Some Extensions of Generalized Morphic Rings and EM-rings

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Abstract

Let R be a commutative ring with unity. The main objective of this article is to study the relationships between PP-rings, generalized morphic rings and EM-rings. Although PP-rings are included in the later rings, the converse is not in general true. We put necessary and sufficient conditions to ensure the converse using idealization and polynomial rings.

1 Introduction

All rings are assumed to be commutative with unity 1. Let \( Z(R) \) be the set of all zero divisors in \( R \), and let \( \text{reg}(R) = R \setminus Z(R) \).

A ring \( R \) is called a morphic ring if for each \( a \in R \), there exists \( b \in R \) such that \( \text{Ann}(a) = bR \) and \( \text{Ann}(b) = aR \). It is known that for reduced commutative rings, morphic rings are equivalent to von Neumann regular rings. A ring \( R \) is called generalized morphic ring if \( \text{Ann}(a) \) is principal for each \( a \in R \), for more details, see [10], [12], [13] and [14]. It is clear that the class of generalized morphic rings includes a wide range of rings such as integral domains, principal ideal rings, von Neumann regular rings, PP-rings, etc. If for each polynomial \( f(x) \in Z(R[x]) \) there exists \( c_f \in R \) and \( f_1(x) \in \text{reg}(R[x]) \) such that \( f(x) = c_f f_1(x) \), then \( R \) is called an EM-ring. Note that in this case \( \text{Ann}_{R[x]}(f) = \text{Ann}_{R[x]}(c_f) \), which simplifies working and characterizing...
zero-divisors in $R[x]$. These rings were defined and characterized in [2], and it was shown there that this class includes a wide range of rings.

It is shown in [2] that if $R$ is a Noetherian ring, then $R$ is generalized morphic if and only if it is an EM-ring. In fact the Noetherian condition is not necessary as will be shown later on.

Recall that if $R$ is a ring, and $M$ is an $R$-module, then the idealization $R(+)M$ is the set of all ordered pairs $(r,m) \in R \times M$, equipped with addition defined by $(r,m) + (s,n) = (r+s, m+n)$ and multiplication defined by $(r,m)(s,n) = (rs, rn + sm)$. It is well-known that $R(+)R \simeq R[x]/(x^2)$. For the general case, we consider the ring $R[x]/(x^{n+1})$, where $n \in \mathbb{N}$. In this case we set $R[x]/(x^{n+1}) = \{ \sum_{i=0}^{n} a_i X^i : a_i \in R, X = x + (x^{n+1}) \}$.

A ring $R$ is called a PP-ring if every principal ideal of $R$ is a projective $R$-module. It is well known that $R$ is a PP-ring if and only if for each $a \in R$, Ann$(a)$ is generated by an idempotent. A ring $R$ is called a PF-ring if every principal ideal of $R$ is a flat $R$-module. It is well known that $R$ is a PF-ring if and only if for each $a \in R$, Ann$(a)$ is pure, i.e. for each $b \in$ Ann$(a)$, there exists $c \in$ Ann$(a)$ such that $b = bc$.

It is clear that a PP-ring is generalized morphic ring, and it was shown in [2] that a PP-ring is also an EM-ring, while $\mathbb{Z}_4$ is generalized morphic EM-ring that is not PP-ring.

In this article we will characterize when some extensions of a generalized morphic ring are generalized morphic. To be more precise; we will characterize when the polynomial ring, the ring $R[x]/(x^{n+1})$ and the idealization of a generalized morphic ring is generalized morphic. We show that the later two rings are generalized morphic if and only if their base ring $R$ is a PP-ring.

We will characterize when the idealization of an EM-ring is an EM-ring. We will also continue the investigation of the polynomial rings of EM-rings we started in [2].

The following two lemmas will be used frequently in the following work.

**Lemma 1.1.** Let $R$ be a reduced ring. If $(a,x), (b,y) \in R(+)R$ such that $(a,x)(b,y) = (0,0)$, then $ab = ay = bx = 0$.

**Proof.** We have $(0,0) = (ab, ay + bx)$, and so,

$$ab = 0,$$

$$ay + bx = 0,$$

$$0 = a(ay + bx) = a^2y + abx = a^2y = 0.$$
Thus, \((ay)^2 = 0\), and since \(R\) is reduced we have \(ay = 0\), whence \(bx = 0\).

**Lemma 1.2.** Let \(R\) be a ring, and let \(S = \{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq (R^+)^n\). Then \(\text{Ann}(S) \neq \{(0, 0)\}\) if and only if \(\text{Ann}(a_1, \ldots, a_n) \neq \{0\}\).

**Proof.** Assume that \((a, b) \neq (0, 0)\) and \((a, b)(a_i, b_i) = (0, 0)\) for all \(i\). Then \(aa_i = 0\) for all \(i\). If \(a = 0\), then \(b \neq 0\) and \(ba_i = 0\) for all \(i\). Thus \(\text{Ann}(a_1, \ldots, a_n) \neq \{0\}\).

Now, if \(a \neq 0\) and \(aa_i = 0\) for all \(i\), then \((0, a)(a_i, b_i) = (0, 0)\) for all \(i\). Thus \(\text{Ann}(S) \neq \{(0, 0)\}\).

Recall that for any ring \(R\), the set \(\text{Min}(R)\) is the set of all minimal prime ideals of \(R\), equipped with the hull kernel topology, and for any set \(I\) of \(R\), \(V(I) = \{P \in \text{Min}(R) : I \subseteq P\}\), and \(\text{Supp}(I) = \text{V}(\text{Ann}(I))\). An ideal \(I\) of \(R\) is called a \(z^0\)−ideal, if whenever \(V(a) \subseteq V(b)\), with \(a \in I\), we have \(b \in I\).

## 2 Generalized Morphic Rings

In this section we will relate reduced generalized morphic rings to complemented rings, and characterize when the polynomial ring of a generalized morphic ring is generalized morphic, and characterize generalized morphic rings using their minimal prime ideals.

A ring \(R\) is called complemented if for each \(a \in R\), there exists \(b \in R\) such that \(ab = 0\) and \(a + b \in \text{reg}(R)\). A reduced ring \(R\) is complemented if and only if for each \(a \in R\) there exists \(b \in R\) such that \(\text{Ann}(\text{Ann}(a)) = \text{Ann}(b)\).

For more properties of complemented reduced rings, see Theorem 2.2 and Proposition 2.5 in [5], and Theorem 4.5 in [9].

It is clear that if \(R\) is a reduced generalized morphic ring, then for any \(a \in R\), there exists \(b \in R\) such that \(\text{Ann}(a) = bR\), and so, \(\text{Ann}(\text{Ann}(a)) = \text{Ann}(b)\). Thus \(R\) is a complemented ring.

For a complemented ring that is not generalized morphic, see Example 5.8 in [7] together with Theorem 1.3 in [8] and Theorem 2.2 below.

Recall that a ring \(R\) is said to be Armendariz if the product of two polynomials in \(R[x]\) is zero if and only if the product of their coefficients is zero.

We now characterize the case at which the polynomial ring of a generalized morphic ring is generalized morphic.

**Theorem 2.1.** If \(R[x]\) is a generalized morphic ring, then \(R\) is generalized morphic. If \(R\) is Armendariz, then the converse is also true.

**Proof.** Assume \(R[x]\) is generalized morphic, and let \(a \in \text{Z}^+(R)\). Then \(\text{Ann}_{R[x]}(a) = f(x)R[x]\), where \(f(x) = \sum_{i=0}^n a_ix^i\). Let \(j\) be the least index such
that $a_j \neq 0$. Then $aa_j = 0$, and so, $a_j R \subseteq \text{Ann}_R(a)$. Let $b \in \text{Ann}_R(a) \setminus \{0\}$. Then $b = f(x)g(x)$ for some $g(x) = \sum_{i=0}^{m} b_i x^i \in R[x]$. Then $b = a_0 b_0 \in a_0 R$, and $a_0 \neq 0$. Thus $j = 0$ and $\text{Ann}_R(a) = a_0 R$ is principal, and hence $R$ is generalized morphic.

For the converse assume $R$ is Armendariz generalized morphic and let $f(x) = \sum_{i=0}^{m} a_i x^i \in Z(R[x])$. Then there exists $a \in R$ such that $aa_i = 0$ for all $i$. Thus $\{0\} \neq \text{Ann}_R(a_0, a_1, ..., a_n)$. Since $R$ is generalized morphic, there exists $b \in R$ such that $\text{Ann}_R(a_0, a_1, ..., a_n) = bR$, see Theorem 5 in [12]. Thus $bR[x] \subseteq \text{Ann}_{R[x]}(f)$. If $g(x) = \sum_{i=0}^{m} c_i x^i \in \text{Ann}(f)$, then $c_i a_j = 0$ for all $i$ and $j$, since $R$ is Armendariz, and so, $c_i \in bR$ for each $i$ and $g(x) \in bR[x]$. Thus $\text{Ann}_{R[x]}(f) = bR[x]$ is principal and $R[x]$ is generalized morphic.

**Question:** While there are non-commutative generalized morphic rings that are non-Armendariz, is it necessary for a commutative generalized morphic ring to be Armendariz?

Next, we will characterize generalized morphic reduced rings using minimal prime ideals, and the concept of $z^0$–ideals, borrowed from the rings of continuous functions.

**Theorem 2.2.** Let $R$ be a reduced ring. Then $R$ is a generalized morphic ring if and only if for each $a \in R$ there exists $b \in R$ such that $\text{Supp}(a) = V(b)$ and $bR$ is a $z^0$–ideal.

**Proof.** Assume $R$ is a generalized morphic ring, and let $a \in R$. Then $\text{Ann}(a) = bR$ for some $b \in R$. So we have $\text{Supp}(a) = V(\text{Ann}(a)) = V(bR) = V(b)$. Moreover, if $V(br) \subseteq V(c)$, then $V(b) \subseteq V(br) \subseteq V(c)$, and so, for each $P \in \text{Min}(R)$, if $b \in P$, then $c \in P$ and hence, $ac \in P$. If $b \notin P$, then $a \in \text{Ann}(b) \subseteq P$, and so, $ac \in P$. Thus, $ac \in \bigcap_{P \in \text{Min}(R)} P = \{0\}$, since $R$ is reduced. Therefore, $c \in \text{Ann}(a) = bR$, and $bR$ is a $z^0$–ideal.

Conversely, assume $a, b \in R$ such that $\text{Supp}(a) = V(b)$ and $bR$ is a $z^0$–ideal. Let $P \in \text{Min}(R)$. If $a \in P$, then $ab \in P$. If $a \notin P$, then $\text{Ann}(a) \subseteq P$, and so, $P \in \text{Supp}(a) = V(b)$. Hence, $b \in P$, and so, $ab \in P$. Thus, $ab \in \bigcap_{P \in \text{Min}(R)} P = \{0\}$, which implies that $bR \subseteq \text{Ann}(a)$. If $c \in \text{Ann}(a)$, then we have $V(b) = \text{Supp}(a) \subseteq V(c)$, and so, $c \in bR$, being a $z^0$–ideal. Hence $\text{Ann}(a) = bR$, and $R$ is a generalized morphic ring. 

\[\square\]
3 When is $R[x]/(x^{n+1})$ Generalized Morphic ring?

In this section we characterize the case at which the idealization of a generalized morphic ring or more generally, the ring $R[x]/(x^{n+1})$ is generalized morphic.

**Theorem 3.1.** Let $R$ be a ring, $M$ an $R$–module and let $S = R(+)M$. If $S$ is generalized morphic ring, then $R$ is generalized morphic ring.

**Proof.** Let $a \in Z(R^*)$. Then $\text{Ann}((a,0)) = (r,m)S$, and hence $(0,0) = (a,0)(r,m) = (ar,am)$. So, $ar = 0$, and thus, $rR \subseteq \text{Ann}(a)$. Now, if $x \in \text{Ann}(a)$, then $(x,0)(a,0) = (xa,0) = (0,0)$.

But in this case, we must have $(x,0) = (r,m)(t,s) = (rt,rs+tm)$, for some $(t,s) \in S$. So, $x \in rR$. Therefore, $\text{Ann}(a) = rR$, and hence, $R$ is generalized morphic ring. $\square$

The converse of the above Theorem needs not be true, since $\mathbb{Z}_4$ is a generalized morphic ring, while $\mathbb{Z}_4(+)\mathbb{Z}_4$ is not.

Now, the question is, for what rings $R$, the converse of this Theorem must be true. In the following, we will give the answer. But first we recall the following proposition which was proved in [12].

**Proposition 3.2.** Let $R$ be a reduced ring. Then the following are equivalent:

1. The ring $R$ is morphic.
2. The ring $R[x]/(x^{n+1})$ is morphic for each $n \in \mathbb{N}$.
3. The ring $R(+)R$ is morphic.
4. The ring $R$ is von Neumann regular ring.

In the following, we will prove an analogue result for the equivalence of PP-rings and generalized morphic idealization.

**Lemma 3.3.** Let $R$ be a reduced ring, and let $f = \sum_{i=0}^{n} a_i X^i \in Z(R[x]/(x^{n+1})) \setminus \{0\}$, $g = \sum_{i=0}^{n} b_i X^i \in \text{Ann}(f)$. Then $b_i \in \text{Ann}(a_0,a_1,...,a_{n-i})$ for $i = 0,1,2,...,n$.

**Proof.** Since $fg = 0$, we have $\sum_{i=0}^{j} a_i b_{j-i} = 0$, for $j = 0,1,2,...,n$. Thus, $a_0 b_0 = 0$, and if $b_0 \in \text{Ann}(a_0,a_1,...,a_j)$, $j < n$, then multiplying the equation $a_0 b_{j+1} + a_1 b_j + ... + a_j b_1 + a_{j+1} b_0 = 0$ by $b_0$ yields $a_{j+1} b_0^2 = 0$, and since $R$ is reduced we have $a_{j+1} b_0 = 0$, i.e. $b_0 \in \text{Ann}(a_0,a_1,...,a_{j+1})$. Hence $b_0 \in \text{Ann}(a_0,a_1,...,a_n)$. Now, assume that $b_i \in \text{Ann}(a_0,a_1,...,a_{n-i})$, for $i = 0,1,...,j < n$, then the equation $a_0 b_{j+1} + a_1 b_j + ... + a_j b_1 + a_{j+1} b_0 = 0$.
reduce to \( a_0 b_{j+1} = 0 \). So, assume that \( a_k b_{j+1} = 0 \), for \( k = 0, 1, \ldots, l < n-j-1 \), then the equation \( \sum_{s+k=l+1+j+1} a_k b_s = 0 \), reduces to \( a_{l+1} b_{j+1} = 0 \), and so we have \( b_{j+1} \in \text{Ann}(a_0, a_1, \ldots, a_{n-j-1}) \).

**Theorem 3.4.** The following are equivalent for a ring \( R \):

1. The ring \( R \) is a PP-ring.
2. The ring \( R[x]/(x^{n+1}) \) is generalized morphic for each \( n \in \mathbb{N} \).
3. The ring \( S = R(+)^R \) is generalized morphic.
4. The ring \( R \) is generalized morphic PF-ring.
5. The ring \( R \) is complemented PF-ring.

**Proof.** (1) \( \Rightarrow \) (2) Assume \( R \) is a PP-ring, and \( f = \sum_{i=0}^{n} a_i X^i \in \text{Z}(R[x]/(x^{n+1})) \backslash \{0\} \), \( g = \sum_{i=0}^{n} b_i X^i \in \text{Ann}(f) \). Then it follows by Lemma 3.3 that \( b_i \in \text{Ann}(a_0, a_1, \ldots, a_{n-i}) = e_i R \), where \( e_i^2 = e_i \) for \( i = 0, 1, 2, \ldots, n \), and in this case we would have \( e_i e_j = e_i \), whenever \( i \leq j \). Let \( e = \sum_{i=0}^{n} e_i X^i \). Then it is clear that \( e \in \text{Ann}(f) \). Let \( K_0 = b_0-b_0 X, K_1 = b_1(1-e_0)-b_1 X+b_2 e_0 X-b_0 e_0 X^2 \).

Then it is clear that \( b_i X^i = e K_i \) for \( i = 0, 1 \).

Now, for \( 1 < m \leq n \), let \( T_m = b_m (1-e_{m-1}) - b_m X + 2b_m e_{m-1} X - b_m e_{m-1} X^2 \). Then routine computations yields \( e T_m = b_m X^m + \sum_{j=0}^{m-2} b_m e_j X^{j+1} - \sum_{j=0}^{m-2} b_m e_j X^{j+2} \). Let \( G_{m,i} = -b_m e_{m-1-i} X^i + 2b_m e_{m-1-i} X^{i+1} - b_m e_{m-1-i} X^{i+2} \), for all \( 1 \leq i \leq m-1 \). Then \( e G_{m,i} = -b_m e_{m-1-i} X^{m-1} + b_m e_{m-1-i} X^m - \sum_{j=0}^{m-2-i} b_m e_j X^{j+i+1} + \sum_{j=0}^{m-2-i} b_m e_j X^{j+i+2} \), for \( 1 \leq i \leq m-2 \), and \( e G_{m,m-1} = -b_m e_0 X^{m-1} + b_m e_0 X^m \). Let \( k_m,r = T_m + \sum_{i=1}^{r} G_{m,i} \). Using finite induction, one can show that \( e K_{m,r} = b_m X^m + \sum_{j=0}^{m-2-r} b_m e_j X^{j+r+1} - \sum_{j=0}^{m-2-r} b_m e_j X^{j+r+2}, \) for \( 1 \leq r \leq m-2 \), and \( e K_{m,m-2} = b_m X^m + b_m e_0 X^{m-1} - b_m e_0 X^m \). Now, let \( K_m = (T_m + \sum_{i=1}^{m-2} G_{m,i} + G_{m,m-1}), \) for \( 1 < m \leq n \). Then \( e K_m = b_m X^m \), and so, \( g = e \sum_{m=0}^{n} K_m \). Thus, \( \text{Ann}(f) = (e) \), and \( R[x]/(x^{n+1}) \) is generalized morphic.

(2) \( \Rightarrow \) (3) Clear, since \( R(+)^R \) is isomorphic to \( R[x]/(x^2) \).
(3)⇒ (1) Assume that $S$ is generalized morphic, and let $a \in Z(R) \setminus \{0\}$. Then $(0, a) \in Z(S) \setminus \{(0, 0)\}$, and so, $Ann(0, a) = (x, y)S$. It is clear that $xR \subseteq Ann(a)$, and if $b \in Ann(a)$, then $(b, 0)(0, a) = (0, 0)$, and hence, $(b, 0) = (x, y)(z, w)$. Thus $b = xz \in xR$, and therefore $Ann(a) = xR$. But $(0, 1)(0, a) = (0, 0)$, and so, $(0, 1) = (x, y)(\alpha, \beta)$. Thus we have:

$$0 = x\alpha,$$

$$1 = x\beta + y\alpha,$$

which yields that

$$x = x^2\beta,$$

and hence, $x\beta = (x\beta)^2$, and $Ann(a) = (x\beta)R$. Thus $R$ is a PP-ring.

(1)⇔ (4) See Corollary 3.12 in [14].


Example 3.5. Let $F$ be a field. Then $R = F[x, y]/(xy)$ is a reduced complemented ring that is not a PP-ring, see Remark 2 in [9], and Theorem 4.5 in [9].

One can see easily that $R$ is a generalized morphic ring, while $R[x]/(x^{n+1})$ is not for any $n \in \mathbb{N}$.

It is immediate that if $R$ is a PF-ring that is not a PP-ring, then $R$ and $R(+)M$ are not generalized morphic for any $R$–module $M$.

Since PP-rings are always reduced, we conclude the following easily.

Corollary 3.6. If $R[x]/(x^{n+1})$ is generalized morphic, then $R$ is reduced.

4 Polynomial rings of EM-rings

In [1], the concept of the annihilating content of a polynomial $f(x)$ was introduced to be a constant $c_f$ such that $f(x) = c_f f_1(x)$ with $f_1(x)$ is not a zero-divisor, and in [2], we called a ring $R$ to be an EM-ring if every zero-divisor polynomial in $R[x]$ has an annihilating content. Many properties of this ring were investigated, and many open problems were posed. We now study the polynomial ring of an EM-ring.

Theorem 4.1. If $R$ is an EM-ring, then $R[x]$ is an EM-ring. If $R$ is a reduced, then the converse is also true.
Proof. Assume $R$ is an EM-ring. To show that $R[x]$ is an EM-ring, we will follow the proof of the result in the unpublished article [2]. Let $f(x, y) = \sum_{i=0}^{n} f_i(x)y^i$ be zero-divisor in $R[x, y] = (R[x])[y]$. Then there exists nonzero $h(x)$ such that $hf_i = 0$ for all $i$. Define

$$g(x) = f_0 + f_1x^{\deg(f_0)+1} + f_2x^{\deg(f_0)+\deg(f_1)+2} + \ldots + f_nx^{\sum_{i=1}^{n-1} \deg(f_i)+n}$$

Since $hg = 0$, there exists $c_g \in Z(R)$ and nonzero-divisor $g_1 = \sum_{i=1}^{m} b_i x^i$ such that $g = c_g g_1$. So, $\cap \operatorname{Ann}(b_i) = \{0\}$, and $f_0 = c_g \sum_{i=0}^{\deg(f_0)} b_i x^i = c_g h_0(x)$, $f_1 = c_g \sum_{i=0}^{\deg(f_1)} b_i x^i = c_g h_1(x)$, and so on. Hence, $f(x, y) = c_g \sum_{i=0}^{n} h_i(x)y^i$. If $\sum_{i=0}^{n} h_i(x)y^i$ is a zero-divisor, then there exists nonzero $k(x)$ such that $k(x)h_i(x) = 0$ for each $i$. Define

$$l(x) = \sum_{i=0}^{n} h_i(x) x^{\sum_{j<i} \deg(f_j) + 1}$$

and so, $k(x)l(x) = 0$, and therefore there exists a nonzero $c \in R$ such that $ch_i(x) = 0$, and so, $ch_i = 0$ for all $i$, a contradiction, since $\cap \operatorname{Ann}(b_i) = \{0\}$. Thus $\sum_{i=0}^{n} h_i(x)y^i$ is nonzero-divisor, and $R[x]$ is an EM-ring.

Assume now that $R$ is a reduced ring, and $R[x]$ is an EM-ring. Let $f(x) = \sum_{i=0}^{l} a_i x^i \in Z(R[x]) \setminus \{0\}$. Then $g(y) = \sum_{i=0}^{j} a_i y^i \in Z((R[x])[y]) \setminus \{0\}$, and so, there exists $h(x) = \sum_{i=0}^{m} h_i x^i \in R[x]$ such that $g(y) = h(x) \sum_{i=0}^{j} k_i(x)y^i$, with $\cap \operatorname{Ann}(k_i(x)) = \{0\}$. Assume that $k_i(x) = \sum_{j=0}^{n} k_{i,j} x^j$, which implies that $\cap \operatorname{Ann}(k_{i,j}) = \{0\}$. Note that $a_i = h(x)k_i(x) = h_0 k_0$. But $h(x)k_i(x) = \sum_{k=0}^{m+n_i} c_k x^k$, with $c_k = \sum_{j=0}^{k} h_j k_{i,k-j}$. Now we have:

$$0 = c_{m+n_i} = h_m k_{i,n_i}$$
$$0 = c_{m+n_i-1} = h_m k_{i,n_i-1} + h_{m-1} k_{i,n_i}$$

which implies that $0 = h_m^2 k_{i,n_i-1}$, and so, $0 = h_m k_{i,n_i-1}$, since $R$ is reduced.

$$0 = c_{m+n_i-2} = h_m k_{i,n_i-2} + h_{m-1} k_{i,n_i-1} + h_{m-1} k_{i,n_i},$$
which implies that 0 = \( h_m^2 k_i, n_i - 2 \), and so, 0 = \( h_m k_i, n_i - 2 \).

Now, assume we have \( h_m k_i, s = 0 \), for \( s = n_i, n_i - 1, \ldots, j + 1 \). Thus we have:

\[ 0 = c_{m+j} = h_m k_i j + h_{m-1} k_i, j+1 + \ldots h_j k_i, m, \]

which implies that 0 = \( h_m^2 k_i, j \), and so, 0 = \( h_m k_i, j \), this shows that \( h_m k_i, s = 0 \), for \( s = 0, 1, 2, \ldots, n_i \).

Thus,

\[ h(x) k_i(x) = (h(x) - h_m x^m) k_i(x). \]

Continue to get \( h(x) = h_0 k_i(x) \), which implies that \( h_0 k_i, j = 0 \) for all \( j \in \{1, 2, \ldots, n_j\} \), and \( i \in \{1, 2, \ldots, l\} \)

Now define \( w(x) = \sum_{i=0}^{n_i} k_{i, 0} x^i + x^{n_0+1} \sum_{j=1}^{n_0} k_{0, j} x^j + x^{n_0+n_1+2} \sum_{j=1}^{n_1} k_{1, j} x^j + \ldots + x^{n_0+n_1+\ldots+n_{l-1}+l} \sum_{j=1}^{n_l} k_{l, j} x^j \). Then \( \text{Ann}(w) = \{0\} \), and \( f(x) = h_0 w(x) \). Hence, \( R \) is an EM-ring.

\[ \Box \]

**Question:** Is the above result true for nonreduced rings?

### 5 Idealization of EM-rings

It was shown in [2] that if \( R \) is a Noetherian ring, then \( R \) is an EM-ring if and only if it is a generalized morphic ring, and an example was given for an EM-ring that is not generalized morphic, but the precise relation between the two concepts was not accomplished. In the following, we will give a partial answer.

We now investigate the idealization of EM-rings, and relate it to generalized morphic rings.

**Theorem 5.1.** Assume \( R \) is a ring such that \( S = R(+)R \) is an EM-ring, then \( R \) is an EM-ring.

**Proof.** Let \( f(x) = \sum_{i=0}^{n} a_i x^i \in Z(R[x]) \setminus \{0\} \). Then there exists \( a \in R \setminus \{0\} \) such that \( a a_i = 0 \) for each \( i \). Let \( g(x) = \sum_{i=0}^{n} (a_i, 0) x^i \in S[x] \). Then \( (a, 0)(a_i, 0) = (0, 0) \) for each \( i \), and so, \( g(x) \in Z(S[x]) \setminus \{(0, 0)\} \). Thus there exists \( (r, m) \in S \) such that \( g(x) = (r, m) \sum_{i=0}^{k} (r_i, m_i) x^i \), with \( \bigcap_{i=0}^{k} \text{Ann}(r_i, m_i) = \{(0, 0)\}, n \leq k \).

Hence, we have \( \bigcap_{i=0}^{k} \text{Ann}(r_i) = \{0\} \), and \( f(x) = r \sum_{i=0}^{k} r_i x^i \). Thus, \( R \) is an EM-ring.

\[ \Box \]
The converse of the above Theorem needs not be true, since \( \mathbb{Z}_4 \) is an EM-
ring, while \( \mathbb{Z}_4(+)\mathbb{Z}_4 \) is not.

In [2], we showed that if \( R \) is a PP-ring, then it is an EM-ring. We now give a more precise result.

**Theorem 5.2.** A ring \( R \) is a PP-ring if and only if \( S = R(+)R \) is an EM-
ring.

**Proof.** Assume that \( R \) is a PP-ring, and \( f(x) = \sum_{i=0}^{n} (a_i, b_i)x^i \in \mathbb{Z}(S[x]) \setminus \{(0,0)\} \). Since \( R \) is a PP-ring, we can write \( a_i = u_ir_i \) and \( b_i = v_is_i \), where \( u_i \) and \( v_i \) are idempotents, \( r_i \) and \( s_i \) are regular elements for each \( i \), see [4, Lemma 2]. Define the idempotents \( u, v \) and \( e \) as follows:

\[
1 - u = \prod_{i=0}^{n} (1 - u_i),
\]

\[
1 - v = \prod_{i=0}^{n} (1 - v_i),
\]

\[
1 - e = (1 - u)(1 - v).
\]

Note that \((u_i,0) = (u,e-u)(a_i,0) \) and \((0,b_i) = (u,e-u)((1-u)(b_i+1-e),b_i) \), and so, \( \sum_{i=0}^{n} (a_i, b_i)x^i = (u,e-u)\sum_{i=0}^{n} (a_i + (1-u)(b_i+1-e), b_i)x^i \). Now, let \( I \) be the ideal in \( R \) generated by the elements \( a_i + (1-u)(b_i+1-e) \). Then \( a_i = u_i(a_i + (1-u)(b_i+1-e)) \in I \) for each \( i \). Also, \( (1-u)(b_i+1-e) = a_i + (1-u)(b_i+1-e) - a_i \in I \) for each \( i \), which implies that \( (1-u)b_i = e(1-u)(b_i+1-e) \in I \), since \( eb_i = b_i \) for each \( i \). Therefore, we have \( 1-e = (1-e)(1-u) = I \). Now, if \( \alpha \in Ann(I) \), then \( 0 = \alpha a_i = \alpha u_ir_i \), and so, \( \alpha u_i = 0 \) for each \( i \), which implies that \( \alpha u = 0 \), and so, \( 0 = \alpha(1-u)b_i = \alpha b_i \) for each \( i \). Thus, \( \alpha v_i = 0 \) for each \( i \). Hence we have \( \alpha u = 0 = \alpha v \), and so, \( \alpha e = 0 \). But we have also \( \alpha(1-e) = 0 \), which implies that \( \alpha = 0 \), i.e. \( Ann(I) = \{0\} \), and so it follows by Lemma 1.2 that \( \sum_{i=0}^{n} (a_i + (1-u)(b_i+1-e), b_i)x^i \in reg(S[x]) \).

Thus \( S \) is an EM-ring.

Now assume that \( S \) is an EM-ring, \( b \in Z(R) \setminus \{0\} \) and let \( a \in Ann(b) \setminus \{0\} \). Then \( f(x) = (0,1) + (b,0)x \in Z(S[x]) \setminus \{(0,0)\} \), since it is annihilated by \((0,a)\).

Thus \( f(x) = (\alpha,\beta) \sum_{i=0}^{n} (n_i,m_i)x^i \), with \( \bigcap Ann(n_i) = \{0\} \). Thus, we have:

\[
0 = \alpha n_0,
\]
\[ 1 = \alpha m_0 + \beta n_0, \]
\[ b = \alpha n_1, \]
\[ 0 = \alpha n_i \text{ for all } i > 1. \]

But \( b = b(\alpha m_0 + \beta n_0) = b\alpha m_0 + \alpha n_1\beta n_0 = b(\alpha m_0) \). Also note that \( \alpha m_0 = (\alpha m_0)^2 + \alpha m_0\beta n_0 = (\alpha m_0)^2 \). Thus, \( \text{Ann}(\alpha m_0) \subseteq \text{Ann}(b) \). Now let \( d \in \text{Ann}(b) \). Then we have:
\[ 0 = (dm_0)0 = (dm_0)\alpha n_0 = (d\alpha m_0)n_0, \]
\[ 0 = (dm_0)b = (dm_0)\alpha n_1 = (d\alpha m_0)n_1, \]
\[ 0 = (dm_0)0 = (dm_0)\alpha n_i = (d\alpha m_0)n_i \text{ for all } i > 1, \]

which implies that \( d\alpha m_0 \in \bigcap_i \text{Ann}(n_i) = \{0\} \). Hence, \( \text{Ann}(b) = \text{Ann}(\alpha m_0) = (1 - \alpha m_0)R \) is generated by an idempotent, and so, \( R \) is a PP-ring. \[ \square \]

Using Theorems 3.4 and 5.2, one can deduce the following:

**Corollary 5.3.** For any ring \( R \), we have \( R(+)R \) is an EM-ring if and only if it is generalized morphic.

**Example 5.4.** The space \( X = \beta N \setminus N \) is an F-space that is not a basically disconnected space nor complemented, see [6, 6W and 14.27], and so, \( C(X) \) is a reduced Bézout ring that is not a PP-ring. Thus \( C(X)(+)C(X) \) is not an EM-ring. Also we have \( C(X) \) is an EM-ring which is not generalized morphic.

**Questions:** It is still an open problem to characterize the relation between EM-rings and generalized morphic rings. Although they are not equivalent, we saw that \( R(+)R \) is an EM-ring if and only if it is generalized morphic, even if \( R \) was not Noetherian. We also don’t know yet what sufficient conditions must be add to an EM-ring to become a PP-ring. It is not difficult to show that if \( R[x]/(x^{n+1}) \) is an EM-ring, then \( R \) is a PP-ring. We are still working for the other direction.
References


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