Depth and Stanley depth of the edge ideals of the powers of paths and cycles

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Abstract

Let $k$ be a positive integer. We compute depth and Stanley depth of the quotient ring of the edge ideal associated to the $k^{th}$ power of a path on $n$ vertices. We show that both depth and Stanley depth have the same values and can be given in terms of $k$ and $n$. If $n \equiv 0, k + 1, k + 2, \ldots, 2k(\text{mod}(2k + 1))$, then we give values of depth and Stanley depth of the quotient ring of the edge ideal associated to the $k^{th}$ power of a cycle on $n$ vertices and tight bounds otherwise, in terms of $n$ and $k$. We also compute lower bounds for the Stanley depth of the edge ideals associated to the $k^{th}$ power of a path and a cycle and prove a conjecture of Herzog for these ideals.

1 Introduction

Let $K$ be a field and $S := K[x_1, \ldots, x_n]$ the polynomial ring over $K$. Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. A Stanley decomposition of $M$ is a presentation of the $K$-vector space $M$ as a finite direct sum $\mathcal{D} : M = \bigoplus_{i=1}^{s} v_i K[W_i]$, where $v_i \in M$, $W_i \subseteq \{x_1, \ldots, x_n\}$, and $v_i K[W_i]$ denotes the $K$-subspace of $M$, which is generated by all elements $v_i w$, where $w$ is a monomial in $K[W_i]$. The $\mathbb{Z}^n$-graded $K$-subspace $v_i K[W_i] \subseteq M$ is called a Stanley space of dimension $|W_i|$, if $v_i K[W_i]$ is a free $K[W_i]$-module, where $|W_i|$ denotes the cardinality of $W_i$. Define $\text{sdepth}(\mathcal{D}) = \min\{|W_i| : i = 1, \ldots, s\}$.
and sdepth($M$) = max{sdepth($D$) : $D$ is a Stanley decomposition of $M$}. The number sdepth($D$) is called the Stanley depth of decomposition $D$ and sdepth($M$) is called the Stanley depth of $M$. Stanley conjectured in [24] that sdepth($M$) $\geq$ depth($M$) for any $\mathbb{Z}^n$-graded $S$-module $M$. This conjecture was disproved by Duval et al. in [8] as was expected due to different nature of these two invariants. However, the relation between Stanley depth and some other invariants has already been established; see [11, 12, 21, 26]. In [11], Herzog, Vladoiu and Zheng proved that the Stanley depth of $M$ can be computed in a finite number of steps, if $M = J/I$, where $I \subset J \subset S$ are monomial ideals. But practically it is too hard to compute Stanley depth by using this method; see for instance, [2, 5, 15, 16]. For computing Stanley depth for some classes of modules we refer the reader to [14, 20, 22, 23]. In this paper we attempt to find values and reasonable bounds for depth and Stanley depth of $I$ and $S/I$, where $I$ is the edge ideal of a power of a path or a cycle. We also compare the values of sdepth($I$) and sdepth($S/I$) and give positive answers to the following conjecture of Herzog.

Conjecture 1.1. [9] Let $I \subset S$ be a monomial ideal then sdepth($I$) $\geq$ sdepth($S/I$).

The above conjecture is proved in some other cases; see [13, 16, 20, 23]. The paper is organized as follows: First two sections are devoted to introduction, definitions, notation, and discussion of some known results. In third section, we compute depth and Stanley depth of $S/I(P^k_n)$, where $I(P^k_n)$ denotes the edge ideal of the $k^{th}$ power of a path $P_n$ on $n$ vertices. Let for $q \in \mathbb{Q}$, $\lceil q \rceil$ denotes the smallest integer greater than or equal to $q$. Then in Theorems 3.8 and 3.14 we prove that

$$\text{depth}(S/I(P^k_n)) = \text{sdepth}(S/I(P^k_n)) = \lceil \frac{n}{2k+1} \rceil.$$ 

Let $I(C^k_n)$ be the edge ideal of the $k^{th}$ power of a cycle $C_n$ on $n$ vertices. In fourth section we give some lower bounds for depth and Stanley depth of $S/I(C^k_n)$; see Theorems 4.5 and 4.7. If $n \geq 2k+2$, then by Corollaries 4.6 and 4.8 we prove that if $n \equiv 0, k+1, \ldots, 2k(\text{mod}(2k+1))$ then depth($S/I(C^k_n)$) = sdepth($S/I(C^k_n)$) = $\lceil \frac{n}{2k+1} \rceil$. Otherwise,

$$\lceil \frac{n}{2k+1} \rceil - 1 \leq \text{depth}(S/I(C^k_n)), \text{sdepth}(S/I(C^k_n)) \leq \lceil \frac{n}{2k+1} \rceil.$$ 

Last section is devoted to Conjecture 1.1 for $I(P^k_n)$ and $I(C^k_n)$. By our Theorem 5.2 we have

$$\text{sdepth}(I(P^k_n)) \geq \lceil \frac{n}{2k+1} \rceil + 1.$$
which shows that $I(P^k_n)$ satisfies Conjecture 1.1. Let $n \geq 2k + 1$. Proposition 5.3 gives a lower bound for $I(C^n_k)/I(P^k_n)$ that is
\[
sdepth(I(C^n_k)/I(P^k_n)) \geq \lceil \frac{n + k + 1}{2k + 1} \rceil.
\]

Corollary 5.5 of this paper proves that $I(C^n_k)$ satisfies Conjecture 1.1.

2 Definitions and notation

Throughout this paper $m$ denotes the unique maximal graded ideal $(x_1, \ldots, x_n)$ of $S$. We set $S_m := K[x_1, x_2, \ldots, x_n]$, supp($v$) := \{i \in \mathbb{N}: x_i|v\}$ and supp($I$) := \{i : x_i|u, \text{ for some } u \in \mathbb{S}(I)\}, where $\mathbb{S}(I)$ denotes the unique minimal set of monomial generators of the monomial ideal $I$. Let $I \subseteq S$ be an ideal. Then we write $I$ instead of $IS$. Thus every ideal will be considered an ideal of $S$ unless otherwise stated. Let $I$ and $J$ be monomial ideals of $S$, then for $I + J$ we write $(I, J)$.

We review some notation and refer the reader to [3] for further details. Let $G$ be a simple graph. For a positive integer $k$, the $k$th power of graph $G$ is another graph $G^k$ on the same set of vertices, such that two vertices are adjacent in $G^k$ when their distance in $G$ is at most $k$. In the whole paper we label the vertices of the graph $G$ by $1, 2, \ldots, n$. We denote the set of vertices of $G$ by $[n] := \{1, 2, \ldots, n\}$ and its edge set by $E(G)$. We assume that all graphs and their powers are simple graphs. We also assume that all graphs have at least two vertices and a non-empty edge set. For a graph $G$, the edge ideal $I(G)$ associated to $G$ is defined as $I(G) := \langle x_i x_j : \{i, j\} \in E(G) \rangle$. For $n \geq 2$, a graph $G$ is called a path if $E(G) = \{\{i, i + 1\} : i \in [n - 1]\}$. A path on $n$ vertices is denoted by $P_n$. For $n \geq 3$, a graph $G$ is called a cycle if $E(G) = \{\{i, i + 1\} : i \in [n - 1]\} \cup \{1, n\}$. A cycle on $n$ vertices is denoted by $C_n$. For $n \geq 2$, the $k$th power of a path, denoted by $P^k_n$, is a graph such that for all $1 \leq i < j \leq n$, $\{i, j\} \in E(P^k_n)$ if and only if $0 < j - i \leq k$. If $n \leq k + 1$, then $P^k_n$ is a complete graph on $n$ vertices. If $n \geq k + 2$, then
\[
E(P^k_n) = \bigcup_{i=1}^{n-k-1} \{\{i, i + 1\}, \{i, i + 2\}, \ldots, \{i, i + k\}\} \cup \bigcup_{j=n-k+1}^{n-1} \{\{j, j + 1\}, \{j, j + 2\}, \ldots, \{j, n\}\}.
\]

For $n \geq 3$, the $k$th power of a cycle, denoted by $C^k_n$, is a graph such that for all vertices $1 \leq i, j \leq n$, $\{i, j\} \in E(C^k_n)$ if and only if $|j - i| \leq k$ or $|j - i| \geq n - k$. If $n \leq 2k + 1$, then $C^k_n$ is a complete graph on $n$ vertices. If $n \geq 2k + 2$, then
\[
E(C^k_n) = E(P^k_n) \cup \bigcup_{l=1}^{k} \{\{l, l+n-k\}, \{l, l+n-k+1\}, \{l, l+n-k+2\}, \ldots, \{l, n\}\}.
\]
For examples of powers of paths and cycles see Figures 1 and 2.

If \( n \leq k + 1 \), then \( I(P_n^k) \) is a squarefree Veronese ideal of degree 2. If \( n \geq k + 2 \), then

\[
G(I(P_n^k)) = \bigcup_{i=1}^{n-k} \{x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_{i+k}\} \cup \bigcup_{j=n-k+1}^{n-1} \{x_j x_{j+1}, x_j x_{j+2}, \ldots, x_j x_n\}.
\]

If \( n \leq 2k + 1 \), then \( I(C_n^k) \) is a squarefree Veronese ideal of degree 2. If \( n \geq 2k + 2 \), then

\[
G(I(C_n^k)) = G(I(P_n^k)) \cup \bigcup_{l=1}^{k} \{x_l x_{l+n-k}, x_l x_{l+n-k+1}, \ldots, x_l x_n\}.
\]

\[\begin{array}{c}
\text{Figure 1: From left to right, } P_{12}^3 \text{ and } P_{12}^4 \text{ respectively.}
\end{array}\]

\[\begin{array}{c}
\text{Figure 2: From left to right, } C_{10}^3 \text{ and } C_{10}^4 \text{ respectively.}
\end{array}\]

**Lemma 2.1** ([17, Lemma 3]). If \( n \geq k + 1 \), then \( |G(I(P_n^k))| = nk - \frac{k(k+1)}{2} \).

**Rem. 2.2.** If \( n \geq 2k + 1 \), then \( |G(I(C_n^k))| = nk \).

Let \( G \) be a graph and \( i \in [n] \), then \( N_G(x_i) := \{x_j : x_i x_j \in G(G)\} \), where \( j \in [n] \setminus \{i\} \). For \( k \geq 2 \), \( 0 \leq i \leq k - 1 \) and \( n \geq 2k + 2 \), let \( A_{n-k-1} \).
Lemma 2.4 (Lemma 2.2). Let 0 → $U \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of modules over a local ring $S$, or a Noetherian graded ring with $S_0$ local, then

1. $\text{depth } M \geq \min\{\text{depth } N, \text{depth } U\}$.
2. $\text{depth } U \geq \min\{\text{depth } M, \text{depth } N + 1\}$.
3. $\text{depth } N \geq \min\{\text{depth } U - 1, \text{depth } M\}$.

Lemma 2.5 (Lemma 2.2). Let 0 → $U \rightarrow V \rightarrow W \rightarrow 0$ be a short exact sequence of $\mathbb{Z}^n$-graded $S$-modules. Then

$sdepth(V) \geq \min\{sdepth(U), sdepth(W)\}$.

The above Lemma can also be seen as an immediate consequence of the result of J. Apel [1, Sec.3].
3 Depth and Stanley of cyclic modules associated to the edge ideals of the powers of a path

We start this section with some results. These results are essential for computations of depth and Stanley depth of $S/I(P^k_n)$.

**Lemma 3.1.** Let $a \geq 2$ be an integer, $\{E_i : 1 \leq i \leq a\}$ and $\{G_i : 0 \leq i \leq a\}$ be some families of $\mathbb{Z}^n$-graded $S$-modules such that we have the following short exact sequences:

\[
0 \rightarrow E_1 \rightarrow G_0 \rightarrow G_1 \rightarrow 0 \\
0 \rightarrow E_2 \rightarrow G_1 \rightarrow G_2 \rightarrow 0 \\
\vdots \\
0 \rightarrow E_{a-1} \rightarrow G_{a-2} \rightarrow G_{a-1} \rightarrow 0 \\
0 \rightarrow E_a \rightarrow G_{a-1} \rightarrow G_a \rightarrow 0
\]

and depth($G_a)$ $\geq$ depth($E_a$), depth($E_i$) $\geq$ depth($E_{i-1}$) for all $2 \leq i \leq a$. Then depth($G_0$) = depth($E_1$).

**Proof.** By assumption, we have depth($G_a)$ $\geq$ depth($E_a$), applying Depth Lemma on the exact sequence (a) we get depth($G_{a-1}$) = depth($E_a$). We also have by assumption

depth($G_{a-1}$) = depth($E_a$) $\geq$ depth($E_{a-1}$).

By applying Depth Lemma on the exact sequence (a-1) we have depth($G_{a-2}$) = depth($E_{a-1}$). We repeat the same steps on all exact sequences one by one from bottom to top and we get depth($G_{i-1}$) = depth($E_i$) for all $i$. Thus if $i = 1$ then we have depth($G_0$) = depth($E_1$).

**Lemma 3.2.** Let $k \geq 2$ and $n \geq 2k+2$. Then

\[
S/(I(P^k_n), A_{n-1}) \cong S_{n-k-1}/I(P^k_{n-k-1})[x_n].
\]

**Proof.** Since

\[
\mathcal{G}(I(P^k_n)) = \bigcup_{i=1}^{n-k} \{x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_{i+k}\} \cup \bigcup_{i=n-k+1}^{n-1} \{x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_n\},
\]
so we have

\[ I(P_n^k) + A_{n-1} = A_{n-1} + \]
\[
\left\lfloor \sum_{i=1}^{n-2k-1} (x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_{i+k}) + \sum_{i=n-2k}^{n-k} (x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_{i+k}) + \right. \\
\left. \sum_{i=n-k+1}^{n-1} (x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_n) \right\rfloor = \sum_{i=1}^{n-2k-1} (x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_{i+k}) + \\
\sum_{i=n-k+1}^{n-2k-2} (x_i x_{i+1}, x_i x_{i+2}, \ldots, x_i x_{n-1}) + A_{n-1} = I(P_{n-k-1}^k) + A_{n-1}.
\]

Thus the required result follows.

Lemma 3.3. Let \( k \geq 2, 0 \leq i \leq k - 1 \) and \( n \geq 3k + 2 \). Then

\[ S/(I(P_n^k) : x_{n-k+i}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].\]

Proof. It is enough to prove that \( (I(P_n^k) : x_{n-k+i}) = (I(P_{n-k-1+i}^k), B_{n-k+i}) \).

Clearly

\[ I(P_{n-2k-1+i}^k) \subset I(P_n^k) \subset (I(P_n^k) : x_{n-k+i}).\]

Let \( u \in B_{n-k+i} \), then by definition of \( I(P_n^k) \), \( u x_{n-k+i} \in I(P_n^k) \) that is \( u \in (I(P_n^k) : x_{n-k+i}) \). Thus \( B_{n-k+i} \subset (I(P_n^k) : x_{n-k+i}) \) and we have \( (I(P_{n-2k-1+i}^k), B_{n-k+i}) \subset (I(P_n^k) : x_{n-k+i}) \). Now for the other inclusion, let \( w \) be a monomial generator of \( (I(P_n^k) : x_{n-k+i}) \), then \( w = \frac{v}{\text{gcd}(v, x_{n-k+i})} \), where \( v \in S(I(P_n^k)) \). If \( \text{supp}(v) \cap \text{supp}(B_{n-k+i}) \neq \emptyset \), then we have \( w \in S(B_{n-k+i}) \) and if \( \text{supp}(v) \cap \text{supp}(B_{n-k+i}) = \emptyset \), then \( w \in S(I(P_n^k)) \cap K[x_1, x_2, \ldots, x_{n-2k-1+i}] = S(I(P_{n-2k-1+i}^k)) \).

Lemma 3.4. Let \( n \geq 3k + 2 \) and \( 0 \leq i \leq k - 1 \), then we have

\[ S/(I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i} \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].\]

Proof. As \( (I(P_n^k), A_{n-k+(i-1)}) : x_{n-k+i} = (I(P_n^k) : x_{n-k+i}), A_{n-k+(i-1)} \).

Now using the proof of Lemma 3.3 we obtain

\[ (I(P_n^k) : x_{n-k+i}, A_{n-k+(i-1)}) = (I(P_{n-2k-1+i}^k), B_{n-k+i}, A_{n-k+(i-1)}) = (I(P_{n-2k-1+i}^k), B_{n-k+i}), \]

as \( A_{n-k+(i-1)} \subset B_{n-k+i} \). Thus the required result follows by Lemma 3.3.
**Remark 3.5.** Let $m \geq 2$ and $I(P_m^{m-1}) \subset S_m = K[x_1, x_2, \ldots, x_m]$ be the edge ideal of the $(m - 1)^{th}$ power of path $P_m$. Then $I(P_m^{m-1})$ is a squarefree Veronese ideal of degree 2 in variables $x_1, x_2, \ldots, x_m$. Thus by [10, Corollary 10.3.7] and Theorem 3.9

\[
\text{depth}(S_n/I(P_m^{m-1})) = \text{sdepth}(S_n/I(P_m^{m-1})) = 1.
\]

**Remark 3.6.** Let $k \geq 2$ and $2k + 2 \leq n \leq 3k + 1$, then it is easy to see that

1. If $n = 2k + 2$, then
   \[
   S/(I(P_n^k) : x_n) = S/(x_2, \ldots, x_{n-k-1}, x_{n-k+1}, \ldots, x_n) \cong K[x_1, x_{n-k}].
   \]

2. If $0 \leq i < k - 1$ and $n > 2k + 2$, then
   \[
   S/(I(P_n^k) : x_{n-k+i}) = S/((I(P_n^k), A_{n-k+i}) : x_{n-k+i})
   \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}]
   \begin{cases}
   S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}], & \text{if } n - 2k - 1 + i \geq k + 1; \\
   S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}], & \text{otherwise}.
   \end{cases}
   \]

We recall a lemma from [11] which is heavily used in this paper.

**Lemma 3.7 ([11, Lemma 3.6]).** Let $J \subset I$ be monomial ideals of $S$, and let $T = S[x_{n+1}]$ be the polynomial ring over $S$ in the variable $x_{n+1}$. Then \( \text{depth}(IT/JT) = \text{depth}(I/J) + 1 \) and \( \text{sdepth}(IT/JT) = \text{sdepth}(I/J) + 1 \).

**Theorem 3.8.** Let $n \geq 2$. Then \( \text{depth}(S/I(P_n^k)) = \lceil \frac{n}{2k+1} \rceil \).

**Proof.**

1. If $n \leq k + 1$, then $I(P_n^k)$ is a squarefree Veronese ideal thus by Remark 3.5, \( \text{depth}(S/I(P_n^k)) = 1 = \lceil \frac{n}{2k+1} \rceil \).

2. For $n \geq k + 2$, we consider the following cases:

   - **(1) If $k = 1$, then by [18, Lemma 2.8] we have \( \text{depth}(S/I(P_1^1)) = \lceil \frac{n}{3} \rceil = \lceil \frac{n}{2k+1} \rceil \).**
   - **(2) If $k \geq 2$ and $2k + 2 \leq n \leq 2k + 1$, then we get \( \text{depth}(S/I(P_n^k)) \geq 1 \) as \( m \not\in \text{Ass}(S/I(P_n^k)) \). Since $x_{k+1} \not\in I(P_n^k)$ and $x_s x_{k+1} \in S(I(P_n^k))$ for all $s \in \{1, 2, \ldots, k, k + 2, \ldots, n\}$, therefore we have \( (I(P_n^k) : x_{k+1}) = (x_1, \ldots, x_k, x_{k+2}, \ldots, x_n) \). By [23, Corollary 1.3], we have
     \[
     \text{depth}(S/I(P_n^k)) \leq \text{depth}(S/(I(P_n^k) : x_{k+1}))
     = \text{depth}(S/(x_1, \ldots, x_k, x_{k+2}, \ldots, x_n)) = 1.
     \]
     Thus \( \text{depth}(S/I(P_n^k)) = 1 = \lceil \frac{n}{2k+1} \rceil \).
(3) For $k \geq 2$, $2k + 2 \leq n \leq 3k + 1$ and $0 \leq i \leq k - 1$, consider the family of short exact sequences

\[
0 \longrightarrow S/(I(P^k_n), A_{n-k+(i-1)}) : x_{n-k+i} \\
S/(I(P^k_n), A_{n-k+(i-1)}) \longrightarrow S/(I(P^k_n), A_{n-k+i}) \longrightarrow 0
\]

By Lemma 3.2, $S/(I(P^k_n), A_{n-1}) \cong S_{n-k-1}/I(P^k_{n-k-1})[x_n]$. Since we are considering the case $2k + 2 \leq n \leq 3k + 1$ which implies that $k + 1 \leq n - k - 1 \leq 2k$. If $n - k - 1 = k + 1$ then $S_{n-k-1}/I(P^k_{n-k-1}) = S_{k+1}/I(P^k_{k+1})$, by Remark 3.5 and Lemma 3.7 we have depth $S/(I(P^k_n), A_{n-1}) = 2$. If $2k + 1 < n - k - 1 \leq 2k$, then by case(b)(2) depth$(S_{n-k-1}/I(P^k_{n-k-1})) = 1$. Thus by Lemma 3.7 we have depth $(S/(I(P^k_n), A_{n-1})) = 2$. Now we show that depth $(S/(I(P^k_n) : x_{n-k})) = 2$. For this we consider two cases:

If $n = 2k + 2$, then by Remark 3.6

\[
S/(I(P^k_n) : x_{n-k}) = \]

\[
S/(x_2, x_3, \ldots, x_{n-k-1}, x_{n-k+1}, \ldots, x_n) \cong K[x_1, x_{n-k}],
\]

and thus depth $(S/(I(P^k_n) : x_{n-k})) = 2$. If $n > 2k + 2$, by Remark 3.6 we have

\[
S/(I(P^k_n) : x_{n-k}) \cong S_{n-2k-1}/I(P^n_{n-2k-2})[x_{n-k}],
\]

where $2 \leq n - 2k - 1 \leq k$. Thus by Remark 3.5 and Lemma 3.7 we get depth $(S/(I(P^k_n) : x_{n-k})) = 2$. Now for $1 \leq i \leq k - 1$, by Remark 3.6 we obtain

\[
S/(I(P^k_n), A_{n-k+(i-1)}) : x_{n-k+i} = S/(I(P^k_n) : x_{n-k+i})
\]

\[
\cong S_{n-2k-1+i}/I(P^{(n-k+i)}_{n-2k-1+i})[x_{n-k+i}].
\]

Let $T := S_{n-2k-1+i}/I(P^{(n-k+i)}_{n-2k-1+i})[x_{n-k+i}]$. We consider the following cases:

(i) If $k + 1 = n - 2k - 1 + i$, then $T = S_{k+1}/I(P^k_{k+1})[x_{n-k+i}]$, thus by case(a) and Lemma 3.7 we have depth$(T) = 2$.

(ii) For $k+1 < n-2k-1+i$, $T = S_{n-2k-1+i}/I(P^n_{n-2k-1+i})[x_{n-k+i}]$. Since $k + 2 \leq n - 2k - 1 + i \leq 2k - 1$, thus by case(b)(2) and Lemma 3.7 we have depth$(T) = 2$.  

(iii) If $2 \leq n - 2k - 1 + i < k + 1$, then

$$T = S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}],$$

by Remark 3.5 and Lemma 3.7 we have $\text{depth}(T) = 2$.

Thus by Lemma 3.1 we have $\text{depth}(S/I(P_n^k)) = 2$.

(4) For $k \geq 2$, $n \geq 3k + 2$ and $0 \leq i \leq k - 1$, consider the family of short exact sequences

$$0 \rightarrow S/((I(P_n^k), A_{n-k+i-1}) : x_{n-k+i}) \xrightarrow{x_{n-k+i}}$$

$$S/(I(P_n^k), A_{n-k+i-1}) \rightarrow S/(I(P_n^k), A_{n-k+i}) \rightarrow 0$$

By Lemma 3.2, $S/(I(P_n^k), A_{n-1}) \cong S_{n-1}/I(P_{n-1}^k)[x_n]$. Thus by induction on $n$ and Lemma 3.7 we have $\text{depth}(S/(I(P_n^k), A_{n-1})) = \lceil \frac{n-k}{2k+1} \rceil + 1$. By Lemma 3.4 we have

$$S/(I(P_n^k), A_{n-k+i-1}) \cong S_{n-2k-1+i}/I(P_{n-2k-1+i}^k)[x_{n-k+i}].$$

Thus by induction on $n$ and Lemma 3.7 we have

$$\text{depth}(S/(I(P_n^k), A_{n-k+i-1}) : x_{n-k+i}) = \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1.$$

Here we can see that

$$\text{depth}(S/(I(P_n^k), A_{n-1})) = \lceil \frac{n-k-1}{2k+1} \rceil + 1 \geq$$

$$\lceil \frac{n-k-2}{2k+1} \rceil + 1 = \text{depth}(S/(I(P_n^k), A_{n-2}) : x_{n-1}),$$

and for all $1 \leq i \leq k - 1$,

$$\text{depth}(S/(I(P_n^k), A_{n-k+i-1}) : x_{n-k+i}) = \lceil \frac{n-2k-1+i}{2k+1} \rceil + 1 \geq$$

$$\lceil \frac{n-2k-2+i}{2k+1} \rceil + 1 = \text{depth}(S/(I(P_n^k), A_{n-k+i-2}) : x_{n-k+i-1}).$$

Thus by Lemma 3.1 we have $\text{depth}(S/I(P_n^k)) = \lceil \frac{n-2k-1}{2k+1} \rceil + 1 = \lceil \frac{n}{2k+1} \rceil$.

Let $d \in [n]$ and $I_{n,d} := (u \in S \text{ square free monomial} : \deg(u) = d)$. Then $I_{n,d}$ is called squarefree Veronese ideal of degree $d$ in the variables $x_1, x_2, \ldots, x_n$. Cimpoeas proved the following theorems:
Theorem 3.9 ([5, Theorem 1.1]).

1. \( \text{sdepth}(S/I_{n,d}) = d - 1. \)
2. \( d \leq \text{sdepth}(I_{n,d}) \leq \frac{n-d}{d+1} + d. \)

**Theorem 3.10** ([7, Theorem 1.4]). Let \( M \) be a \( \mathbb{Z}^n \)-graded \( S \)-module. If \( \text{sdepth}(M) = 0 \), then \( \text{depth}(M) = 0 \). Conversely, if \( \text{depth}(M) = 0 \) and \( \dim_K(M_a) = 1 \) for any \( a \in \mathbb{Z}^n \), then \( \text{sdepth}(M) = 0 \).

**Lemma 3.11** ([25, Lemma 4]). Let \( n \geq 2 \), then \( \text{sdepth}(S/I(P^1_n)) = \lceil \frac{n}{3} \rceil \).

**Example 3.12.** Let \( n \geq 2 \), and \( n \leq 2k + 1 \), then \( \text{sdepth}(S/I(P^k_n)) = 1 \).

**Proof.** If \( n \leq k + 1 \), then by Theorem 3.9 \( \text{sdepth}(S/I(P^k_n)) = 1 \). Now if \( k + 2 \leq n \leq 2k + 1 \), then \( \text{sdepth}(S/I(P^k_n)) \geq 1 \) as \( m \notin \text{Ass}(S/I(P^k_n)) \), thus by Theorem 3.10 \( \text{sdepth}(S/I(P^k_n)) \geq 1 \). Since \( x_{k+1} \notin I(P^k_n) \) and \( x_kx_{k+1} \in S/I(P^k_n) \) for all \( i \in \{1, \ldots, k, k+2, \ldots, n\} \), therefore \( (I(P^k_n) : x_{k+1}) = (x_1, \ldots, x_k, x_{k+2}, \ldots, x_n) \). Thus by [4, Proposition 2.7] \( \text{sdepth}(S/I(P^k_n)) \leq \text{sdepth}(S/(I(P^k_n) : x_{k+1})) = \text{sdepth}(S/(x_1, \ldots, x_k, x_{k+2}, \ldots, x_n)) = 1. \)

**Proposition 3.13.** Let \( k \geq 2 \) and \( n \geq 2k + 2 \). Then

\[
\text{sdepth}(S/I(P^k_n)) \geq \lceil \frac{n}{2k+1} \rceil.
\]

**Proof.** (1) If \( 2k + 2 \leq n \leq 3k + 1 \), then by applying Lemma 2.4 on the exact sequences in case(b)(3) of Theorem 3.8 we get \( \text{sdepth}(S/I(P^k_n)) \geq 2 = \lceil \frac{n}{2k+1} \rceil \).

(2) If \( n \geq 3k + 2 \), then the proof is similar to Theorem 3.8. We apply Lemma 2.4 on the exact sequences in case(b)(4) of Theorem 3.8 and obtain

\[
\text{sdepth}(S/I(P^k_n)) \geq \min \left\{ \text{sdepth}(S/I(P^k_n, A_{n-1})), \right.
\]

\[
\min_{i=0}^{k-1} \{\text{sdepth}(S/(I(P^k_n, A_{n-k+i-1}) : x_{n-k+i}))\} \geq \lceil \frac{n}{2k+1} \rceil.
\]

**Theorem 3.14.** Let \( n \geq 2 \), then \( \text{sdepth}(S/I(P^k_n)) = \lceil \frac{n}{2k+1} \rceil \).

**Proof.** If \( k = 1 \), then the result follows by Lemma 3.11. Let \( k \geq 2 \). If \( n \leq 2k + 1 \), then by Example 3.12 we have the required result. If \( n \geq 2k + 2 \), then by Proposition 3.13 we have

\[
\text{sdepth}(S/I(P^k_n)) \geq \lceil \frac{n}{2k+1} \rceil.
\]

We need to prove that \( \text{sdepth}(S/I(P^k_n)) \leq \lceil \frac{n}{2k+1} \rceil \), for this we consider the following three cases:
(1) If \( n = (2k + 1)l \), where \( l \geq 1 \). We see that
\[
v = x_{k+1}x_{3k+2}x_{5k+3} \cdots x_{(2k+1)l-k} \in S/I(P_n^k),
\]
but \( x_{t_1}v \in I(P_n^k) \) for all \( t_1 \in [n] \setminus \text{supp}(v) \), thus by Lemma 2.5,
\[
\text{sdepth}(S/I(P_n^k)) \leq l = \lceil \frac{n}{2k+1} \rceil.
\]

(2) If \( n = (2k + 1)l + r \), where \( r \in \{1, 2, 3, \ldots, k+1\} \) and \( l \geq 1 \), then we have
\[
v = x_{k+1}x_{3k+2}x_{5k+3} \cdots x_{(2k+1)l-k}x_{(2k+1)l+r} \in S/I(P_n^k),
\]
and \( x_{t_2}v \in I(P_n^k) \) for all \( t_2 \in [n] \setminus \text{supp}(v) \), so by Lemma 2.5,
\[
\text{sdepth}(S/I(P_n^k)) \leq l + 1 = \lceil \frac{n}{2k+1} \rceil.
\]

(3) If \( n = (2k + 1)l + s \), where \( s \in \{k+2, k+3, \ldots, 2k\} \) and \( l \geq 1 \), since
\[
v = x_{k+1}x_{3k+2}x_{5k+3} \cdots x_{(2k+1)l+k+1} \in S/I(P_n^k),
\]
but \( x_{t_3}v \in I(P_n^k) \) for all \( t_3 \in [n] \setminus \text{supp}(v) \), by Lemma 2.5, we get
\[
\text{sdepth}(S/I(P_n^k)) \leq l + 1 = \lceil \frac{n}{2k+1} \rceil.
\]

4 Depth and Stanley depth of cyclic modules associated to the edge ideals of the powers of a cycle

In this section, we compute bounds for depth and Stanley depth of cyclic modules associated to the edge ideals of powers of a cycle. In order to complete the main task of this section we prove the following three lemmas.

**Lemma 4.1.** Let \( k \geq 2 \) and \( n \geq 3k+2 \), then \( S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}^k) \).

**Proof.** Since \( S(I(C_n^k)) \cap \bigcup_{l=1}^{k-1} \{ x_1x_{l+n-k}, x_1x_{l+n-k+1}, \ldots, x_1x_{n-1} \} \cup \{ x_1x_n, x_2x_n, \ldots, x_kx_n \} \), we have
\[
I(C_n^k) + A_{n-1} = \\
I(P_n^k) + \sum_{l=1}^{k-1} (x_1x_{l+n-k}, x_1x_{l+n-k+1}, \ldots, x_1x_{n-1}) + (x_1x_n, x_2x_n, \ldots, x_kx_n) + A_{n-1}.
\]
Thus by the proof of Lemma 3.2, we obtain $I(P_n^k) + A_{n-1} = I(P_{n-k}^k) + A_{n-1}$.

As
\[ \sum_{i=1}^{k-1} (x_1 x_{i+n-k}, x_1 x_{i+n-k+1}, \ldots, x_1 x_{n-1}) + A_{n-1} = A_{n-1}. \]

Therefore $S/(I(C_n^k), A_{n-1}) = S/(I(P_{n-k}^k), A_{n-1}, (x_1 x_n, x_2 x_n, \ldots, x_k x_n))$
\[ \cong K[x_1, x_2, \ldots, x_{n-k}, x_n]/(I(P_{n-k}^k), (x_1 x_n, x_2 x_n, \ldots, x_k x_n)). \]

After renumbering the variables, we have $K[x_1, \ldots, x_{n-k-1}, x_n]/(I(P_{n-k}^k), (x_1 x_n, x_2 x_n, \ldots, x_k x_n)) \cong S_{n-k}/I(P_{n-k}^k)$. □

**Lemma 4.2.** Let $k \geq 2$ and $n \geq 3k + 2$ and $0 \leq i \leq k - 1$, then
\[ S/(I(C_n^k) : x_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}]. \]

**Proof.** Let $w$ be a monomial generator of $(I(C_n^k) : x_{n-k+i})$. Then $w = \frac{v}{\gcd(v, x_{n-k+i})}$, where $v \in S/(I(C_n^k))$. If $\text{supp}(v) \cap \mathfrak{g}(D_{n-k+i}) \neq \emptyset$, then we have $w \in \mathfrak{g}(D_{n-k+i})$ and if $\text{supp}(v) \cap \mathfrak{g}(D_{n-k+i}) = \emptyset$ then $w \in E := \mathfrak{g}(I(C_n^k)) \cap K[x_{i+1}, x_{i+2}, \ldots, x_{n-2k+1+i}]$. So we obtain $(I(C_n^k) : x_{n-k+i}) \subset E + D_{n-k+i}$. The other inclusion being trivial we get $(I(C_n^k) : x_{n-k+i}) = E + D_{n-k+i}$, which further implies that $S/(I(C_n^k) : x_{n-k+i}) = S/(E + D_{n-k+i})$. After renumbering the variables, we have $S/(I(C_n^k) : x_{n-k+i}) = S/(E, D_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].$ □

**Lemma 4.3.** Let $k \geq 2$, $n \geq 3k + 2$ and $0 \leq i \leq k - 1$. Then
\[ S/(I(C_n^k), A_{n-k+(i-1)}, A_{n-k+(i+1)}) : x_{n-k+i} \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}]. \]

**Proof.** As $(I(C_n^k), A_{n-k+(i+1)}) : x_{n-k+i} = (I(C_n^k) : x_{n-k+i}, A_{n-k+(i-1)})$. By using the same arguments as in the proof of Lemma 4.2 we have
\[ ((I(C_n^k), A_{n-k+(i-1)}), A_{n-k+i}) = (E, D_{n-k+i}, A_{n-k+(i-1)}) = (E, D_{n-k+i}) \]
\[ A_{n-k+(i-1)} \subset D_{n-k+i}. \] Thus the required result follows by Lemma 4.2. □

**Corollary 4.4 ([10, Corollary 10.3.7]).** Let $2 \leq d < n$. Then
\[ \text{depth}(S/I_{n,d}^t) = \max\{0, n - t(n - d) - 1\}. \]
Theorem 4.5. Let \( n \geq 3 \), then
\[
\text{depth}(S/I(C_n^k)) = \begin{cases} 
1, & \text{if } n \leq 2k+1; \\
\left\lceil \frac{n-k}{2k+1} \right\rceil, & \text{if } n \geq 2k+2.
\end{cases}
\]

Proof. (a) If \( n \leq 2k+1 \), then \( I(C_n^k) \) is a squarefree Veronese ideal of degree 2. Thus by Corollary 4.4, \( \text{depth}(S/I(C_n^k)) = 1 \).

(b) For \( n \geq 2k+2 \), we consider the following cases:

(1) If \( k = 1 \), then by [6, Proposition 1.3] \( \text{depth}(S/I(C_1^1)) = \left\lceil \frac{n-1}{3} \right\rceil \).

(2) If \( k \geq 2 \) and \( 2k+2 \leq n \leq 3k+1 \), then we have \( \text{depth}(S/I(C_n^k)) \geq 1 = \left\lceil \frac{n-k}{2k+1} \right\rceil \) as \( m \notin \text{Ass}(S/I(C_n^k)) \).

(3) For \( k \geq 2 \), \( n \geq 3k+2 \) and \( 0 \leq i \leq k-1 \), consider the family of short exact sequences
\[
0 \longrightarrow S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \longrightarrow S/(I(C_n^k), A_{n-k}) \longrightarrow S/(I(C_n^k), A_{n-k+i}) \longrightarrow 0
\]

By Lemma 4.1 we have \( S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}') \). Now by Lemma 4.3, we get
\[
S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}'][x_{n-k+i}].
\]

By Theorem 3.8 and Lemma 3.7, we obtain
\[
\text{depth}(S/((I(C_n^k), A_{n-k+(i-1)}) : x_{n-k+i})) = \left\lceil \frac{n-2k-1}{2k+1} \right\rceil + 1 = \left\lceil \frac{n}{2k+1} \right\rceil.
\]

Again by Theorem 3.8, we have \( \text{depth}(S/(I(C_n^k), A_{n-1})) = \left\lceil \frac{n-k}{2k+1} \right\rceil \). Thus by applying Lemma 2.3(1) on the family of short exact sequences we get
\[
\text{depth}(S/I(C_n^k)) \geq \left\lceil \frac{n-k}{2k+1} \right\rceil.
\]

Corollary 4.6. Let \( n \geq 3 \). If \( n \geq 2k+2 \), then
\[
\text{depth}(S/I(C_n^k)) = \begin{cases} 
\left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } n \equiv 0, k+1, \ldots, 2k \mod(2k+1); \\
\left\lceil \frac{n}{2k+1} \right\rceil - 1 \leq \text{depth}(S/I(C_n^k)) \leq \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } n \equiv 1, \ldots, k \mod(2k+1).
\end{cases}
\]

Proof. By Theorem 4.5, it is enough to prove that \( \text{depth}(S/I(C_n^k)) \leq \left\lceil \frac{n}{2k+1} \right\rceil \) for \( k \geq 2 \) and \( n \geq 2k+2 \). Since \( x_{n-k} \notin I(C_n^k) \), thus by [23, Corollary 1.3] we have \( \text{depth}(S/I(C_n^k)) \leq \text{depth}(S/(I(C_n^k) : x_{n-k})) \). Now we consider two cases:
(1) Let \( 2k + 2 \leq n \leq 3k + 1 \), then \( S/(I(C_n^k) : x_{n-k}) = S/(I(P_n^k) : x_{n-k}) \) so by the proof of Theorem 3.8 we have \( \text{depth}(S/(I(P_n^k) : x_{n-k})) = 2 = \left\lceil \frac{n}{2k+1} \right\rceil \). Therefore
\[
\text{depth}(S/(I(C_n^k))) \leq \text{depth}(S/(I(C_n^k) : x_{n-k})) = 2 = \left\lceil \frac{n}{2k+1} \right\rceil.
\]

(2) Let \( n \geq 3k + 2 \), then by Lemma 4.2,
\[
S/(I(C_n^k) : x_{n-k}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}].
\]
By Lemma 3.7 and Theorem 3.8,\( \text{depth}(S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k}]) = \left\lceil \frac{n}{2k+1} \right\rceil \). Thus \( \text{depth}(S/(I(C_n^k))) \leq \text{depth}(S/(I(C_n^k) : x_{n-k})) = \left\lceil \frac{n}{2k+1} \right\rceil \).

\[ \square \]

**Theorem 4.7.** Let \( n \geq 3 \), then
\[
\text{sdepth}(S/I(C_n^k)) = \begin{cases} 1, & \text{if } n \leq 2k + 1; \\ \left\lceil \frac{n-k}{2k+1} \right\rceil, & \text{if } n \geq 2k + 2. 
\end{cases}
\]

**Proof.**
(a) If \( n \leq 2k + 1 \), then \( \text{sdepth}(S/I(C_n^k)) = 1 \) by Theorem 3.9.
(b) For \( n \geq 2k + 2 \), consider the following cases:

(1) If \( k = 1 \), then by [6, Proposition 1.8] \( \text{sdepth}(S/I(C_n^1)) \geq \left\lceil \frac{n+1}{3} \right\rceil \).

(2) If \( k \geq 2 \) and \( 2k + 2 \leq n \leq 3k + 1 \), then \( \text{depth}(S/I(C_n^k)) \geq 1 \) as \( m \notin \text{Ass}(S/I(C_n^k)) \), thus by Theorem 3.10, \( \text{sdepth}(S/I(C_n^k)) \geq 1 = \left\lceil \frac{n-k}{2k+1} \right\rceil \).

(3) For \( k \geq 2 \), \( n \geq 3k + 2 \) and \( 0 \leq i \leq k - 1 \), consider the family of short exact sequences
\[
0 \longrightarrow S/(I(C_n^k), A_{n-k+i}) : x_{n-k+i}) \longrightarrow S/(I(C_n^k), A_{n-k+i}) \longrightarrow S/(I(C_n^k), A_{n-k+i}) \longrightarrow 0.
\]
By Lemma 4.1 we have \( S/(I(C_n^k), A_{n-1}) \cong S_{n-k}/I(P_{n-k}^k) \). Now by Lemma 4.3, we get
\[
S/(I(C_n^k), A_{n-k+i}) \cong S_{n-2k-1}/I(P_{n-2k-1}^k)[x_{n-k+i}].
\]
By Theorem 3.14 and Lemma 3.7, we obtain
\[
\text{sdepth}(S/(I(C_n^k), A_{n-k+i}) : x_{n-k+i})) = \left\lceil \frac{n-2k-1}{2k+1} \right\rceil + 1 = \left\lceil \frac{n}{2k+1} \right\rceil.
\]
Again by Theorem 3.14, we have \( \text{sdepth}(S/(I(C_n^k), A_{n-1})) = \lceil \frac{n-k}{2k+1} \rceil \).

By applying Lemma 2.4 on the above family of short exact sequences we get \( \text{sdepth}(S/I(C_n^k)) \geq \lceil \frac{n}{2k+1} \rceil \).

\[ \text{Corollary 4.8.} \text{ Let } n \geq 3, \text{ if } n \geq 2k + 2, \text{ then} \]

\[
\begin{align*}
\text{sdepth}(S/I(C_n^k)) &= \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } n \equiv 0, k+1, \ldots, 2k \pmod{2k+1}; \\
\left\lceil \frac{n}{2k+1} \right\rceil - 1 &\leq \text{sdepth}(S/I(C_n^k)) \leq \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } n \equiv 1, \ldots, k \pmod{2k+1}.
\end{align*}
\]

\[ \text{Proof.} \text{ When } k = 1, \text{ then by [6, Theorem 1.9], } \text{sdepth}(S/I(C_n^k)) \leq \left\lceil \frac{n}{2k+1} \right\rceil. \text{ By Theorem 4.7 it is enough to prove that } \text{sdepth}(S/I(C_n^k)) \leq \left\lceil \frac{n}{2k+1} \right\rceil \text{ for } k \geq 2 \text{ and } n \geq 2k + 2. \text{ Since } x_{n-k} \notin I(C_n^k), \text{ thus by [4, Proposition 2.7] we have} \]

\[ \text{sdepth}(S/I(C_n^k)) \leq \text{sdepth}(S/(I(C_n^k) : x_{n-k})). \]

Now we consider two cases:

(1) Let \( 2k + 2 \leq n \leq 3k + 1 \), then \( S/(I(C_n^k) : x_{n-k}) = S/(I(P_n^k) : x_{n-k}) \) so by the proof of Theorem 3.14 we have \( \text{sdepth}(S/(I(P_n^k) : x_{n-k})) = 2 = \left\lceil \frac{n}{2k+1} \right\rceil \). Therefore

\[ \text{sdepth}(S/I(C_n^k)) \leq \text{sdepth}(S/(I(C_n^k) : x_{n-k})) = 2 = \left\lceil \frac{n}{2k+1} \right\rceil. \]

(2) Let \( n \geq 3k + 2 \), then by Lemma 4.2

\[ S/(I(C_n^k) : x_{n-k}) \cong S_{n-2k-1}/(I(P_n^{k-2k-1})[x_{n-k}]). \]

By Lemma 3.7 and Theorem 3.14, \( \text{sdepth}(S_{n-2k-1}/(I(P_n^{k-2k-1})[x_{n-k}]) = \left\lceil \frac{n}{2k+1} \right\rceil \). Thus \( \text{sdepth}(S/I(C_n^k)) \leq \text{sdepth}(S/(I(C_n^k) : x_{n-k})) = \left\lceil \frac{n}{2k+1} \right\rceil \).

\[ \square \]

5 Lower bounds for Stanley depth of edge ideals of \( k \)th powers of paths and cycles and a conjecture of Herzog

In this section we compute some lower bounds for Stanley depth of \( I(P_n^k) \) and \( I(C_n^k) \). These bounds are good enough to prove that Conjecture 1.1 is true for \( I(P_n^k) \) and \( I(C_n^k) \). Let \( 0 \leq i \leq k - 1 \), define

\[ R_{n-k+i} := K[\{x_1, x_2, \ldots, x_n\} \setminus \{x_{n-k}, x_{n-k+1}, \ldots, x_{n-k+i}\}] \]
and

\[ B'_{n-k+i} := \{ x_j : x_j \in N_{P_n^k}^+(x_{n-k+i}) \} \setminus \{ x_{n-k}, x_{n-k+1}, \ldots, x_{n-k+(i-1)} \} \].

Thus \( R_{n-k+i} \) is a subring of \( S \) and \( B'_{n-k+i} \) is a monomial prime ideal of \( S \). Let \( I \subset Z = K[x_{i_1}, x_{i_2}, \ldots, x_{i_r}] \) be a monomial ideal and \( Z' := Z[x_{i_r+1}] \). Then we write \( IZ' = I[x_{i_r+1}] \). Now we recall a useful remark of Cimpoeas.

**Remark 5.1.** [4, Remark 1.7] Let \( I \) be a monomial ideal of \( S \), and \( I' = (I, x_{n+1}, x_{n+2}, \ldots, x_{n+m}) \) be a monomial ideal of \( S' = S[x_{n+1}, x_{n+2}, \ldots, x_{n+m}] \). Then

\[ \operatorname{sdepth}_S(I') \geq \min\{ \operatorname{sdepth}_S(I) + m, \operatorname{sdepth}_S(S/I) + \left\lceil \frac{m}{2} \right\rceil \}. \]

**Theorem 5.2.** Let \( n \geq 2 \), then \( \operatorname{sdepth}(I(P_n^k)) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1 \).

**Proof.**

(a) If \( n \leq 2k + 1 \), then as the minimal generators of \( I(P_n^k) \) have degree 2, by [15, Lemma 2.1] we have \( \operatorname{sdepth}(I(P_n^k)) \geq 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 1 \).

(b) For \( n \geq 2k + 2 \), if \( k = 1 \), then by [19, Theorem 2.3], \( \operatorname{sdepth}(I(P_n^1)) \geq n - \lfloor \frac{n-1}{2} \rfloor = \left\lceil \frac{n-1}{2} \right\rceil + 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1 \). Now for \( k \geq 2 \), we prove this result by induction on \( n \). We consider the following decomposition of \( I(P_n^k) \) as a vector space:

\[ I(P_n^k) = I(P_n^k) \cap R_{n-k} \oplus x_{n-k}(I(P_n^k) : x_{n-k})S. \]

Similarly, we can decompose \( I(P_n^k) \cap R_{n-k} \) as follows:

\[ I(P_n^k) \cap R_{n-k} = I(P_n^k) \cap R_{n-k} \oplus x_{n-k+1}(I(P_n^k) \cap R_{n-k} : x_{n-k+1})R_{n-k}. \]

Continuing in the same way for \( 1 \leq i \leq k-1 \) we have

\[ I(P_n^k) \cap R_{n-k+i} = I(P_n^k) \cap R_{n-k+(i+1)} \oplus x_{n-k+(i+1)}(I(P_n^k) \cap R_{n-k+i} : x_{n-k+(i+1)})R_{n-k+i}. \]

Finally we get the following decomposition of \( I(P_n^k) \):

\[ I(P_n^k) = I(P_n^k) \cap R_{n-1} \oplus \bigoplus_{i=1}^{k-1} x_{n-k+i}(I(P_n^k) \cap R_{n-k+(i-1)} : x_{n-k+i})R_{n-k+i} \oplus x_{n-k}(I(P_n^k) : x_{n-k})S. \]

Therefore

\[ \operatorname{sdepth}(I(P_n^k)) \geq \min \left\{ \operatorname{sdepth}(I(P_n^k) \cap R_{n-1}), \operatorname{sdepth}(I(P_n^k) : x_{n-k})S, \min_{i=1}^{k-1} \operatorname{sdepth}(I(P_n^k) \cap R_{n-k+i} : x_{n-k+i})R_{n-k+i}) \right\}. \]
As $I(P_n^k) \cap R_{n-1} = \mathcal{S}(I(P_{n-k}^k))[x_n]$, thus by induction on $n$ and Lemma 3.7 we have $\text{sdepth}(I(P_n^k) \cap R_{n-1}) \geq \left\lceil \frac{n-k-1}{2k+1} \right\rceil + 1 \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1$. Now we need to show that $\text{sdepth}(I(P_n^k) : x_{n-k})S \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1$ and

$$\text{sdepth}(I(P_n^k) \cap R_{n-k+1} : x_{n-k+i}R_{n-k+i}) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1.$$ 

For this we consider the following cases:

1. Let $2k + 2 \leq n \leq 3k + 1$. If $n = 2k + 2$, then $(I(P_n^k) : x_{n-k})S = (x_2, x_{n-k-1}, x_{n-k+1}, \ldots, x_n)S$, thus by [2, Theorem 2.2] and Lemma 3.7 we have

$$\text{sdepth}(I(P_n^k) : x_{n-k})S = \left\lceil \frac{n-2}{2} \right\rceil + 2 \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1.$$ 

If $2k + 3 \leq n \leq 3k + 1$, then by Remark 3.6, we get

$$(I(P_n^k) : x_{n-k})S = (\mathcal{S}(I(P_{n-2k-1}^k)), B_{n-k})[x_{n-k}].$$ 

Since $\text{sdepth}(I(P_{n-2k-1}^k)(n-k)) + |\mathcal{S}(B_{n-k})| \geq 2$, by Remark 3.5 we have

$$\text{sdepth}(S_{n-2k-1}/I(P_{n-2k-1}^k) + \left\lceil \frac{|\mathcal{S}(B_{n-k})|}{2} \right\rceil \geq 2,$n$$

then by Remark 5.1, $\text{sdepth}(\mathcal{S}(I(P_{n-2k-1}^k)), B_{n-k}) \geq 2$, and by Lemma 3.7 we have $\text{sdepth}(I(P_n^k) : x_{n-k})S \geq 3 = \left\lceil \frac{n}{2k+1} \right\rceil + 1$. Now since

$$(I(P_n^k) \cap R_{n-k+1} : x_{n-k+i}R_{n-k+i}) = (\mathcal{S}(I(P_{n-2k-1}^k)), B_{n-k+i}'[x_{n-k+i}].$$ 

So by the same arguments we have

$$\text{sdepth}(I(P_n^k) \cap R_{n-k+1} : x_{n-k+i}R_{n-k+i}) \geq 3 = \left\lceil \frac{n}{2k+1} \right\rceil + 1.$$ 

2. If $n \geq 3k + 2$, then by the proof of Lemma 3.3 ($I(P_n^k) : x_{n-k})S = (\mathcal{S}(I(P_{n-2k-1}^k)), B_{n-k})[x_{n-k}]$ and

$$(I(P_n^k) \cap R_{n-k+1} : x_{n-k+i}R_{n-k+i}) = (\mathcal{S}(I(P_{n-2k-1+i}^k)), B_{n-k+i}'[x_{n-k+i}].$$
By Remark 5.1 we have

\[
\text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1)), B_{n-k}) \geq \min \left\{ \text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1))) + (\mathcal{G}(B_{n-k})), \text{sdepth}(S_{n-2k-1}/I(P^k_n - 2k - 1)) + \left\lfloor \frac{|\mathcal{G}(B_{n-k})|}{2} \right\rfloor \right\}
\]

By induction on \( n \) we have \( \text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1))) \geq \left\lceil \frac{n-2k-1}{2k+1} \right\rceil + 1 = \left\lceil \frac{n}{2k+1} \right\rceil \), and by Theorem 3.14, \( \text{sdepth}(S_{n-2k-1}/I(P^k_n - 2k - 1)) = \left\lceil \frac{n}{2k+1} \right\rceil - 1 \). Therefore \( \text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1))), B_{n-k}) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1 \).

Thus by Lemma 3.7 we have \( \text{sdepth}((I(P^k_n) : x_{n-k})S) > \left\lceil \frac{n}{2k+1} \right\rceil + 1 \).

Now using Remark 5.1 again, we get

\[
\text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1+i)), B'_{n-k+i}) \geq \min \left\{ \text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1+i))) + |\mathcal{G}(B'_{n-k+i})|, \text{sdepth}(S_{n-2k-1+i}/I(P^k_n - 2k - 1+i)) + \left\lfloor \frac{|\mathcal{G}(B'_{n-k+i})|}{2} \right\rfloor \right\}
\]

By induction on \( n \) we have \( \text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1+i))) \geq \left\lceil \frac{n-2k-1+i}{2k+1} \right\rceil + 1 \), and by Theorem 3.14 we have \( \text{sdepth}(S_{n-2k-1+i}/I(P^k_n - 2k - 1+i)) = \left\lceil \frac{n-2k-1+i}{2k+1} \right\rceil \). Therefore

\[
\text{sdepth}(\mathcal{G}(I(P^k_n - 2k - 1+i)), B'_{n-k+i}) \geq \left\lceil \frac{n-2k-1+i}{2k+1} \right\rceil + 1.
\]

Thus by Lemma 3.7

\[
\text{sdepth}((I(P^k_n) \cap R_{n-k+i-1} : x_{n-k+i})R_{n-k+i}) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1.
\]

This completes the proof.

\[\Box\]

**Proposition 5.3.** Let \( n \geq 2k + 1 \), then \( \text{sdepth}(I(C^k_n)/I(P^k_n)) \geq \left\lceil \frac{n+k+1}{2k+1} \right\rceil \).

**Proof.** When \( k = 1 \), then by [6, Proposition 1.10] we have the required result.

Now assume that \( k \geq 2 \) and consider the following cases:

\( (1) \). If \( 2k + 1 \leq n \leq 3k + 1 \), then as \( I(C^k_n) \) is a monomial ideal generated by degree 2 so by [11, Theorem 2.1] \( \text{sdepth}(I(C^k_n)/I(P^k_n)) \geq 2 = \left\lceil \frac{n+k+1}{2k+1} \right\rceil \).
Thus by Theorem 3.14 and Lemma 3.7, we have

\[
\text{sdepth}(I(C_n^k)/I(P_n^k)) \geq \min_{s=1}^k \left\lfloor \frac{n - (j_s + s + 2k)}{2k + 1} \right\rfloor + 2.
\]
It is easy to see that \( \max\{j_s + s\} = k + 1 \). Therefore

\[
sdepth(I(C^k_n)/I(P^k_n)) \geq \left\lceil \frac{n - (3k + 1)}{2k + 1} \right\rceil + 2 = \left\lceil \frac{n + k + 1}{2k + 1} \right\rceil.
\]

\[\square\]

**Theorem 5.4.** Let \( n \geq 3 \), then

\[
sdepth(I(C^k_n)) \geq 2, \quad \text{if} \quad n \leq 2k + 1;
\]

\[
sdepth(I(C^k_n)) \geq \left\lceil \frac{n - k}{2k + 1} \right\rceil + 1, \quad \text{if} \quad n \geq 2k + 2.
\]

**Proof.** (a) If \( n \leq 2k + 1 \), then as the minimal generators of \( I(C^k_n) \) have degree 2, so by [15, Lemma 2.1] \( sdepth(I(C^k_n)) \geq 2 \).

(b) If \( n \geq 2k + 2 \), then consider the short exact sequence

\[
0 \longrightarrow I(P^k_n) \longrightarrow I(C^k_n) \longrightarrow I(C^k_n)/I(P^k_n) \longrightarrow 0,
\]

by Lemma 2.4 we have

\[
sdepth(I(C^k_n)) \geq \min\{sdepth(I(P^k_n)), sdepth(I(C^k_n)/I(P^k_n))\}.
\]

By Theorem 5.2, \( sdepth(I(P^k_n)) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1 \), and by Proposition 5.3, we obtain \( sdepth(I(C^k_n)/I(P^k_n)) \geq \left\lceil \frac{n+k+1}{2k+1} \right\rceil = \left\lceil \frac{n-k}{2k+1} \right\rceil + 1 \).

\[\square\]

**Corollary 5.5.** Let \( n \geq 3 \), if \( n \leq 2k + 1 \), then \( sdepth(I(C^k_n)) \geq 2 = sdepth(S/I(C^k_n)) + 1 \). If \( n \geq 2k + 2 \), then

\[
sdepth(I(C^k_n)) \geq sdepth(S/I(C^k_n)), \quad \text{if} \quad n \equiv 1, \ldots, k \pmod{(2k + 1)};
\]

\[
sdepth(I(C^k_n)) \geq sdepth(S/I(C^k_n)) + 1, \quad \text{if} \quad n \equiv 0, k + 1, \ldots, 2k \pmod{(2k + 1)}.
\]

**Proof.** Proof follows by Corollary 4.8, Theorem 4.7 and Theorem 5.4.

**Acknowledgement**

The authors would like to thank the referee for a careful reading of the paper and for valuable comments. This research is partially supported by HEC Pakistan.
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