Generalized 2-absorbing submodules

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Abstract

In this paper, we will introduce the concepts of generalized 2-absorbing submodules of modules over a commutative ring as generalizations of 2-absorbing submodules and obtain some related results.

1 Introduction

Throughout this paper, $R$ will denote a commutative ring with identity, $\mathbb{Z}$ and $\mathbb{N}$ will denote respectively the ring of integers and the set of natural numbers. Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be prime if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [7].

Badawi gave a generalization of prime ideals in [3] and said such ideals 2-absorbing ideals. A proper ideal $I$ of $R$ is a 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that $I$ is a 2-absorbing ideal of $R$ if and only if whenever $I_1$, $I_2$, and $I_3$ are ideals of $R$ with $I_1 I_2 I_3 \subseteq I$, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$. In [4], the authors introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal $I$ of $R$ is called a 2-absorbing primary ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The authors in [6] and [12], extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule $N$ of $M$ is called a 2-absorbing submodule...
of $M$ if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

The purpose of this paper is to introduce the concepts of generalized 2-absorbing submodules of an $R$-module $M$ as a generalizations of 2-absorbing submodules of $M$ and investigate some properties of this class of modules.

2 Generalized 2-absorbing submodules

Definition 2.1. We say that a proper submodule $N$ of an $R$-module $M$ is a generalized 2-absorbing submodule or G2-absorbing submodule of $M$ if whenever $a, b \in R, m \in M$ and $abm \in N$, then $a \in \sqrt{(N :_R m)}$ or $b \in \sqrt{(N :_R m)}$ or $ab \in (N :_R M)$.

Example 2.2. Clearly every 2-absorbing submodule is a G2-absorbing submodule. But the converse is not true in general. For example, the submodule $8\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a G2-absorbing submodule which is not a 2-absorbing submodule. Also, the submodule $(1/p + \mathbb{Z})$ of $\mathbb{Z}_{p^\infty}$, where $p$ is a prime number, is a G2-absorbing submodule which is not a 2-absorbing submodule.

Example 2.3. Consider the submodule $N = 0$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_{42}$. We have $2, 3, 7 = 0$ while $2 \cdot 3 \cdot 7 \neq 0$, $3 \cdot 7 \neq 0$, and $2, 3 \notin (0 :_{\mathbb{Z}} M) = 42\mathbb{Z}$ for all $i, j \in \mathbb{N}$. Thus the submodule $N$ of $M$, is not G2-absorbing submodule.

Lemma 2.4. Let $I$ be an ideal of $R$ and $N$ be a G2-absorbing submodule of $M$. If $a \in R, m \in M$ and $Iam \subseteq N$, then $a \in \sqrt{(N :_R m)}$ or $I \subseteq \sqrt{(N :_R m)}$ or $Ia \subseteq (N :_R M)$.

Proof. Let $a \notin \sqrt{(N :_R m)}$ and $Ia \notin (N :_R M)$. Then there exists $b \in I$ such that $ba \notin (N :_R M)$. Now, $bam \in N$ implies that $b \in \sqrt{(N :_R m)}$, since $N$ is a G2-absorbing submodule of $M$. We have to show that $I \subseteq \sqrt{(N :_R m)}$. Let $c$ be an arbitrary element of $I$. Thus $(b + c)a \in N$. Hence, either $b + c \in \sqrt{(N :_R m)}$ or $(b + c)a \in (N :_R M)$. If $b + c \in \sqrt{(N :_R m)}$, then by $b \in \sqrt{(N :_R m)}$ it follows that $c \in \sqrt{(N :_R m)}$. If $(b + c)a \in (N :_R M)$, then $ca \notin (N :_R M)$, but $cam \in N$. Thus $c \in \sqrt{(N :_R m)}$. Hence, we conclude that $I \subseteq \sqrt{(N :_R m)}$.

Lemma 2.5. Let $I, J$ be ideals of $R$ and $N$ be a G2-absorbing submodule of $M$. If $m \in M$ and $IJm \subseteq N$, then $I \subseteq \sqrt{(N :_R m)}$ or $J \subseteq \sqrt{(N :_R m)}$ or $IJ \subseteq (N :_R M)$.

Proof. Let $I \notin \sqrt{(N :_R m)}$ or $J \notin \sqrt{(N :_R m)}$. We have to show that $IJ \subseteq (N :_R M)$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $a \notin \sqrt{(N :_R m)}$ but $am \subseteq N$. Now, Lemma 2.4, shows that
Let $aJ \subseteq (N :_R M)$ and so $(I \setminus \sqrt{(N :_R m)})J \subseteq (N :_R M)$, similarly there exists $b \in J \setminus \sqrt{(N :_R m)}$ such that $IB \subseteq (N :_R M)$ and also $I(J \setminus \sqrt{(N :_R m)}) \subseteq (N :_R M)$. Thus we have the following.

By $a + c \in I$ and $b + d \in J$ it follows that $(a + c)(b + d)m \in N$. Therefore, $a + c \in \sqrt{(N :_R m)}$ or $b + d \in \sqrt{(N :_R m)}$ or $(a + c)(b + d) \in (N :_R M)$. If $a + c \in \sqrt{(N :_R m)}$, then $c \notin \sqrt{(N :_R m)}$ hence, $c \in I \setminus \sqrt{(N :_R m)}$ which implies that $cd \in (N :_R M)$. Similarly by $(b + d) \in \sqrt{(N :_R m)}$, we can deduce that $cd \in (N :_R M)$. If $(a + c)(b + d) \in (N :_R M)$, then $ab + ad + cb + cd \in (N :_R M)$ and so $cd \in (N :_R M)$. Therefore, $IJ \subseteq (N :_R M)$. \hfill \Box

**Theorem 2.6.** Let $N$ be a proper submodule of $M$. The following statement are equivalent:

(a) $N$ is a $G2$-absorbing submodule of $M$;

(b) If $IJL \subseteq N$ for some ideals $I, J$ of $R$ and a submodule $L$ of $M$, then $I \subseteq \sqrt{(N :_R L)}$ or $J \subseteq \sqrt{(N :_R L)}$ or $IJ \subseteq (N :_R M)$.

**Proof.** (a) $\Rightarrow$ (b) Let $IJL \subseteq N$ for some ideals $I, J$ of $R$, a submodule $L$ of $M$ and $IJ \not\subseteq (N :_R M)$. Then by Lemma 2.5, for all $m \in L$ either $I \subseteq \sqrt{(N :_R m)}$ or $J \subseteq \sqrt{(N :_R m)}$. If $I \not\subseteq \sqrt{(N :_R m)}$, for all $m \in L$ we are done. Similarly if $J \not\subseteq \sqrt{(N :_R m)}$, for all $m \in L$ we are done. Suppose that $m, m_0 \in L$ are such that $I \not\subseteq \sqrt{(N :_R m)}$ and $J \not\subseteq \sqrt{(N :_R m_0)}$. Thus $J \subseteq \sqrt{(N :_R m)}$ and $I \subseteq \sqrt{(N :_R m_0)}$. Since $IJ(m + m_0) \subseteq N$ we have either $I \subseteq \sqrt{(N :_R m + m_0)}$ or $J \subseteq \sqrt{(N :_R m + m_0)}$. By $I \subseteq \sqrt{(N :_R m + m_0)}$, it follows that $I \subseteq \sqrt{(N :_R m)}$ which is a contradiction, similarly by $J \subseteq \sqrt{(N :_R m + m_0)}$ we get a contradiction. Therefore either $I \subseteq \sqrt{(N :_R L)}$ or $J \subseteq \sqrt{(N :_R L)}$.

(b) $\Rightarrow$ (a) This is obvious. \hfill \Box

**Proposition 2.7.** Let $N$ be a $G2$-absorbing submodule of an $R$-module $M$. Then we have the following.

(a) If $K$ is a submodule of $M$ such that $K \not\subseteq N$, then $(N :_R K)$ is a $2$-absorbing primary ideal of $R$.

(b) $(N :_R M)$ is a $2$-absorbing primary ideal of $R$.

**Proof.** (a) Let $a, b, c \in R$ and $abc \in (N :_R K)$. Then $a^t cK \subseteq N$ for some positive integer $t$ or $b^s cK \subseteq N$ for some positive integer $s$ or $abM \subseteq N$ since $N$ is a $G2$-absorbing submodule of $M$. Therefore, $(ac)^t K \subseteq N$ or $(bc)^s K \subseteq N$ or $abK \subseteq N$ as needed.

(b) Since $N$ is a proper submodule of $M$, this follows from part (a). \hfill \Box
Corollary 2.8. Let $N$ be a $G_2$-absorbing submodule of an $R$-module $M$. Then $\sqrt{(N :_R M)}$ is a 2-absorbing ideal of $R$.

Proof. By Proposition 2.7 (b), $(N :_R M)$ is a 2-absorbing primary ideal of $R$. Thus, by [4, Theorem 2.2], we have $\sqrt{(N :_R M)}$ is a 2-absorbing ideal of $R$. □

An $R$-module $M$ is said to be a **multiplication module** if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$ [5].

Corollary 2.9. Let $M$ be a multiplication $R$-module. If $N$ is a $G_2$-absorbing submodule of $M$ such that $\sqrt{(N :_R M)} = (N :_R M)$, then $N$ is a 2-absorbing submodule of $M$.

Proof. By Proposition 2.7 (b), $(N :_R M)$ is a 2-absorbing primary ideal of $R$. Thus $\sqrt{(N :_R M)} = (N :_R M)$ is a 2-absorbing ideal of $R$ by [4, 2.2]. Now the result follows from [2, 3.9]. □

Let $N$ be a submodule of an $R$-module $M$. The intersection of all prime submodules of $M$ containing $N$ is said to be the (prime) radical of $N$ and denote by $\text{rad}(N)$. In case $N$ does not contained in any prime submodule, the radical of $N$ is defined to be $M$ [10].

A proper submodule $N$ of an $R$-module $M$ is said to be a **2-absorbing primary submodule** of $M$ if whenever $a, b \in R$, $m \in M$, and $abm \in N$, then $am \in \text{rad}(N)$ or $bm \in \text{rad}(N)$ or $ab \in (N :_R M)$ [11].

Theorem 2.10. Let $M$ be a multiplication $R$-module and $N$ be a $G_2$-absorbing submodule of $M$. Then $N$ is a 2-absorbing primary submodule of $M$.

Proof. Let $a, b \in R$, $m \in M$, and $abm \in N$. Then we have $a^t m \in N$ for some positive integer $t$ or $b^s m \in N$ for some positive integer $s$ or $abm \in N$. If $abm \subseteq N$, then we are done. Suppose that $a^t m \in N$ for some positive integer $t$. As $M$ is a multiplication $R$-module, $Rm = IM$ for some ideal $I$ of $R$. Thus $a^t IM \subseteq N$. This implies that $Ia \subseteq \sqrt{(N :_R M)}$. Thus

$$aRm = aIM \subseteq \sqrt{(N :_R M)}M \subseteq (\text{rad}(N) :_R M)M \subseteq \text{rad}(N).$$

Hence $am \in \text{rad}(N)$, as needed. □

Corollary 2.11. Let $M$ be a finitely generated multiplication $R$-module. If $N$ is a $G_2$-absorbing submodule of $M$, then $\text{rad}(N)$ is a 2-absorbing submodule of $M$.

Proof. By Theorem 2.10, $N$ is a is a 2-absorbing primary submodule of $M$. Now the result follows from [11, 2.6]. □
**Proposition 2.12.** Let $N$ be a $G2$-absorbing submodule of an $R$-module $M$. Then $(N :_M r)$ is a $G2$-absorbing submodule of $M$ containing $N$ for any $r \in R \setminus (N :_R M)$.

**Proof.** Let $r \in R \setminus (N :_R M)$. Suppose that $a, b \in R$ and $m \in M$ such that $abm \in (N :_M r)$. Then $rabm \in N$. Since $N$ is a $G2$-absorbing submodule of $M$, either $a^s rm \in N$ or $b^s rm \in N$ for some $t, s \in \mathbb{N}$ or $ab \in (N :_R M)$. Thus $a^t m \in (N :_M r)$ or $b^t m \in (N :_M r)$ or $ab \in (N :_R M) \subseteq ((N :_M r) :_R M)$ as required.

**Proposition 2.13.** Let $M$ and $\hat{M}$ be $R$-modules and $f : M \rightarrow \hat{M}$ be an epimorphism. Then we have the following.

(a) If $N$ is a $G2$-absorbing submodule of $M$ such that $\ker(f) \subseteq N$, then $f(N)$ is a $G2$-absorbing submodule of $\hat{M}$.

(b) If $\hat{N}$ is a $G2$-absorbing submodule of $\hat{M}$, then $f^{-1}(\hat{N})$ is a $G2$-absorbing submodule of $M$.

**Proof.** (a) If $f(N) = \hat{M}$, then
\[ \ker(f) + N = f^{-1}(f(N)) = f^{-1}(\hat{M}) = f^{-1}(f(M)) = M. \]
Thus $N = M$ a contradiction. Hence $f(N) \neq \hat{M}$. Now let $a, b \in R$ and $y \in \hat{M}$ such that $aby \in f(N)$. Then there exists $n \in N$ such that $aby = f(n)$. Since $f$ is an epimorphism, we have $f(m) = y$ for some $m \in M$. Thus $abf(m) = f(n)$. This implies that $f(abm - n) = 0$ which gives that $abm - n \in \ker(f) \subseteq N$. So $abm \in N$. Since $N$ is a $G2$-absorbing submodule of $M$, $a^t m \in N$ or $b^t m \in N$ for some $t, s \in \mathbb{N}$ or $ab \in (N :_R M)$. Therefore, $a^t y \in f(N)$ or $b^t y \in f(N)$ or $ab \in (f(N) :_R \hat{M})$. Thus $f(N)$ is a $G2$-absorbing submodule of $M$.

(b) If $f^{-1}(\hat{N}) = M$, then
\[ f(M) \cap \hat{N} = f(f^{-1}(\hat{N})) = f(M) = \hat{M}. \]
Thus $\hat{N} = \hat{M}$ a contradiction. Hence $f^{-1}(\hat{N}) \neq M$. Now let $a, b \in R$ and $m \in M$ such that $abm \in f^{-1}(\hat{N})$. Then $abf(m) \in f(f^{-1}(\hat{N})) = \hat{N}$. Since $\hat{N}$ is a $G2$-absorbing submodule of $\hat{M}$, $a^t f(m) \in \hat{N}$ or $b^t f(m) \in \hat{N}$ for some $t, s \in \mathbb{N}$ or $abM \subseteq \hat{N}$. Therefore, $a^t m \in f^{-1}(\hat{N})$ or $b^t m \in f^{-1}(\hat{N})$ or $abM \subseteq f^{-1}(\hat{N})$. Thus $f^{-1}(\hat{N})$ is a $G2$-absorbing submodule of $M$.

Recall that the set of zero divisors of an $R$-module $M$; denoted by $Z(M)$ is defined by $Z(M) = \{ r \in R : \exists x \in M, rx = 0 \}$.

**Theorem 2.14.** Let $S$ be a multiplicatively closed subset of $R$ and $S^{-1}M$ be the module of fraction of an $R$-module $M$. Then the we have the following.
If \( N \) is a \( G_{2} \)-absorbing submodule of \( M \) such that \((N :_{R} M) \cap S = \emptyset\), then \( S^{-1}N \) is a \( G_{2} \)-absorbing submodule of \( S^{-1}M \).

(b) If \( S^{-1}N \) is a \( G_{2} \)-absorbing submodule of \( S^{-1}M \) such that \( Z(M/N) \cap S = \emptyset \), then \( N \) is a \( G_{2} \)-absorbing submodule of \( M \).

Proof. (a) Assume that \( a, b \in R, s, t, l \in S, m \in M \) and \((a/s)(b/t)(m/l) \in S^{-1}N \) which implies \( uabm \in N \) for some \( u \in S \). Since \( N \) is a \( G_{2} \)-absorbing submodule of \( M \), \( a^{p}um \in N \) or \( b^{q}um \in N \) for some \( p, q \in \mathbb{N} \) or \( ab \in (N :_{R} M) \). Hence \((a/s)^{p}(m/l) = (a^{p}mu)/(s^{p}lu) \in S^{-1}N \) or \((b/t)^{q}(m/l) = (b^{q}mu)/(t^{q}lu) \in S^{-1}N \) since \( ab/st \in S^{-1}(N :_{R} M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M) \). Therefore, \( S^{-1}N \) is a \( G_{2} \)-absorbing submodule of \( S^{-1}M \).

(b) First note that \((S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_{R} M) \) because \( Z(M/N) \cap S = \emptyset \). Let \( a, b \in R \) and \( m \in M \) be such that \( abm \in N \). Then \( abm/1 \in S^{-1}N \). Since \( S^{-1}N \) is a \( G_{2} \)-absorbing submodule of \( S^{-1}M \), either \((a/1)^{p}(m/1) \in S^{-1}N \) or \((b/1)^{q}(m/1) \in S^{-1}N \) for some \( p, q \in \mathbb{N} \) or \( ab/1 \in (S^{-1}N :_{S^{-1}R} S^{-1}M) \) if \( ab/1 \in (S^{-1}N :_{S^{-1}R} S^{-1}M) \), then \( ab/1 \in S^{-1}(N :_{R} M) \) and we are done. Otherwise, there exists \( s \in S \) such that \( sa^{p}m = N \) or \( b^{q}m \in N \). This implies \( a^{p}m \in N \) or \( b^{q}m \in N \), since \( S \cap Z(M/N) = \emptyset \). Hence \( N \) is a \( G_{2} \)-absorbing submodule of \( M \).

\[ \square \]

### 3 \ G_{2} \)-Absorbing submodules over Noetherian rings

A submodule \( N \) of an \( R \)-module \( M \) is said to be idempotent if \( N = (N :_{R} M)^{2}M \). Also, \( M \) is said to be fully idempotent if every submodule of \( M \) is idempotent [1]. Clearly, every fully idempotent \( R \)-module is a multiplication \( R \)-module.

**Theorem 3.1.** Let \( R \) be a Noetherian ring and \( N \) be a submodule of a fully idempotent \( R \)-module \( M \). If \((N :_{R} M) \) is a \( 2 \)-absorbing primary ideal of \( R \), then \( N \) is a \( G_{2} \)-absorbing submodule of \( M \).

**Proof.** Let \( a, b \in R, K \) be a submodule of \( M \), and \( abK \subseteq N \). Then we have \( ab(K :_{R} M)M \subseteq N \). Thus by [4, 2.18], either \( a(K :_{R} M)M \subseteq \sqrt{(N :_{R} M)} \) or \( b(K :_{R} M)M \subseteq \sqrt{(N :_{R} M)} \) and \( ab \in (N :_{R} M) \) since \( (N :_{R} M) \) is a \( 2 \)-absorbing primary ideal of \( R \). If \( ab \in (N :_{R} M) \), then we are done. Otherwise, since \( R \) is Noetherian, \( (a(K :_{R} M))^{t}M \subseteq N \) for some positive integer \( t \) or \( (b(K :_{R} M))^{s}M \subseteq N \) for some positive integer \( s \). Thus \( (a(K :_{R} M))^{t}M \subseteq N \) or \( (b(K :_{R} M))^{s}M \subseteq N \) since \( (N :_{R} M) \) is a \( 2 \)-absorbing primary ideal of \( R \). If \( ab \in (N :_{R} M) \), then we are done. Otherwise, \( a(K :_{R} M)^{t}M \subseteq N \) or \( b(K :_{R} M)^{s}M \subseteq N \) since \( (N :_{R} M)M = N \) by multiplication \( R \)-module. Hence, \( a^{t}K \subseteq N \) or \( b^{s}K \subseteq N \) since \( M \) is a fully idempotent \( R \)-module. Therefore, \( N \) is a \( G_{2} \)-absorbing submodule of \( M \).

\[ \square \]
The following example shows that Theorem 3.1 (a) is not satisfied in general.

**Example 3.2.** The \( Z \)-module \( M = \mathbb{Q} \) is not a fully idempotent \( Z \)-module. Set \( N = \mathbb{Z} \). Then we have 3.2. (1/6) \( \in \mathbb{Z} \) while 3.2. (1/6) \( \notin \mathbb{Z} \), 2.3. (1/6) \( \notin \mathbb{Z} \), and 2.3 \( \notin (\mathbb{Z} :_{\mathbb{Z}} \mathbb{Q}) = 0 \) for all \( i, j \in \mathbb{N} \). Thus the submodule \( N \) of \( M \) is not \( G2 \)-absorbing submodule. But \( (N :_{\mathbb{Z}} M) = 0 \) is a 2-absorbing primary ideal of \( Z \).

Let \( R_i \) be a commutative ring with identity and \( M_i \) be an \( R_i \)-module, for \( i = 1, 2 \). Let \( R = R_1 \times R_2 \). Then \( M = M_1 \times M_2 \) is an \( R \)-module and each submodule of \( M \) is in the form of \( N = N_1 \times N_2 \) for some submodules \( N_1 \) of \( M_1 \) and \( N_2 \) of \( M_2 \).

**Lemma 3.3.** Let \( R = R_1 \times R_2 \) and \( M = M_1 \times M_2 \). Then \( M_i \) is a fully idempotent \( R_i \)-module, for \( i = 1, 2 \) if and only if \( M \) is a fully idempotent \( R \)-module.

**Proof.** First suppose that \( M \) is a fully idempotent \( R \)-module and \( N_1 \) is a submodule of an \( R_1 \)-module \( M_1 \). Then \( N = N_1 \times 0 \) is a submodule of \( M \). Thus \( N = (N :_R M)^2 M = (N_1 :_{R_1} M_1)^2 M_1 \times (0 :_{R_2} M_2)^2 M_2 \). Hence \( N_1 = (N_1 :_{R_1} M_1)^2 M_1 \). Therefore, \( M_1 \) is a fully idempotent \( R_1 \)-module. Similarly, \( M_2 \) is a fully idempotent \( R_2 \)-module. Conversely, let \( N \) be a submodule of \( M \). Then \( N = N_1 \times N_2 \) for some submodules \( N_1 \) of \( M_1 \) and \( N_2 \) of \( M_2 \). By assumption, \( N_i = (N_i :_{R_i} M_i)^2 M_i \) for \( i = 1, 2 \). So

\[
N = (N_1 :_{R_1} M_1)^2 M_1 \times (N_2 :_{R_2} M_2)^2 M_2 = (N :_R M)^2 M,
\]

as request.

**Theorem 3.4.** Let \( R = R_1 \times R_2 \) be a Noetherian ring and \( M = M_1 \times M_2 \), where \( M_1 \) is a fully idempotent \( R_1 \)-module and \( M_2 \) is a fully idempotent \( R_2 \)-module. Then we have the following.

(a) A proper submodule \( K_1 \) of \( M_1 \) is a \( G2 \)-absorbing submodule if and only if \( N = K_1 \times M_2 \) is a \( G2 \)-absorbing submodule of \( M \).

(b) A proper submodule \( K_2 \) of \( M_2 \) is a \( G2 \)-absorbing submodule if and only if \( N = M_1 \times K_2 \) is a \( G2 \)-absorbing submodule of \( M \).

(c) If \( K_1 \) is a primary submodule of \( M_1 \) and \( K_2 \) is a primary submodule of \( M_2 \), then \( N = K_1 \times K_2 \) is a \( G2 \)-absorbing submodule of \( M \).

**Proof.** (a) Let \( K_1 \) be a \( G2 \)-absorbing submodule of \( M_1 \). Then \( (K_1 :_{R_1} M_1) \) is a 2-absorbing primary ideal of \( R_1 \) by Proposition 2.7. Now since \( (N :_{R} M) \)
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Conversely, let \( N = K_1 \times M_2 \) be a G2-absorbing submodule of \( M \). Then 
\[ (N :_{R} M) = (K_1 :_{R_1} M_1) \times R_2 \] is a primary ideal of \( R \) by Proposition 2.7. Thus \( (K_1 :_{R_1} M_1) \) is a primary ideal of \( R_1 \) by [4, 2.23]. Hence by Theorem 3.1, \( K_1 \) is a G2-absorbing submodule of \( M_1 \).

(b) This is proved similar to the part (a).
(c) Let \( K_1 \) be a primary submodule of \( M_1 \) and \( K_2 \) be a primary submodule of \( M_2 \). Then \( (K_1 :_{R_1} M_1) \) and \( (K_2 :_{R_2} M_2) \) are primary ideals of \( R_1 \) and \( R_2 \), respectively. Now since 
\[ (N :_{R} M) = (K_1 :_{R_1} M_1) \times (K_2 :_{R_2} M_2) \]
we have \( (N :_{R} M) \) is a 2-absorbing primary ideal of \( R \) by [4, 2.23]. Thus the result follows from Theorem 3.1.

\[ \square \]

**Theorem 3.5.** Let \( R = R_1 \times R_2 \) be a Noetherian ring and \( M = M_1 \times M_2 \) be a fully idempotent \( R \)-module, where \( M_1 \) is an \( R_1 \)-module and \( M_2 \) is an \( R_2 \)-module. Suppose that \( N = N_1 \times N_2 \) is a proper submodule of \( M \). Then the following conditions are equivalent:

(a) \( N \) is a G2-absorbing submodule of \( M \);

(b) Either \( N_1 = M_1 \) and \( N_2 = M_2 \) and \( N_1 \) is a G2-absorbing submodule of \( M_1 \) or \( N_1, N_2 \) are primary submodules of \( M_1, M_2 \), respectively.

**Proof.** 
(a) \( \Rightarrow \) (b). Let \( N = N_1 \times N_2 \) be a G2-absorbing submodule of \( M \). Then 
\[ (N :_{R} M) = (K_1 :_{R_1} M_1) \times (K_2 :_{R_2} M_2) \]
is a 2-absorbing primary ideal of \( R \) by Proposition 2.7. By [4, 2.23], we have \( (K_1 :_{R_1} M_1) = R_1 \) and \( (K_2 :_{R_2} M_2) \) is a 2-absorbing primary ideal of \( R_2 \) or \( (K_2 :_{R_2} M_2) = R_2 \) and \( (K_1 :_{R_1} M_1) \) is a 2-absorbing primary ideal of \( R_1 \) or \( (K_1 :_{R_1} M_1) \) and \( (K_2 :_{R_2} M_2) \) are primary ideals of \( R_1 \) and \( R_2 \), respectively. Suppose that \( (K_1 :_{R_1} M_1) = R_1 \) and \( (K_2 :_{R_2} M_2) = R_2 \). Then \( N_1 = M_1 \) and \( N_2 = M_2 \) is a G2-absorbing submodule of \( M_2 \) by Theorem 3.4 and Lemma 3.3. Similarly if \( (K_2 :_{R_2} M_2) = R_2 \) and \( (K_1 :_{R_1} M_1) \) is a 2-absorbing primary ideal of \( R_1 \). Then \( N_2 = M_2 \) and \( N_1 \) is a G2-absorbing submodule of \( M_1 \). If the last case hold, then as \( M_1 \) (resp. \( M_2 \)) is a multiplication \( R_1 \)-(resp. \( R_2 \)) module, \( N_1 \) (resp. \( N_2 \)) is a primary submodule of \( M_1 \) (resp. \( M_2 \)) by [8, Corollary 2].

(b) \( \Rightarrow \) (a). This can be proved easily by using Theorem 3.4. \( \square \)

**Theorem 3.6.** Let \( R \) be a Noetherian ring, \( N \) be a G2-absorbing submodule of an \( R \)-module \( M \), and \( m \in M \setminus N \). Then \( (N :_{R} m) \) is a prime ideal of \( R \) or there exists an element \( a \in R \) such that \( (N :_{R} a^n m) \) is a prime ideal of \( R \) for some positive integer \( n \).
Proof. By Corollary 2.8, \( \sqrt{(N : R M)} \) is a 2-absorbing ideal of \( R \), therefore by [4, Theorem 2.3], we have either \( \sqrt{(N : R M)} = p \) or \( \sqrt{(N : R M)} = p \cap q \), where \( p \) and \( q \) are distinct prime ideals of \( R \). Suppose that \( \sqrt{(N : R M)} = p \). Then \( p = \sqrt{(N : R M)} \subseteq \sqrt{(N : R m)} \). We show that \( \sqrt{(N : R m)} \) is a prime ideal of \( R \). Let \( ab \in \sqrt{(N : R m)} \) for some \( a, b \in R \). Then \( (ab)^n \in (N : R m) \) implies \( (ab)^m \in N \). As \( N \) is a G2-absorbing submodule of \( M \), then either \( a^m \in N \) or \( b^m \in N \) for some \( t, s \in \mathbb{N} \) or \( (ab)^m \in (N : R m) \). If \( a^m \in N \) or \( b^m \in N \), then \( a \in \sqrt{(N : R m)} \) or \( b \in \sqrt{(N : R m)} \). If \( (ab)^m \in (N : R M) \), then \( ab \in p \). Since \( p \) is prime ideal of \( R \), then either \( a \in p \subseteq \sqrt{(N : R m)} \) or \( b \in p \subseteq \sqrt{(N : R m)} \). Therefore, \( \sqrt{(N : R m)} \) is a prime ideal of \( R \). Now suppose that \( \sqrt{(N : R M)} = p \cap q \). If \( p \not\subseteq \sqrt{(N : R m)} \), then by previous argument, we have \( \sqrt{(N : R m)} \) is a prime ideal of \( R \). If \( p \not\subseteq \sqrt{(N : R m)} \), then there exists \( a \in p \setminus \sqrt{(N : R m)} \). Also,

\[
pq \subseteq \sqrt{pq} = \sqrt{p \cap q} = \sqrt{(N : R M)} \subseteq \sqrt{(N : R m)}.
\]

Now since \( R \) is Noetherian, there exists \( n \in \mathbb{N} \) such that \( (pq)^n \subseteq (N : R m) \). This implies that \( q \subseteq \sqrt{(N : R a^m)} \) and the result follows by a similar argument. \( \square \)

Now, we study G2-absorbing avoidance Theorem for submodules. We first define an efficient covering of submodules: let \( N, N_1, N_2, ..., N_n \) be submodules of an \( R \)-module \( M \). An efficient covering of \( N \) is a covering \( N \subseteq N_1 \cup N_2 \cup ... \cup N_n \) in which no \( N_k \) (where \( 1 \leq k \leq n \)) satisfies \( N \subseteq N_k \). In other words, a covering \( N \subseteq N_1 \cup N_2 \cup ... \cup N_n \) is efficient if no \( N_k \) is superfluous. Analogously, we say that \( N = N_1 \cup N_2 \cup ... \cup N_n \) is an efficient union if none of the \( N_i \) may be excluded. Any cover or union consisting of submodules of \( M \) can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

To proceed further, we require the following lemma.

**Lemma 3.7.** [9, Lemma 2.1]. Let \( N = N_1 \cup ... \cup N_n \) be an efficient union of submodules of an \( R \)-module \( M \) for \( n > 1 \). Then \( \cap_{j \neq k} N_j = \cap_{j=1}^n N_j \) for all \( k \).

**Theorem 3.8.** Let \( R \) be a Noetherian ring and \( N \subseteq N_1 \cup N_2 \cup ... \cup N_n \) be an efficient covering consisting of submodules of an \( R \)-module \( M \), where \( n > 2 \). If \( \sqrt{(N_j : R M)} \not\subseteq \sqrt{(N_k : R m)} \) for all \( m \in M \setminus N_k \) whenever \( j \neq k \), then no \( N_i \) is a G2-absorbing submodule of \( M \).

**Proof.** Suppose \( N_k \) is a G2-absorbing submodule of \( M \) for some \( 1 \leq k \leq n \), and look for a contradiction. Since \( N \subseteq N_1 \cup N_2 \cup ... \cup N_n \) is an efficient covering, \( N \not\subseteq N_k \), so there exists an element \( m_k \in N \setminus N_k \). It is clear
that \( N = (N_1 \cap N) \cup (N_2 \cap N) \cup \ldots \cup (N_n \cap N) \) is an efficient union. By Lemma 3.7, we have \( \cap_j \neq k (N \cap N_j) \subseteq N \cap N_k \). By using Theorem 3.6, we have either \( \sqrt{(N_k : R \ m_k)} \) is a prime ideal of \( R \) or there exists \( a \in R \) such that \( \sqrt{(N_k : R \ a^m m_k)} \) is a prime ideal of \( R \). First, suppose that \( \sqrt{(N_k : R \ m_k)} \) is a prime ideal. By the given hypothesis \( \sqrt{(N_j : R \ M)} \nsubseteq \sqrt{(N_k : R \ m_k)} \) for \( j \neq k \). So, there exists \( s_j \in \sqrt{(N_j : R \ M)} \) but \( s_j \notin (\sqrt{N_k : R \ m_k}) \), where \( j \neq k \). This implies that \( s_j^{n_j} \in (N_j : R \ M) \) but \( s_j^{n_j} \notin (N_k : R \ m_k) \) where \( j \neq k \) and \( n_j \in \mathbb{N} \). Let \( s = \prod_{j \neq k} s_j \). Then \( s \in \sqrt{(N_j : R \ M)} \) but \( s \notin \sqrt{(N_k : R \ m_k)} \) where \( j \neq k \). Therefore, \( s^m \in (N_j : R \ M) \) for all \( j \neq k \) but \( s^m \notin (N_k : R \ m_k) \), where \( m = \max\{n_1, n_2, \ldots, n_k-1, n_k+1, \ldots, n_n\} \). Thus \( s^m m_k \in \cap_j \neq k (N \cap N_j) \setminus (N \cap N_k) \), since \( s^m m_k \in N \cap N_k \) implies \( s^m \in (N_k : R \ m_k) \), a contradiction. So, no \( N_k \) is a \( G2 \)-absorbing submodule of \( M \). Now, consider the case when \( \sqrt{(N_k : R \ a^m m_k)} \) is a prime ideal, where \( n \) is positive integer and \( a \in R \). Clearly, \( s_j \in \sqrt{(N_j : R \ M)} \) but \( s_j \notin \sqrt{(N_k : R \ a^m m_k)} \), where \( j \neq k \). Therefore, \( s^m a^m m_k \in \cap_j \neq k (N \cap N_j) \setminus (N \cap N_k) \), since \( s^m a^m m_k \in N \cap N_k \) implies \( s^m \in (N_k : R \ a^m m_k) \), a contradiction. So, no \( N_k \) is \( G2 \)-absorbing submodule of \( M \).

**Theorem 3.9.** (\( G2 \)-Absorbing Avoidance Theorem). Let \( R \) be a Noetherian ring and \( N, N_1, \ldots, N_n \) (\( n \geq 2 \)) be submodules of an \( R \)-module \( M \) such that at most two of \( N_1, N_2, \ldots, N_n \) are \( G2 \)-absorbing submodules. If \( N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n \) and \( \sqrt{(N_j : R \ M)} \nsubseteq \sqrt{(N_k : R \ m)} \) for all \( m \in M \setminus N_k \) whenever \( j \neq k \), then \( N \subseteq N_i \) for some \( 1 \leq i \leq n \).

**Proof.** If \( n = 2 \), then it is obvious. Now, take \( n > 2 \) and \( N \nsubseteq N_i \) for all \( 1 \leq i \leq n \). Then \( N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n \) is an efficient covering. Using Theorem 3.8, no \( N_i \) is a \( G2 \)-absorbing submodule, which is a contradiction. Hence \( N \subseteq N_i \) for some \( 1 \leq i \leq n \).

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