On A General Huygens-Wilker Inequality*

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Abstract

This note will present an extension of a general Wilker type inequality. The proofs rely basically on iteration of derivations for real functions.

1 Introduction

We set
\[ f(x) := a \left( \frac{x}{\sin(x)} \right)^m + b \left( \frac{x}{\tan(x)} \right)^n \]

for any \( x \in [0, \frac{\pi}{2}] \) where \( a \) and \( b \) are two positive real numbers, \( m \) and \( n \neq 0 \). The inequality \( f(x) > a + b \) for \( a = 2, b = 1, m = -1 \) and \( n = -1 \) is known as Huygens inequality and for \( a = b = 1, m = -2, n = 1 \) we obtain Wilker’s inequality ([2, 4, 5]). These and more related inequalities were extensively studied, reproved and generalized see [9, 3, 1, 8, 10, 6, 7].

Our main focus is on the general inequality \( f(x) > a + b \) where it is proved that \( f(x) \) is strictly increasing on \( [0, \frac{\pi}{2}] \) under some conditions on the parameters \( a, b, m \) and \( n \). Inverse inequality cases of \( f(x) < a + b \) are also derived.

Lemma 1 The derivative \( f'(x) \) is equal to:

\[
P(x) \left[ am (\sin(x) - x \cos(x)) - bn \left( \frac{x}{\sin(x)} \right)^{n-m} \cos(x)^{n-1} (x - \cos(x) \sin(x)) \right]
\]

where \( P(x) = \frac{1}{\sin(x) \cos(x)^n} \left( \frac{x}{\sin(x)} \right)^m \) and \( f'(x) = 0 \) on \( ]0, \frac{\pi}{2}[ \) if and only if:

1. \[
\frac{am}{bn} = \left( \frac{x}{\sin(x)} \right)^{n-m} \cos(x)^{n-1} \left( \frac{x - \cos(x) \sin(x)}{\sin(x) - x \cos(x)} \right) = L(x),
\]

2. \[
\frac{am}{bn} = \left( \frac{x}{\tan(x)} \right)^{-n-1} \left( \frac{x}{\sin(x)} \right)^{1-m} \left( \frac{x - \cos(x) \sin(x)}{\sin(x) - x \cos(x)} \right) = L(x),
\]

3. \[
\frac{am}{bn} = \left( \frac{x}{\sin(x)} \right)^{n-m} \cos(x) \left( \frac{x}{\cos(x)} - \sin(x) \right) \left( \frac{\sin(x)}{\sin(x) - x \cos(x)} \right) = H(x),
\]

4. \[
\frac{am}{bn} = \left( \frac{x}{\tan(x)} \right)^{n} \left( \frac{\sin(x)}{x} \right)^{n} \left( \frac{x}{\cos(x)} - \sin(x) \right) \left( \frac{\sin(x)}{\sin(x) - x \cos(x)} \right) = H(x).
\]

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Of course the four expressions are all equivalent but it is mandatory to separate them to conclude.

It is worth mentioning that when $0 < n < 1$, $f(x)$ isn’t an increasing function on $[0, \frac{\pi}{2}]$ as it can be shown that $f(x)$ is at least decreasing on $[\xi, \frac{\pi}{2}]$ for some $\xi$ (from Lemma 1). However with the boundary condition $a(m) - b > 2bn$ the inequality $f(x) > a + b$ on $[0, \frac{\pi}{2}]$ seems to hold for any $m$ and $n$ of same sign. In fact with $am \geq 2bn > 0$ studying the case $am = 2bn$ is sufficient and for $a = b$ the inequality is already proven in [11]. Some special cases for particular values of $a, b, m$ and $n$ are proved among others in [12] and [13].

2 Main Results

Before stating the main theorem we have the following:

Lemma 2 The function $D(x) := \frac{x - \cos(x)\sin(x)}{\sin(x) - x \cos(x)}$ is strictly decreasing on $]0, \frac{\pi}{2}[$.

Proof. First by applying a succession of Hospital’s rule one can show that

$$\lim_{x \to 0} D(x) = 2 \quad \text{and} \quad D'(x) = \frac{-\sin(x)(-2 + x^2 + 2\cos(x)^2 + \sin(x)x \cos(x))}{(\sin(x) - x \cos(x))^2}.$$ 

Then

$$S(x) := -2 + x^2 + 2\cos(x)^2 + \sin(x)x \cos(x) > 0 \quad \text{for} \quad x \in ]0, \frac{\pi}{2}[$$

since $S(0) = S'(0) = S''(0) = 0$ and $S'''(0) = 2(\sin(2x) - 2x \cos(2x)) > 0$ on $]0, \frac{\pi}{2}[$. ■

Lemma 3 The function $I(x) := \frac{x}{\cos(x)} \sin(x) - \sin(x) - x \cos(x)$ is strictly increasing on $]0, \frac{\pi}{2}[$.

Proof. Similarly to the precedent proof we have $\lim_{x \to 0} I(x) = 2$ and

$$I'(x) = \frac{\sin(x)(\cos(x)^3 - \cos(x) + 2x^2 \cos(x) - x \sin(x))}{\cos(x)^2(\sin(x) - x \cos(x))^2}.$$ 

If $C(x) := \cos(x)^3 - \cos(x) + 2x^2 \cos(x) - x \sin(x)$, then we need to show that $x \tan(x) + 1 - \cos(x)^2 - 2x^2 \geq 0$ for all $x \in ]0, \frac{\pi}{4}[$. Set $R(x) := x \tan(x) + 1 - \cos(x)^2 - 2x^2$, $R(0) = R'(0) = R''(0) = 0$, upon computing $R^{(3)}(x) > 0$ on $]0, \frac{\pi}{4}[$ since

$$3 \tan(x) + 3x \tan(x)^2 + x > 3 \cos(x)^3 \sin(x) + \cos(x)^3 \sin(x)$$

and the result follows. ■

Theorem 1 Let $a \geq 0$ and $b \geq 0$. If $am \geq 2bn$, $m$ and $n$ are of same sign not equal to zero and $0 > \min(m, n) \text{ or } \min(m, n) \geq 1$, then $f(x)$ is strictly increasing on $]0, \frac{\pi}{2}[$ consequently:

$$f(x) := a \left(\frac{x}{\sin(x)}\right)^m + b \left(\frac{x}{\tan(x)}\right)^n > a + b \quad \text{for all} \quad x \in ]0, \frac{\pi}{2}[$.$$

Proof. The inequality when $0 < \min(m, n)$, $m < 0$ and $n < 0$ was already proved in [1], $\frac{am}{bn} \leq 2$ and $H(x)$ as in (3) or (4) is strictly increasing on $]0, \frac{\pi}{2}[$ with $\lim_{x \to 0} H(x) = 2$, to see this from Lemma 3 consider (3) when $n \geq m$ and (4) for any $m < 0$, $n < 0$. If $\min(m, n) \geq 1$, $\frac{am}{bn} \geq 2$ but $L(x) < 2$ on $]0, \frac{\pi}{2}[$ as in (1) when $m \geq n \geq 1$. Also $L(x) < 2$ on $]0, \frac{\pi}{2}[$ as in (2) when $n \geq m \geq 1$ (by Lemma 2). ■
Corollary 1  For the function \( f(x) \) given, if \( 0 < n < 1, \ m \geq 1, \ am \geq 2b, \) then \( f(x) > a + b \) on \( |0, \frac{\pi}{2}|. \)

Corollary 2  Let \( a \geq 0 \) and \( b \geq 0. \) If \( am \leq 2bn, \ a((\frac{x}{2})^m - 1) \leq b, \ m \) and \( n \) are of same sign not equal to zero and \( 0 > \min(m, n) \) or \( \min(m, n) \geq 1, \) then

\[
f(x) := a \left( \frac{x}{\sin(x)} \right)^m + b \left( \frac{x}{\tan(x)} \right)^n < a + b \text{ for all } x \in [0, \frac{\pi}{2}].
\]

Proof. From Lemma 1 and Theorem 1, it is easy to see that: under stated conditions \( f \) has at most one single critical point (minimum) on \( |0, \frac{\pi}{2}|; \) by the regularity of \( f \) and its boundary limit values \( f(x) < a + b \) for all \( x \in [0, \frac{\pi}{2}]. \)

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References


