hat{\otimes} B)^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ for all $a \in A$ and $b \in B$. Then, with this notion,

\[ \Delta(A \hat{\otimes} B) = \{\varphi \otimes \psi : \varphi \in \Delta(A), \psi \in \Delta(B)\}. \]

Theorem 2.9. Let $A$ and $B$ be Banach algebras and let $\varphi \in \Delta(A) \cup \{0\}$ and $\psi \in \Delta(B) \cup \{0\}$. Then $A \hat{\otimes} B$ is $\Delta$-weak $(\varphi \otimes \psi)$-amenable if and only if $A$ is $\Delta$-weak $\varphi$-amenable and $B$ is $\Delta$-weak $\psi$-amenable.

Proof. Suppose that $A \hat{\otimes} B$ is $\Delta$-weak $(\varphi \otimes \psi)$-amenable. So, there exists $m \in (A \hat{\otimes} B)^{**}$ such that

\[ m(\varphi \otimes \psi) = 0, \]

for all $a \otimes b \in A \hat{\otimes} B$ and $(\varphi' \otimes \psi') \in \Delta(A \hat{\otimes} B) \setminus \{\varphi \otimes \psi\}$. Choose $b_0 \in B$ such that $\psi(b_0) = 1$, and define $m_\psi \in A^{**}$ by $m_\psi(f) = m(f \otimes \psi)$ $(f \in A^*)$. Then $m_\psi(\varphi) = m(\varphi \otimes \psi) = 0$ and for every $a \in A$ and $\varphi' \in \Delta(A) \setminus \{\varphi\}$, we have

\[ m_\psi(\varphi' \cdot a) = m(\varphi' \cdot a \otimes \psi) = m(\varphi' \cdot a \otimes \psi \cdot b_0) = m((\varphi' \otimes \psi) \cdot (a \otimes b_0)) = \varphi' \otimes \psi(a \otimes b_0) = \varphi'(a). \]

Thus $A$ is $\Delta$-weak $\varphi$-amenable. By a similar argument one can prove that $B$ is $\Delta$-weak $\psi$-amenable.

Conversely, assume that $A$ is $\Delta$-weak $\varphi$-amenable and $B$ is $\Delta$-weak $\psi$-amenable. By Theorem 2.5 there are bounded nets $(a_\alpha)$ and $(b_\beta)$ in $\ker(\varphi)$ and $\ker(\psi)$, respectively, such that $|\varphi'(aa_\alpha) - \varphi'(a)| \to 0$ and $|\psi'(bb_\beta) - \psi'(b)| \to 0$ for all $a \in A$, $b \in B$, $\varphi' \in \Delta(A) \setminus \{\varphi\}$ and $\psi' \in \Delta(B) \setminus \{\psi\}$. Consider the bounded net
Corollary 2.10. Let \( A \) and \( B \) be Banach algebras. Then \( A \hat{\otimes} B \) is \( \Delta \)-weak character amenable if and only if both \( A \) and \( B \) are \( \Delta \)-weak character amenable.

By using above corollary and Theorem 2.9, we can proof following proposition.

Proposition 2.11. Let \( A \) and \( B \) be Banach algebras. Then \( A \hat{\otimes} B \) is \( \Delta \)-weak character amenable if and only if \( B \hat{\otimes} A \) is \( \Delta \)-weak character amenable.

3. \( \Delta \)-weak character amenability of \( A \times_\theta B \)

Let \( A \) and \( B \) be Banach algebras with \( \Delta(B) \neq \emptyset \) and \( \theta \in \Delta(B) \). Then the set \( A \times B \) equipped with the multiplication

\[
(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 + \theta(b_2)a_1 + \theta(b_1)a_2, b_1b_2) \quad (a_1, a_2 \in A, b_1, b_2 \in B),
\]

and the norm \( \|(a, b)\| = \|a\| + \|b\| \) \((a \in A, b \in B)\), is a Banach algebra which is called the \( \theta \)-Lau product of \( A \) and \( B \) and is denoted by \( A \times_\theta B \). Lau product was introduced by Lau [8] for certain class of Banach algebras and was extended by Monfared [9] for the general case.
We note that the dual space \((A \times_\theta B)^*\) can be identified with \(A^* \times B^*\), via
\[
\langle(f,g),(a,b)\rangle = (a,f) + \langle b,g \rangle \quad (a \in A, f \in A^*, b \in B, g \in B^*).
\]
Moreover, \((A \times_\theta B)^*\) is a \((A \times_\theta B)\)-bimodule with the module operations given by
\[
(f,g) \cdot (a,b) = (f \cdot a + \theta(b)f, f(a)\theta + g \cdot b),
\]
and
\[
(a,b) \cdot (f,g) = (a.f + \theta(b)f, f(a)\theta + b \cdot g),
\]
for all \(a \in A, b \in B, f \in A^*\) and \(g \in B^*\).

**Proposition 3.1.** Let \(A\) be a unital Banach algebra and \(B\) be a Banach algebra and \(\theta \in \Delta(B)\). Then \(A \times_\theta B\) has a \(\Delta\)-weak approximate identity if and only if \(B\) has a \(\Delta\)-weak approximate identity.

**Proof.** Let \((\langle a_\alpha, b_\alpha \rangle)_\alpha\) be a \(\Delta\)-weak approximate identity for \(A \times_\theta B\). For every \(\psi \in \Delta(B)\) and \(b \in B\) we have,
\[
|\psi(bb_\alpha) - \psi(b)| = |(0, \psi)((0,b)(a_\alpha, b_\alpha)) - (0, \psi)(0,b)| \to 0.
\]
Then \((b_\alpha)_\alpha\) is a \(\Delta\)-weak approximate identity for \(B\).

Conversely, let \(e_A\) be the identity of \(A\) and \((b_\beta)_\beta\) be a \(\Delta\)-weak approximate identity for \(B\). We claim that \((e_A - \theta(b_\beta)e_A, b_\beta)\) is a \(\Delta\)-weak approximate identity for \(A \times_\theta B\). In fact for every \(a \in A, b \in B\) and \(\varphi \in \Delta(A)\), we have
\[
|(\varphi, \theta)((a,b)(e_A - \theta(b_\beta)e_A, b_\beta)) - (\varphi, \theta)(a,b)|
= |(\varphi, \theta)(a + \theta(b)e_A - \theta(bb_\beta)e_A, bb_\beta)) - (\varphi, \theta)(a,b)|
= 0.
\]
Also for every \(a \in A, b \in B\) and \(\psi \in \Delta(B)\), we have
\[
|(0, \psi)((a,b)(e_A - \theta(b_\beta)e_A, b_\beta)) - (0, \psi)(a,b)| = |\psi(bb_\beta) - \psi(b)| \to 0.
\]
Therefore \((e_A - \theta(b_\beta)e_A, b_\beta)\) is a \(\Delta\)-weak approximate identity for \(A \times_\theta B\). \(\square\)

**Theorem 3.2.** Let \(A\) be a unital Banach algebra and \(B\) be a Banach algebra and \(\theta \in \Delta(B)\). Then \(A \times_\theta B\) is \(\Delta\)-weak character amenable if and only if both \(A\) and \(B\) are \(\Delta\)-weak character amenable.

**Proof.** Suppose that \(A \times_\theta B\) is \(\Delta\)-weak character amenable. Let \(\varphi \in \Delta(A) \cup \{0\}\). Then there exists \(m \in (A \times_\theta B)^{**}\) such that \(m(\varphi, \theta) = 0\) and \(m(h,(a,b)) = h(a,b)\) for all \((a,b) \in A \times_\theta B\) and \(h \in \Delta(A \times_\theta B)\), where \(h \neq (\varphi, \theta)\). Let \(e_A\) be the identity of \(A\) and define \(m_\psi \in A^{**}\) by \(m_\psi(f) = m(f, f(e_A)\theta)(f \in A^*)\). For every \(a \in A\) and \(\varphi' \in \Delta(A) \setminus \{\varphi\}\), we have
\[
m_\psi(\varphi' \cdot a) = m(\varphi' \cdot a, (\varphi' \cdot a)(e_A)\theta)
= m(\varphi' \cdot a, \varphi'(a)\theta)
= m((\varphi', \theta) \cdot (a, 0))
= (\varphi', \theta)(a, 0)
= \varphi'(a).
\]
Also \( m\varphi(\varphi) = m(\varphi, \theta) = 0 \). Thus \( A \) is a \( \Delta \)-weak \( \varphi \)-amenable. Therefore \( A \) is \( \Delta \)-weak character amenable.

Let \( \psi \in \Delta(B) \setminus \{0\} \). From the \( \Delta \)-weak character amenability of \( A \times_\theta B \) it follows that there exists a \( m \in (A \times_\theta B)^{**} \) such that \( m(0, \psi) = 0 \) and \( m(h \cdot (a, b)) = h(a, b) \) for all \( (a, b) \in A \times_\theta B \) and \( h \in \Delta(A \times_\theta B) \), where \( h \neq (0, \psi) \). Define \( m_\varphi \in B^{**} \) by \( m_\varphi(g) = m(0, g) \). So \( m_\varphi(\psi) = m(0, \psi) = 0 \) and

\[
m_\varphi(\psi \cdot b) = m(0, \psi' \cdot b) = m((0, \psi') \cdot (0, b)) = (0, \psi')(0, b') = \psi'(b),
\]

for all \( b \in B \) and \( \psi' \in \Delta(B) \setminus \{\psi\} \). Therefore \( B \) is \( \Delta \)-weak character amenable.

Conversely, let \( A \) and \( B \) be \( \Delta \)-weak character amenable. We show that for every \( h \in \Delta(A \times_\theta B) \), \( A \times_\theta B \) is \( \Delta \)-weak \( h \)-amenable. To see this we first assume that \( h = (0, \psi) \), where \( \psi \in \Delta(B) \). Since \( B \) is \( \Delta \)-weak character amenable, by Theorem 2.5 there exists a net \( (b_\beta)_\beta \subseteq \ker \psi \) such that \( |\psi'(b_\beta) - \psi'(b)| \longrightarrow 0 \), for all \( b \in B \) and \( \psi' \in \Delta(B) \), where \( \psi' \neq \psi \). Consider the bounded net \( ((e_A - \theta(b_\beta)e_A, b_\beta))_\beta \subseteq A \times_\theta B \). A similar argument as in the proof of Proposition 3.1 shows that

\[
|(\varphi, \theta)((a, b)(e_A - \theta(b_\beta)e_A, b_\beta)) - (\varphi, \theta)(a, b)| \longrightarrow 0,
\]

for all \( \varphi \in \Delta(A), \psi \in \Delta(B) \) and \( a \in A, b \in B \). Also one can easily check that \( ((e_A - \theta(b_\beta)e_A, b_\beta))_\beta \subseteq \ker h \). So, by Theorem 2.5, \( A \times_\theta B \) is \( \Delta \)-weak \( (0, \psi) \)-amenable.

Now let \( h = (\varphi, \theta) \), where \( \varphi \in \Delta(A) \). Since \( A \) is \( \Delta \)-weak \( \varphi \)-amenable, by Theorem 2.5 there exists a net \( (a_\alpha)_\alpha \subseteq \ker \varphi \) such that \( |\varphi'(aa_\alpha) - \varphi'(a)| \longrightarrow 0 \), for all \( a \in A \) and \( \varphi' \in \Delta(A) \), where \( \varphi' \neq \varphi \). Also since \( B \) is \( \Delta \)-weak \( \theta \)-amenable again by Theorem 2.5, there exists a net \( (b_\beta)_\beta \subseteq \ker(\theta) \) such that \( |\psi'(bb_\beta) - \psi'(b)| \longrightarrow 0 \), for all \( b \in B \) and \( \psi' \in \Delta(B) \), where \( \psi' \neq \theta \). Consider the bounded net \( ((a_\alpha, b_\beta))_{(\alpha, \beta)} \subseteq A \times_\theta B \). It is easy to see that \( ((a_\alpha, b_\beta))_{(\alpha, \beta)} \subseteq \ker(\varphi, \theta) \). For every \( a \in A, b \in B \) and \( \psi' \in \Delta(B) \), we have

\[
|(0, \psi')(a, b)(a_\alpha, b_\beta)) - (0, \psi')(a, b)| = |\psi'(bb_\beta) - \psi'(b)| \longrightarrow 0,
\]

and for every \( \varphi' \in \Delta(A) \),

\[
|(\varphi', \theta)((a, b)(a_\alpha, b_\beta)) - (\varphi', \theta)((a, b))|
\]

\[
= |\varphi'(aa_\alpha) + \theta(b_\beta)\varphi'(a) + \theta(b)\varphi'(a_\alpha) + \theta(bb_\beta) - \varphi'(a) - \theta(b)|
\]

\[
= |\varphi'(aa_\alpha) + \theta(b)\varphi'(a_\alpha) - \varphi'(a) - \theta(b)|
\]

\[
\leq |\varphi'(aa_\alpha) - \varphi'(a)| + |\theta(b)||\varphi'(a_\alpha e_A) - \varphi'(e_A)| \longrightarrow 0.
\]

So, Theorem 2.5 yields that \( A \times_\theta B \) is \( \Delta \)-weak \( (\varphi, \theta) \)-amenable. Therefore \( A \times_\theta B \) is \( \Delta \)-weak character amenable.

4. \( \Delta \)-weak character amenability of abstract Segal algebras

We start this section with the basic definition of abstract Segal algebra; see [2] for more details. Let \( (A, \| \cdot \|_A) \) be a Banach algebra. A Banach algebra \( (B, \| \cdot \|_B) \) is an abstract Segal algebra with respect to \( A \) if:

(i) \( B \) is a dense left ideal in \( A \);
(ii) there exists $M > 0$ such that $\|b\|_A \leq M \|b\|_B$ for all $b \in B$;
(iii) there exists $C > 0$ such that $\|ab\|_B \leq C \|a\|_A \|b\|_B$ for all $a, b \in B$.

Several authors have studied various notions of amenability for abstract Segal algebras; see, for example, [11, 12].

To prove our next result we need to quote the following lemma from [1].

**Lemma 4.1.** Let $A$ be a Banach algebra and let $B$ be an abstract Segal algebra with respect to $A$. Then $\Delta(B) = \{ \varphi_B : \varphi \in \Delta(A) \}$.

**Theorem 4.2.** Let $A$ be a Banach algebra and let $B$ be an abstract Segal algebra with respect to $A$. If $B$ is $\Delta$-weak character amenable, then so is $A$. In the case that $B^2$ is dense in $B$ and $B$ has a bounded approximate identity the converse is also valid.

**Proof.** Let $\varphi \in \Delta(A)$. Since $B$ is $\Delta$-weak character amenable, by Lemma 4.1 $B$ is $\Delta$-weak $\varphi|_B$-amenable. Now from the Theorem 2.5, it follows that there exists a bounded net $(b_\alpha)_\alpha$ in ker($\varphi|_B$) such that

$$\left| \psi|_B(b_\alpha b) - \psi|_B(b) \right| \to 0,$$

for all $b \in B$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi|_B$. Let $\psi \in \Delta(A)$ and $a \in A$. From the density of $B$ in $A$ it follows that there exists a set $(b_i)_i \subseteq B$ such that $\lim_i b_i = a$. So

$$\left| \psi(ab_\alpha) - \psi(a) \right| = \lim_i \left| \psi|_B(b_i b_\alpha) - \psi|_B(b_i) \right| \to 0.$$

Then Theorem 2.5 implies that $A$ is $\Delta$-weak $\varphi$-amenable. Therefore $A$ is $\Delta$-weak character amenable.

Conversely, suppose that $A$ is $\Delta$-weak character amenable. Let $\varphi|_B \in \Delta(B)$. By Theorem 2.5, there exists a bounded net $(a_\alpha)_\alpha$ in ker($\varphi$) such that $\left| \psi(aa_\alpha) - \psi(a) \right| \to 0$, for all $a \in A$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi$. Let $(e_i)_i$ be a bounded approximate identity for $B$ with bound $M > 0$. Set $b_\alpha = \lim_i (e_i a_\alpha e_i)$, for all $\alpha$. From the fact that $B^2$ is dense in $B$ and the continuity of $\varphi$, we infer that $b_\alpha \subseteq \ker(\varphi|_B)$. Moreover, for every $b \in B$ and $\psi|_B \in \Delta(B)$, with $\psi \neq \varphi$, we have

$$\left| \psi|_B(b_\alpha b) - \psi|_B(b) \right| = \lim_i \left| \psi|_B(be_i a_\alpha e_i) - \psi|_B(b) \right|$$

$$= \lim_i \left| \psi|_B(be_i^2 a_\alpha) - \psi|_B(b) \right|$$

$$= \left| \psi|_B(b_\alpha a) - \psi|_B(b) \right| \to 0.$$

Hence, $B$ is $\Delta$-weak $\varphi|_B$-amenable by Theorem 2.5. Therefore $B$ is $\Delta$-weak character amenable.

\hfill $\Box$

5. $\Delta$-WEAK CHARACTER AMENABILITY OF MODULE EXTENSION BANACH ALGEBRAS

Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. The $l^1$-direct sum of $A$ and $X$, denoted by $A \oplus_1 X$, with the product defined by

$$(a, x)(a', x') = (aa', a \cdot x' + x \cdot a') \quad (a, a' \in A, x, x' \in X),$$
is a Banach algebra that is called the module extension Banach algebra of $A$ and $X$.

Using the fact that the element $(0, x)$ is nilpotent in $A \oplus_1 X$ for all $x \in X$, it is easy to verify that
\[
\Delta(A \oplus_1 X) = \{ \tilde{\varphi} : \varphi \in \Delta(A) \},
\]
where $\tilde{\varphi}(a, x) = \varphi(a)$ for all $a \in A$ and $x \in X$.

**Theorem 5.1.** Let $A$ be a Banach algebra and $X$ be a Banach $A$-bimodule. Then $A \oplus_1 X$ is $\Delta$-weak character amenable if and only if $A$ is $\Delta$-weak character amenable.

**Proof.** Suppose that $A$ is $\Delta$-weak character amenable. Let $\tilde{\varphi} \in \Delta(A \oplus_1 X)$. By Theorem 2.5, there exists a bounded net $(a_\alpha)_\alpha$ in $\ker(\varphi)$ such that $|\psi(aa_\alpha) - \tilde{\varphi}(a)| \to 0$, for all $a \in A$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi$. Choose a bounded net $(a_\alpha, 0)_\alpha$ in $A \oplus_1 X$. Clearly, $(a_\alpha, 0)_\alpha \subseteq \ker(\tilde{\varphi})$. For every $a \in A$, $x \in X$ and $\psi \in \Delta(A \oplus_1 X)$, we have
\[
|\tilde{\psi}((a, x)(a_\alpha, 0)) - \tilde{\varphi}(a, x)| = |\tilde{\psi}(aa_\alpha, x \cdot a_\alpha) - \tilde{\psi}(a, x)|
= |\psi(aa_\alpha) - \psi(a)| \to 0.
\]
So, Theorem 2.5 implies that $A \oplus_1 X$ is $\Delta$-weak $\tilde{\varphi}$-amenable. Therefore $A \oplus_1 X$ is $\Delta$-weak character amenable.

For the converse, let $\varphi \in \Delta(A)$. Again by Theorem 2.5, there exists a bounded net $(a_\alpha, x_\alpha)_\alpha$ in $\ker(\tilde{\varphi})$ such that $|\tilde{\psi}(a_\alpha)(x_\alpha) - \tilde{\varphi}(a, x)| \to 0$, for all $a \in A$, $x \in X$ and $\tilde{\psi} \in \Delta(A \oplus_1 X)$, with $\psi \neq \tilde{\varphi}$. So,
\[
|\psi(aa_\alpha) - \psi(a)| = |\tilde{\psi}(aa_\alpha, a \cdot x_\alpha + x \cdot a_\alpha) - \tilde{\psi}(a, x)|
= |\tilde{\psi}((a, x)(a_\alpha, x_\alpha)) - \tilde{\psi}(a, x)| \to 0,
\]
for all $a \in A$ and $\psi \in \Delta(A)$. Moreover, $\varphi(a_\alpha) = \tilde{\varphi}(a_\alpha, x_\alpha) = 0$, for all $\alpha$. Thus $(a_\alpha)_\alpha \subseteq \ker(\varphi)$. By Theorem 2.5, $A$ is $\Delta$-weak $\varphi$-amenable. Therefore $A$ is $\Delta$-weak character amenable. \(\square\)

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