THE ORLICZ SPACE OF $\chi^\pi$

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Abstract. In this paper we introduced the Orlicz space of $\chi^\pi$. We establish some inclusion relations, topological results and we characterize the duals of the Orlicz of $\chi^\pi$ sequence spaces.

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1. Introduction

A complex sequence, whose $k^{th}$ terms is $x_k$ is denoted by $\{x_k\}$ or simply $x$. Let $w$ be the set of all sequences $x = (x_k)$ and $\phi$ be the set of all finite sequences. Let $\ell_\infty, c, c_0$ be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of $\ell_\infty, c, c_0$ we have $\|x\| = \sup_k |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\Lambda$. A sequence $x$ is called entire sequence if $\lim_{k \to \infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by $\Gamma$. $\chi$ was discussed in Kamthan [19]. Matrix transformation involving $\chi$ were characterized by Sridhar [20] and Sirajiudeen [21]. Let $\chi$ be the set of all those sequences $x = (x_k)$ such that $(k!|x_k|)^{1/k} \to 0$ as $k \to \infty$. Then $\chi$ is a metric space with the metric

$$d(x, y) = \sup_k \left\{ (k!|x_k - y_k|)^{1/k} : k = 1, 2, 3, \ldots \right\}$$

Orlicz [4] used the idea of Orlicz function to construct the space $(L^M)$. Lindenstrauss and Tzafriri [5] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p(1 \leq p < \infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary[6], Mursaleen et al,[7], Bektas and Altin[8], Tripathy et al.[9], Rao and subramanian[10] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref[11].
Recall([4],[11]) an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by $M(x+y) \leq M(x)+M(y)$ then this function is called modulus function, introduced by Nakano[18] and further discussed by Ruckle[12] and Maddox[13] and many others.

An Orlicz function $M$ is said to satisfy $\Delta_2$- condition for all values of $u$, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)(u \geq 0)$. The $\Delta_2$- condition is equivalent to $M(\ell u) \leq K \ell M(u)$, for all values of $u$ and for $\ell > 1$. Lindenstrauss and Tzafriri[5] used the idea of Orlicz function to construct Orlicz sequence space.

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$  \hspace{1cm} (1)

The space $\ell_M$ with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}$$ \hspace{1cm} (2)

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \leq p < \infty$, the space $\ell_M$ coincide with the classical sequence space $\ell_p$. Given a sequence $x = \{x_k\}$ its $n^{th}$ section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$ $\delta^{(n)} = (0, 0, ..., 1, 0, 0, ..., 1$ in the $n^{th}$ place and zero’s else where; and $s^{(k)} = (0, 0, ..., 1, -1, 0, ...), 1$ in the $n^{th}$ place -1 in the $(n+1)^{th}$ place and zero’s else where.

An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k \ (k = 1, 2, 3, ...)$ are continuous. We recall the following definitions [see [15]].

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric-space $(X, d)$ is said to have AK (or sectional convergence) if and only if $d(x^{(n)}, x) \to 0$ as $n \to \infty$. [see[15]] The space is said to have AD (or) be an AD space if $\phi$ is dense in $X$. We note that AK implies AD by [14].

If $X$ is a sequence space, we define

(i)$X^\prime$ - the continuous dual of $X$.

(ii)$X^\alpha = \{ a = (a_k) : \sum_{k=1}^{\infty} |a_kx_k| < \infty, \text{ for each } x \in X \}$;

(iii)$X^\beta = \{ a = (a_k) : \sup_{k=1}^{\infty} a_kx_k \text{ is convergent, for each } x \in X \}$;

(iv)$X^\gamma = \{ a = (a_k) : \sup_1^n |a_kx_k| < \infty, \text{ for each } x \in X \}$;

(v)Let $X$ be an FK-space $\supset \phi$. Then $X^f = \left\{ f(\delta^{(n)}) : f \in X^\prime \right\}$.

$X^\alpha, X^\beta, X^\gamma$ are called the $\alpha-$ (or Kô the-T ôplitz) dual of $X$, $\beta-$ (or generalized Kô the-T ôplitz) dual of $X$, $\gamma-$ dual of $X$. Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$ then $Y^\mu \subset X^\mu$, for $\mu = \alpha, \beta, \text{ or } \gamma$. 

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Lemma 1.1. (See (15, Theorem 7.27)). Let $X$ be an FK-space $\supset \phi$. Then

(i) $X^\gamma \subset X^f$.
(ii) If $X$ has AK, $X^\beta = X^f$.
(iii) If $X$ has AD, $X^\beta = X^\gamma$.

2. Definitions and Preliminaries

Let $w$ denote the set of all complex double sequences $x = (x_k)_{k=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function. Let

$$
\chi^\pi_M = \left\{ x \in w : \lim_{k \to \infty} \left( M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right) = 0 \text{ for some } \rho > 0 \right\},
$$

$$
\Gamma^\pi_M = \left\{ x \in w : \lim_{k \to \infty} \left( M \left( \frac{|x_k|^{1/k}}{\pi_k^{1/k} \rho} \right) \right) = 0 \text{ for some } \rho > 0 \right\},
$$

and

$$
\Lambda^\pi_M = \left\{ x \in w : \sup_k \left( M \left( \frac{|x_k|^{1/k}}{\pi_k^{1/k} \rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\}.
$$

The space $\chi^\pi_M$ is a metric space with the metric

$$
d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{(k! |x_k - y_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right) \leq 1 \right\}
$$

The space $\Gamma^\pi_M$ and $\Lambda^\pi_M$ is a metric space with the metric

$$
d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{|x_k - y_k|^{1/k}}{\pi_k^{1/k} \rho} \right) \right) \leq 1 \right\}
$$

3. Main Results

Proposition 3.1. $\chi^\pi_M \subset \Gamma^\pi_M$, with the hypothesis that $M \left( \frac{|x_k|}{\pi_k^{1/k} \rho} \right) \leq M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right)$

Proof. Let $x \in \chi^\pi_M$. Then we have the following implications

$$
M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \to 0 \text{ as } k \to \infty
$$
But $M \left( \frac{|x_k|}{\pi_k^1/\rho} \right) \leq M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right)$, by our assumption, implies that
$$\Rightarrow M \left( \frac{|x_k|^{1/k}}{\pi_k^1/\rho} \right) \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ by (5)}.$$ 
$$\Rightarrow x \in \Gamma^\pi_M$$ 
$$\Rightarrow \chi^\pi_M \subset \Gamma^\pi_M.$$ This completes the proof.

**Proposition 3.2.** $\chi^\pi_M$ has AK whenever $M$ is a modulus function.

**Proof.** Let $x = \{x_k\} \in \chi^\pi_M$, but then $\left\{ M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \right\} \in \chi$, and hence
$$\sup_{k \geq n+1} M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{6}$$ 

$$d(x, x^{[n]}) = \sup_{k \geq n+1} \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ by using (6)}.$$ 
$$\Rightarrow x^{[n]} \rightarrow x \text{ as } n \rightarrow \infty, \text{ implying that } \chi^\pi_M \text{ has AK. This completes the proof.}$$

**Proposition 3.3.** $\chi^\pi_M$ is solid.

**Proof.** Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \chi^\pi_M$. 
$$M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \leq M \left( \frac{(k!|y_k|)^{1/k}}{\pi_k^1/\rho} \right), \text{ because } M \text{ is non-decreasing}.$$ 
But $M \left( \frac{(k!|y_k|)^{1/k}}{\pi_k^1/\rho} \right) \in \chi$, because $y \in \chi^\pi_M$. That is $M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \rightarrow 0 \text{ as } k \rightarrow \infty$ and $M \left( \frac{(k!|y_k|)^{1/k}}{\pi_k^1/\rho} \right) \rightarrow 0 \text{ as } k \rightarrow \infty$. Therefore $x = \{x_k\} \in \chi^\pi_M$. This completes the proof.

**Proposition 3.4.** Let $M$ be an Orlicz function which satisfies $\Delta_2$-condition. Then $\chi \subset \chi^\pi_M$.

**Proof.** Let 
$$x \in \chi \tag{7}$$ 
Then $(k!|x_k|)^{1/k} \leq \epsilon$ sufficiently large $k$ and every $\epsilon > 0$. But then by taking $\rho \geq \frac{2}{\epsilon}$
$$M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \leq M \left( \frac{2}{\epsilon} \right) \leq M(2\epsilon) \text{ (because } M \text{ is non-decreasing)}.$$ 
$$M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \leq K \epsilon \text{ by } \Delta_2 - \text{condition, for some } K > 0 \leq \epsilon \tag{8}$$ 
$$\Rightarrow M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^1/\rho} \right) \rightarrow 0 \text{ as } k \rightarrow \infty \text{ (by defining } M(\epsilon) < \frac{2}{\epsilon}). \text{ Hence } x \in \chi^\pi_M. \text{ From (7) }
and since
\[ x \in \chi_M^{\pi} \]  
we get \( x \subset \chi_M^{\pi} \). This completes the proof.

**Proposition 3.5.** If \( M \) is a modulus function, then \( \chi_M^{\pi} \) is linear set over the set of complex numbers \( \mathbb{C} \).

**Proof.** Let \( x, y \in \chi_M^{\pi} \) and \( \alpha, \beta \in \mathbb{C} \). In order to prove the result we need to find some \( \rho_3 \) such that
\[
M \left( \frac{(k! |x_k|^{1/k})}{\pi_k^{1/k} \rho_3} \right) \to 0 as k \to \infty. 
\]  
(10)

Since \( x, y \in \chi_M^{\pi} \), there exists some positive \( \rho_1 \) and \( \rho_2 \) such that
\[
M \left( \frac{(k! |x_k|^{1/k})}{\pi_k^{1/k} \rho_1} \right) \to 0 as k \to \infty \quad \text{and} \quad M \left( \frac{(k! |y_k|^{1/k})}{\pi_k^{1/k} \rho_2} \right) \to 0 as k \to \infty. 
\]  
(11)

Since \( M \) is a non decreasing modulus function, we have
\[
M \left( \frac{(k! |x_k + \beta y_k|^{1/k})}{\pi_k^{1/k} \rho_3} \right) \leq M \left( \frac{(k! |x_k|^{1/k})}{\pi_k^{1/k} \rho_1} + \frac{(k! |\beta y_k|^{1/k})}{\pi_k^{1/k} \rho_2} \right) \leq M \left( \frac{|\alpha| (k! |x_k|^{1/k})}{\pi_k^{1/k} \rho_3} + \frac{|\beta| (k! |y_k|^{1/k})}{\pi_k^{1/k} \rho_3} \right). 
\]

Take \( \rho_3 \) such that \( \frac{1}{\rho_3} = \min \left\{ \frac{1}{|\alpha| \rho_1}, \frac{1}{|\beta| \rho_2} \right\} \). Then
\[
M \left( \frac{(k! |x_k + \beta y_k|^{1/k})}{\pi_k^{1/k} \rho_3} \right) \leq M \left( \frac{(k! |x_k|^{1/k})}{\pi_k^{1/k} \rho_1} + \frac{(k! |y_k|^{1/k})}{\pi_k^{1/k} \rho_2} \right) \to 0 \quad \text{(by (11))}. 
\]

Hence \( M \left( \frac{(k! |x_k + \beta y_k|^{1/k})}{\pi_k^{1/k} \rho_3} \right) \to 0 as k \to \infty \). So \( (\alpha x + \beta y) \in \chi_M^{\pi} \). Therefore \( \chi_M^{\pi} \) is linear. This completes the proof.

**Definition 3.6.** Let \( p = (p_k) \) be any sequence of positive real numbers. Then we define \( \chi_M^{\pi}(p) = \left\{ x = (x_k) \mid M \left( \frac{(k! |x_k|^{1/k})}{\pi_k^{1/k} p_k} \right) \to 0 as k \to \infty \right\} \). Suppose that \( p_k \) is a constant for all \( k \), then \( \chi_M^{\pi}(p) = \chi_M^{\pi} \).

**Proposition 3.7.** Let \( 0 \leq p_k \leq q_k \) and let \( \left\{ \frac{q_k}{p_k} \right\} \) be bounded. Then \( \chi_M^{\pi}(q) \subset \chi_M^{\pi}(p) \).

**Proof.** Let
\[
M \left( \frac{(k! |x_k|^{1/k})}{\pi_k^{1/k} \rho_3} \right) \to 0 as k \to \infty. 
\]  
(12)


Let \( t_k = \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{q_k} \) and \( \lambda_k = \frac{p_k}{q_k} \). Since \( p_k \leq q_k \), we have \( 0 \leq \lambda_k \leq 1 \). Take \( 0 < \lambda < \lambda_k \). Define

\[
u_k = \begin{cases} t_k, (t_k \geq 1) \\ 0, (t_k < 1) \end{cases}
\]

\[ \text{and } \nu_k = \begin{cases} 0 (t_k \geq 1) \\ t_k, (t_k < 1) \end{cases} \quad (14) \]

\[ t_k = u_k + \nu_k; t_k^\lambda_k = u_k^\lambda_k + \nu_k^\lambda_k. \]

Now it follows that \( u_k^\lambda_k \leq u_k \leq t_k \) and \( \nu_k^\lambda_k \leq \nu_k \). Since \( t_k^\lambda = u_k^\lambda_k + \nu_k^\lambda_k \), then \( t_k^\lambda \leq t_k + \nu_k^\lambda \)

\[ \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{\lambda_k} \leq \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{q_k} \]

\[ \Rightarrow \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{\frac{p_k}{q_k}} \leq \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{q_k} \]

But \( \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{q_k} \rightarrow 0 \) as \( k \rightarrow \infty \). (by (13))

Therefore \( \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_k} \rightarrow 0 \) as \( k \rightarrow \infty \). Hence

\[ x \in \chi_M^\pi (p) \quad (15) \]

From (12) and (15) we get \( \chi_M^\pi (q) \subset \chi_M^\pi (p) \). This completes the proof.

**Proposition 3.8.** (a) Let \( 0 < \inf p_k \leq p_k \leq 1 \). Then \( \chi_M^\pi (p) \subset \chi_M^\pi \)

(b) Let \( 1 \leq p_k \leq \sup \rho_k < \infty \). Then \( \chi_M^\pi \subset \chi_M^\pi (p) \)

**Proof.** (a) Let \( x \in \chi_M^\pi (p) \)

\[ \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_k} \rightarrow 0 \) as \( k \rightarrow \infty \). \quad (16) \]

Since \( 0 < \inf p_k \leq p_k \leq 1 \)

\[ \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_k} \leq \left( M \left( \frac{\left| k! \right| x_k \right)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_k} \quad (17) \]

From (16) and (17) it follows that \( x \in \chi_M^\pi \).

Thus \( \chi_M^\pi (p) \subset \chi_M^\pi \). We have thus proven (a). (b) Let \( p_k \geq 1 \) for each \( k \) and
supp_k < ∞
Let \( x \in \chi^\pi_M \)

\[
M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \to 0 \text{ as } k \to \infty.
\]  
(18)

Since \( 1 \leq p_k \leq \text{supp}_k < \infty \) we have

\[
M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_k} \leq \left( M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{p_k}
\]  
(19)

Therefore \( x \in \chi^\pi_M (p) \). This completes the proof.

**Proposition 3.9.** Let \( 0 < p_k \leq q_k < \infty \) for each \( k \). Then \( \chi^\pi_M (p) \subseteq \chi^\pi_M (q) \).

**Proof.** Let \( x \in \chi^\pi_M (p) \)

\[
M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_k} \to 0 \text{ as } k \to \infty.
\]  
(20)

This implies that \( M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right)^{q_k} \leq \left( M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{p_k} \)

(21)

Therefore \( x \in \chi^\pi_M (q) \). This completes the proof.

**Proposition 3.10.** \( \chi^\pi_M (p) \) is a \( r \)-convex for all \( r \) where \( 0 \leq r \leq \inf p_k \).

Moreover if \( p_k = p \leq 1 \forall k \), then they are \( p \)-convex.

**Proof.** We shall prove the Theorem for \( \chi^\pi_M (p) \).

Let \( x \in \chi^\pi_M (p) \) and \( r \in (0, \lim_{n \to \infty} p_n) \)

Then, there exists \( k_0 \) such that \( r \leq p_k \forall k > k_0 \).

Now, define

\[
g^* (x) = \inf \left\{ \rho : M \left( \frac{(k! |x_k - y_k|)^{1/k}}{\pi_k^{1/k} \rho} \right)^r + M \left( \frac{(k! |x_k - y_k|)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_n} \right\}
\]  
(22)
Since $r \leq p_k \leq 1 \forall k > k_0$
g* is subadditive: Further, for $0 \leq |\lambda| \leq 1; |\lambda|^{p_k} \leq |\lambda|^r \forall k > k_0$,

$$g^* (\lambda x) \leq |\lambda|^r \cdot g^* (x)$$  \hfill (23)

Now, for $0 < \delta < 1$,

$$U = \{x : g^* (x) \leq \delta\}, \text{ which is an absolutely } r - \text{convex set, for }$$

$$|\lambda|^r + |\mu|^r \leq 1 x, y \in U$$  \hfill (24)

Now

$$g^* (\lambda x + \mu y) \leq g^* (\lambda x) + g^* (\mu y)$$

$$\leq |\lambda|^r g^* (x) + |\mu|^r g^* (y)$$

$$\leq |\lambda|^r \delta + |\mu|^r \delta \text{ using (23) and (24)}$$

$$\leq 1 - \delta, \text{ by using (25)}$$

$$\leq \delta. \text{ If } p_k = p \leq 1 \forall k \text{ then for } 0 < r < 1,$$

$$U = \{x : g^* (x) \leq \delta\} \text{ is an absolutely } p - \text{convex set.}$$

This can be obtained by a similar analysis and therefore we omit the details. This completes the proof.

**Proposition 3.11.** $(\chi_M^\pi)^\beta = \Lambda$

*Proof: Step 1:* $\chi_M^\pi \subset \Gamma_M^\pi$ by Proposition 3.1;

$\Rightarrow (\Gamma_M^\pi)^\beta \subset (\chi_M^\pi)^\beta$. But $(\Gamma_M^\pi)^\beta = \Lambda$

$$\Lambda \subset (\chi_M^\pi)^\beta$$  \hfill (26)

**Step 2:** Let $y \in (\chi_M^\pi)^\beta$ we have $f (x) = \sum_{k=1}^\infty x_k y_k$ with $x \in \chi_M^\pi$.

We recall that $s^{(k)}$ has $\frac{\tau_k}{\tau_{k+1}}$ in the $k^{th}$ place and zero’s elsewhere, with

$$x = s^{(k)} (M (\frac{(k \parallel x_k)^{1/k}}{\tau_k^{1/k}} \frac{1}{\rho} \cdot 0, \cdots M \frac{(1)^{1/k}}{\tau_k^{1/k}})) = \{0, 0, \cdots M \frac{(1)^{1/k}}{\tau_k^{1/k}} \cdot 0, \cdots \}$$

which converges to zero. Hence $s^{(k)} \in \chi_M^\pi$. Hence $d (s^{(k)}, 0) = 1$.

But $|y_k| \leq \|f\| d(s^{(k)}, 0) < \infty \forall k$. Thus $(y_k)$ is a bounded sequence and hence an analytic sequence. In other words $y \in \Lambda$.

$$(\chi_M^\pi)^\beta \subset \Lambda$$  \hfill (27)

**Step 3** From (26) and (27) we obtain $(\chi_M^\pi)^\beta = \Lambda$.

This completes the proof.
Proposition 3.12. \( (\chi^\pi_M)^\mu = \Lambda \) for \( \mu = \alpha, \beta, \gamma, f \).

Proof. Step 1: \( \chi^\pi_M \) has AK by Proposition 3.2. Hence by Lemma 1.1 (i) we get \( (\chi^\pi_M)^\beta = (\chi^\pi_M)^f \). But \( (\chi^\pi_M)^\beta = \Lambda \). Hence 
\[
(\chi^\pi_M)^f = \Lambda \tag{28}
\]

Step 2: Since AK \( \Rightarrow \) AD. Hence by Lemma 1.1 (iii) we get \( (\chi^\pi_M)^\beta = (\chi^\pi_M)^\gamma \). Therefore 
\[
(\chi^\pi_M)^\gamma = \Lambda \tag{29}
\]

Step 3: \( \chi^\pi_M \) is normal by Proposition 3.3. Hence by Proposition 2.7 \[16\], we get 
\[
(\chi^\pi_M)^\alpha = (\chi^\pi_M)^\gamma = \Lambda \tag{30}
\]

From (28), (29) and (30) we have \( (\chi^\pi_M)^\alpha = (\chi^\pi_M)^\beta = (\chi^\pi_M)^\gamma = (\chi^\pi_M)^f = \Lambda \).

Proposition 3.13. The dual space of \( \chi^\pi_M \) is \( \Lambda \). In other words \( (\chi^\pi_M)^* = \Lambda \).

Proof. We recall that \( s^{(k)} \) has \( \frac{s_k^{1/k}}{k!} \) has the \( k^{th} \) place zero’s else where, with 
\[
x = s^{(k)}, \left( M \left( \frac{(k!|x_k|)^{1/k}}{s_k^{1/k} \rho} \right) \right) = \{0, 0, \cdots M \left( \frac{(1)^{1/k}}{\rho} \right), 0, \cdots \}
\]
Hence \( s^{(k)} \in \chi^\pi_M \). We have \( f(x) = \sum_{k=1}^\infty x_k y_k \) with \( x \in \chi^\pi_M \) and \( f \in (\chi^\pi_M)^* \), where \( (\chi^\pi_M)^* \) is the dual space of \( \chi^\pi_M \). Take \( x = s^{(k)} \in \chi^\pi_M \). Then 
\[
|y_k| \leq \|f\| d \left( s^{(k)}, 0 \right) < \infty \text{ for all } k. \tag{31}
\]
Thus \( (y_k) \) is a bounded sequence and hence an analytic sequence. In other words, \( y \in \Lambda \). Therefore \( (\chi^\pi_M)^* = \Lambda \). This completes the proof.

Lemma 3.14. \[15, \text{Theorem 8.6.1}\] \( Y \supset X \iff Y^f \subset X^f \) where \( X \) is an AD-space and \( Y \) an FK-space.

Proposition 3.15. Let \( Y \) be any FK-space \( \supset \) \( \phi \). Then \( Y \supset \chi^\pi_M \) if and only if the sequence \( s^{(k)} \) is weakly analytic.

Proof. The following implications establish the result.
\( Y \supset X \iff Y^f \subset (\chi^\pi_M)^f \), since \( \chi^\pi_M \) has AD and by Lemma 3.14.
\( \iff Y^f \subset \Lambda \), since \( (\chi^\pi_M)^f = \Lambda \).
\( \iff \) for each \( f \in Y^' \), the topological dual of \( Y \). Therefore \( f \left( s^{(k)} \right) \in \Lambda \).
\( \iff f \left( s^{(k)} \right) \) is analytic
\( \iff s^{(k)} \) is weakly analytic.
This completes the proof.
Proposition 3.16. $\chi^n_M$ is a complete metric space under the metric
\[
d(x, y) = \sup_k \left\{ M \left( \frac{(k!|x_k - y_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) : k = 1, 2, 3, \ldots \right\}
\]
where $x = (x_k) \in \chi^n_M$ and $y = (y_k) \chi^n_M$.

Proof. Let $\{x^{(n)}\}$ be a Cauchy sequence in $\chi^n_M$. Then given any $\epsilon > 0$ there exists a positive integer $N$ depending on $\epsilon$ such that
\[
d(x^{(n)}, x^{(m)}) < \epsilon \text{ for all } n \geq N \text{ and for all } m \geq N.
\]
Consequently $M \left( \frac{(k!|x^{(n)}_k - x^{(m)}_k|)^{1/k}}{\pi_k^{1/k} \rho} \right)$ is a Cauchy sequence in the metric space $\mathbb{C}$ of complex numbers. But $\mathbb{C}$ is complete. So,
\[
M \left( \frac{(k!|x^{(n)}_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \to M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \text{ as } n \to \infty.
\]

Hence there exists a positive integer $n_0$ such that
\[
\sup_k \left\{ M \left( \frac{(k!|x^{(n)}_k - x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right\} < \epsilon \text{ for all } n \geq n_0.
\]
In particular, we have
\[
\left\{ M \left( \frac{(k!|x^{(n)}_k - x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right\} < \epsilon. \text{ Now}
\]
\[
\left\{ M \left( \frac{(k!|x^{(n)}_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right\} \leq M \left( \frac{(k!|x^{(n)}_k - x^{(n_0)}_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) + \left\{ M \left( \frac{(k!|x^{(n_0)}_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right\} < \epsilon + 0 \text{ as } k \to \infty.
\]
Thus
\[
\left\{ M \left( \frac{(k!|x^{(n)}_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right\} < \epsilon \text{ as } k \to \infty.
\]

That is $x \in \chi^n_M$. Therefore, $\chi^n_M$ is a complete metric space. This completes the proof.
REFERENCES


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