UNIT-REGULARITY AND REPRESENTABILITY FOR SEMIARTINIAN ∗-REGULAR RINGS

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Abstract. We show that any semiartinian ∗-regular ring $R$ is unit-regular; if, in addition, $R$ is subdirectly irreducible then it admits a representation within some inner product space.

1. Introduction

The motivating examples of ∗-regular rings, due to Murray and von Neumann, were the ∗-rings of unbounded operators affiliated with finite von Neumann algebra factors; to be subsumed, later, as ∗-rings of quotients of finite Rickart $C^*$-algebras. All the latter have been shown to be ∗-regular and unit-regular (Handelman [5]). Representations of these as ∗-rings of endomorphisms of suitable inner product spaces have been obtained first, in the von Neumann case, by Luca Giudici (cf. [7]), in general in joint work with Marina Semenova [9]. The existence of such representations implies direct finiteness [8]. In the present note we show that every semiartinian ∗-regular ring is unit-regular and a subdirect product of representables. This might be a contribution to the question, asked by Handelman (cf. [3, Problem 48]), whether all ∗-regular rings are unit-regular. We rely heavily on the result of Baccella and Spinosa [1] that a semiartinian regular ring is unit-regular provided that all its homomorphic images are directly finite. Also, we rely on the theory of representations of ∗-regular rings developed by Florence Micol [12] (cf. [9, 10]). Thanks are due to the referee for a timely, concise, and helpful report.

2. Preliminaries: Regular and ∗-regular rings

We refer to Berberian [2] and Goodearl [3]. Unless stated otherwise, rings will be associative, with unit 1 as constant. A (von Neumann) regular ring $R$ is such that for each $a \in R$ there is $x \in R$ such that $axa = a$; equivalently, every right (left) principal ideal is generated by an idempotent. The socle $\text{Soc}(R)$ is the sum of all minimal right ideals. A regular ring $R$ is semiartinian if each of its homomorphic images has non-zero socle; that is, $R$ has Loewy length $\xi + 1$ for some ordinal $\xi$. A ring $R$ is directly finite if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. A ring $R$ is

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unit-regular if for any \( a \in R \) there is a unit \( u \) of \( R \) such that \( auu = a \). The crucial fact to be used, here, is the following result of Baccella and Spinosa [1].

**Theorem 1.** A semiartinian regular ring is unit-regular provided all its homomorphic images are directly finite.

A \(*\)-ring is a ring \( R \) endowed with an involution \( r \mapsto r^* \). Such \( R \) is \(*\)-regular if it is regular and \( rr^* = 0 \) only for \( r = 0 \). A projection is an idempotent \( e \) such that \( e = e^* \); we write \( e \in P(I) \) if \( e \in I \). A \(*\)-ring is \(*\)-regular if and only if for any \( a \in R \) there is a projection \( e \) with \( aR = eR \); such \( e \) is unique and obtained as \( aa^+ \) where \( a^+ \) is the pseudo-inverse of \( a \). In particular, for \(*\)-regular \( R \), each ideal \( I \) is a \(*\)-ideal, that is, closed under the involution. Thus, \( R/I \) is a \(*\)-ring with involution \( a + I \mapsto a^* + I \) and a homomorphic image of the \(*\)-ring \( R \). In particular, \( R/I \) is regular; and \(*\)-regular since \( aa^* + I \) is a projection generating \( (a + I)(R/I) \).

If \( R \) is a \(*\)-regular ring and \( e \in P(R) \) then the corner \( eRe \) is a \(*\)-regular ring with unit \( e \), operations inherited from \( R \), otherwise. For a \(*\)-regular ring, \( P(R) \) is a modular lattice, with partial order given by \( e \leq f \iff fe = e \), which is isomorphic to the lattice \( L(R) \) of principal right ideals of \( R \) via \( e \mapsto eR \). In particular, \( eRe \) is artinian if and only if \( e \) is contained in the sum of finitely many minimal right ideals.

A \(*\)-ring is subdirectly irreducible if it has a unique minimal ideal, denoted by \( M(R) \). Observe that \( Soc(R) \neq 0 \) implies \( M(R) \subseteq Soc(R) \) since \( Soc(R) \) is an ideal. For the following see Lemma 2 and Theorem 3 in [9].

**Fact 2.** If \( R \) is a subdirectly irreducible \(*\)-regular ring then \( eRe \) is simple for all \( e \in P(M(R)) \) and \( R \) a homomorphic image of a \(*\)-regular sub-\(*\)-ring of some ultraproduction of the \( eRe \), \( R \in P(M(R)) \).

### 3. Preliminaries: Representations

We refer to Gross [4] and Sections 1 of [9], 2–4 of [10]. By an inner product space \( V_F \) we will mean a right vector space (also denoted by \( V_F \)) over a division \(*\)-ring \( F \), endowed with a sesqui-linear form \( \langle \cdot, \cdot \rangle \) which is anisotropic (\( \langle v \mid v \rangle = 0 \) only for \( v = 0 \)) and orthosymmetric, that is, \( \langle v \mid w \rangle = 0 \) if and only if \( \langle w \mid v \rangle = 0 \). Let \( \text{End}^*(V_F) \) denote the \(*\)-ring consisting of those endomorphisms \( \varphi \) of the vector space \( V_F \) which have an adjoint \( \varphi^* \) w.r.t. \( \langle \cdot, \cdot \rangle \).

A representation of a \(*\)-ring \( R \) within \( V_F \) is an embedding of \( R \) into \( \text{End}^*(V) \). \( R \) is representable if such exists. The following is well known, cf. [11] Chapter IV.12

**Fact 3.** Each simple artinian \(*\)-regular ring is representable.

The following two facts are consequences of Propositions 13 and 25 in [9] (cf. Micol [12, Corollary 3.9]) and, respectively, [8 Theorem 3.1] (cf. [6] Theorem 4).

**Fact 4.** A \(*\)-regular ring is representable provided it is a homomorphic image of a \(*\)-regular sub-\(*\)-ring of an ultraproduct of representable \(*\)-regular rings.

**Fact 5.** Every representable \(*\)-regular ring is directly finite.
4. Main results

Theorem 6. If \( R \) is a subdirectly irreducible \(-\)regular ring such that \( \text{Soc}(R) \neq 0 \), then \( \text{Soc}(R) = M(R) \), each \( eRe \) with \( e \in P(M(R)) \) is artinian, and \( R \) is representable.

Proof. Consider a minimal right ideal \( aR \). As \( R \) is subdirectly irreducible, \( M(R) \) is contained in the ideal generated by \( a \); that is, for any \( 0 \neq e \in P(M(R)) \) one has \( e = \sum_i r_i a s_i \) for suitable \( r_i, s_i \in R, r_i a s_i \neq 0 \). By minimality of \( aR \), one has \( a s_i R = a R \) and \( r_i a s_i R = r_i a R \) is minimal, too. Thus, \( e \in \sum_i r_i a R \) means that \( e Re \) is artinian. By Facts 3, 2, and 4, \( \text{Soc}(e Re) = 0 \), \( e Re \) is semiartinian if and only if so is \( R \).

It remains to show that \( \text{Soc}(R) \subseteq M(R) \). Recall that the congruence lattice of \( L(R) \) is isomorphic to the ideal lattice of \( R \) ([13] Theorem 4.3) with an isomorphism \( \theta \mapsto I \) such that \( a R / 0 \in \theta \) if and only if \( a \in I \). In particular, since \( R \) is subdirectly irreducible so is \( L(R) \). Choose \( e \in M(R) \) with \( e R \) minimal. Then for each minimal \( aR \) one has \( e R / 0 \) in the lattice congruence \( \theta \) generated by \( a R / 0 \). Since both quotients are prime, by modularity this means that they are projective to each other. Thus, \( a R / 0 \) is the ideal generated by \( e \), that is, \( R \) is semiartinian.

Theorem 7. Every semiartinian \(-\)regular ring \( R \) is unit-regular and a subdirect product of representable homomorphic images.

Proof. Consider an ideal \( I \) of \( R \). Then \( I = \bigcap_{x \in X} I_x \) with completely meet irreducible \( I_x \), that is, subdirectly irreducible \( R / I_x \). Since \( R \) is semiartinian one has \( \text{Soc}(R / I_x) \neq 0 \), whence \( R / I_x \) is representable by Fact 6 and directly finite by Theorem 4. Then \( R / I \) is directly finite, too, being a subdirect product of the \( R / I_x \). By Theorem 1 it follows that \( R \) is unit-regular.

5. Examples

It appears that semiartinian \(-\)regular rings form a very special subclass of the class of unit-regular \(-\)regular rings, even within the class of those which are subdirect products of representables. E.g. the \(-\)ring of unbounded operators affiliated to the hyperfinite von Neumann algebra factor is representable, unit-regular, and \(-\)regular with zero socle. On the other hand, due to the following, for every simple artinian \(-\)regular ring \( R \) and any natural number \( n > 0 \) there is a semiartinian \(-\)regular ring having ideal lattice an \( n \)-element chain and \( R \) as a homomorphic image.

Proposition 8. Every representable \(-\)regular ring \( R \) embeds into some subdirectly irreducible representable \(-\)regular ring \( \hat{R} \) such that \( R \cong \hat{R} / M(\hat{R}) \). In particular, \( \hat{R} \) is semiartinian if and only if so is \( R \).

The proof needs some preparation. Call a representation \( \iota : R \to \text{End}^*(V_F) \) large if for all \( a, b \in R \) with \( \text{im} \iota(b) \subseteq \text{im} \iota(a) \) and finite \( \text{dim}(\text{im} \iota(a) / \text{im} \iota(b))_F \) one has \( \text{im} \iota(a) = \text{im} \iota(b) \).

Lemma 9. Any representable \(-\)regular ring admits some large representation.
Proof. Inner product spaces can be considered as 2-sorted structures $V_F$ with sorts $V$ and $F$. In particular, the class of inner product spaces is closed under formation of ultraproducts. Representations of $\ast$-rings $R$ can be viewed as $R$-$F$-bimodules $RF$, that is as 3-sorted structures, with $R$ acting faithfully on $V$. It is easily verified that the class of representations of $\ast$-rings is closed under ultraproducts cf. [9, Proposition 13].

Now, given a representation $\eta$ of $R$ in $W_F$, form an ultrapower $\iota$, that is $SV_F'$, such that $\dim F'$ is infinite (recall that $F'$ is an ultrapower of $F$). Observe that $\text{End}^\ast(V_F')$ is a sub-$\ast$-ring of $\text{End}^\ast(V_F)$ and $\dim(U/W)_F$ is infinite for any subspaces $U \supseteq W$ of $V_F'$. Also, $S$ is an ultrapower of $R$ with canonical embedding $\varepsilon: R \to S$. Thus, $\varepsilon \circ \iota$ is a large representation of $R$ in $V_F$.

Proof of Proposition [8] In view of Lemma [9] we may assume a large representation $\iota$ of $R$ in $V_F$. Identifying $R$ via $\iota$ with its image, we have $R$ a $\ast$-regular sub-$\ast$-ring of $\text{End}^\ast(V_F)$. Let $I$ denote the set of all $\varphi \in \text{End}(V_F)$ such that $\dim(\text{im} \varphi)_F$ is finite. According to Micol [12, Proposition 3.12] (cf. Propositions 4.4(i), (iii) and 4.5 in [10]) $R + I$ is a $\ast$-regular sub-$\ast$-ring of $\text{End}^\ast(V_F)$, with unique minimal ideal $I$. By Theorem [6] one has $I = \text{Soc}(R + I)$. Moreover, $R \cap I = \{0\}$ since the representation $\iota$ of $R$ in $V_F$ is large. Hence, $R \cong (R + I)/I$. □

References

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