COEFFICIENT ESTIMATES FOR STARLIKE FUNCTIONS OF ORDER $\beta$

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ABSTRACT. In this paper, we consider the subclass of starlike functions of order $\beta$ denoted by $SL^*(\beta)$ and determine the coefficient estimates for this subclass. In addition, the Fekete-Szegő functional $|a_3 - \mu a_2^2|$ for this class is obtained when $\mu$ is real.

1. Introduction

Let $H$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{ z : |z| < 1 \}$ on the complex plane $\mathbb{C}$. Robertson introduced in [6] the class $S^*(\beta)$ of starlike functions of order $\beta \leq 1$, which is defined by $S^*(\beta) = \{ f \in A : \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > \beta, \quad (z \in \mathbb{U}) \}$. If (0 $\leq \beta < 1$), then a function in either of this set is univalent, if $\beta < 0$ it may fail to be univalent. If $f$ and $g$ are analytic functions in $\mathbb{U}$. Then the function $f$ is said to be subordinate to $g$, and can be written as $f \prec g$ and $f(z) \prec g(z)$ ($z \in \mathbb{U}$) if and only if there exists the Schwarz function $w$, analytic in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. Furthermore, if $g$ is univalent in $\mathbb{U}$ we have the following equivalence $f \prec g \iff f(0) = g(0)$ and $f(u) \subseteq g(u)$. The class $SS^*(\beta)$ of strongly starlike functions of order $\beta$

$$SS^*(\beta) = \{ f \in A : |\text{Arg} \frac{zf'(z)}{f(z)}| < \frac{\beta \pi}{2} \}, 0 < \beta \leq 1,$$

which was introduced in [4]. Moreover, $K - ST \subset SL^*$, for $K = 2 + \sqrt{2}$, where $K - ST$ is the class of $k$-starlike functions introduced in [5], such that $K - ST := \{ f \in A : \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > K \frac{zf'(z)}{f(z)} - 1 \}, K \geq 0$. Let us consider $Q(f, z) = \frac{zf'(z)}{f(z)}$. In this way many interesting classes of analytic functions have been defined (see for instance [1]). In this paper we consider the class $SL^*(\beta)$ such that $SL^*(\beta) = \{ f \in A : |Q(f, z) - (1 - \beta)| < 1 - \beta \}$. It is easy to see that $f \in SL^*(\beta)$ if and only if $\frac{zf'(z)}{f(z)} < q_0(z) = \sqrt{(1 - \beta)(1 + z)}$. $q_0(0) = 1 - \beta$. 149
Corollary 1 If the function \( f(z) = z + a_3z^3 + \cdots \), belongs to the class \( SL^*(\beta) \), then

\[
|a_k| \leq \sqrt{\frac{1 - \beta}{(k^2 - 2(1 - \beta))}}
\]

for \( k \geq 2 \).
Theorem 3. If the function \( f(z) = \sum_{k=1}^{\infty} a_k z^k \) belongs to class \( SL^*(\beta) \), then \( |a_2| \leq \frac{1-\beta}{2(1+\beta)} \), \( |a_3| \leq \frac{1-\beta}{2(4+2\beta)} \), and \( |a_4| \leq \frac{1-\beta}{2(\beta+3)} \).

These estimates are sharp.

Proof. If \( f(z) = \sum_{k=1}^{\infty} a_k z^k \) belongs to class \( SL^*(\beta) \), then \( (1-\beta)f^2(z) = z^2f'^2(z) - (1-\beta)f^2(z)w(z) \), where \( w \) satisfies \( w(0) = 0, |w(z)| < 1 \). Let us denote \( (zf')(z))^2 = \sum_{k=2}^{\infty} A_k z^k \), \( f^2(z) = \sum_{k=2}^{\infty} B_k z^k \), \( w(z) = \sum_{k=1}^{\infty} C_k z^k \).

Then we have \( A_k = \sum_{l=1}^{k-1} (k-l)a_la_{k-l} \), \( B_k = \sum_{l=1}^{k-1} a_la_{k-l} \) and

\[
\sum_{k=2}^{\infty} (A_k - (1-\beta)B_k)z^k = (1-\beta)(\sum_{k=2}^{\infty} C_k z^k)(\sum_{k=2}^{\infty} B_k z^k).
\]

Thus we have

\[
A_2 = a_1 = 1, \quad A_3 = 4a_2a_1 = 4a_2, \quad A_4 = 6a_3 + 4a_2^2,
\]

and

\[
A_5 = 8a_1a_4 + 12a_2a_3,
\]

also

\[
B_2 = a_1 = 1, \quad B_3 = 2a_2, \quad B_4 = 2a_3 + a_2^2
\]

and

\[
B_5 = 2a_1a_4 + 2a_2a_3.
\]

Equating the second and third coefficients of both side of (2) we obtain:

(i) \( A_3 - (1-\beta)B_3 = C_1B_2 \)

(ii) \( A_4 - (1-\beta^2)B_4 = C_1B_3 + C_2B_2 \)

(iii) \( A_5 - (1-\beta^2)B_5 = C_1B_4 + C_2B_3 + C_3B_5 \).

It is well known that \( |C_k| \leq 1 \) and \( \sum_{k=1}^{\infty} |C_k|^2 \leq 1 \), therefore we obtain by (3) and (4) that

\[
|a_2| \leq \frac{1-\beta}{2(1+\beta)}, \quad |a_3| \leq \frac{1-\beta}{4+2\beta}, \quad |a_4| \leq \frac{1-\beta}{2(\beta+3)}.
\]

Conjecture. Let \( f \in SL^*(\beta) \) and \( f(z) = \sum_{k=1}^{\infty} a_k z^k \). Then \( |a_{n+1}| \leq \frac{1-\beta}{2(1+n)} \). This is yet to be proven.

Next, we refer to a classical result of Fekete and Szegö [3] to determine the maximum value of \( |a_3 - \mu a_2^2| \) for functions \( f \) belonging to \( H \) whenever \( \mu \) is real. Other work related to the functional of Fekete and Szegö can be found in [7].
3. FEKETE-SZEGÖ FOR THE CLASS $SL^*(\beta)$

In order to prove our result we have to recall the following lemma:

**Lemma 1** [2] Let $h$ be analytic in $U$ with $\text{Re } h(z) > 0$ and be given by

$$h(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

for $z \in U$, then

$$|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

**Theorem 4** Let $f$ be given by (1) and belongs to the class $SL^*(\beta)$. Then, for $0 \leq \beta < 1$ and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\beta}{\beta+2}, & \text{if } \mu \leq \frac{1+3\beta}{2(\beta+2)}, \\ \frac{1-\beta}{2(\beta+2)} \left(1 + \frac{1-\beta}{2(\beta+2)}\right), & \text{if } \mu \geq \frac{1+3\beta}{2(\beta+2)}. \end{cases}$$

**Proof.**

$$a_3 - \mu a_2^2 = \frac{(1-\beta)^2(1+3\beta)}{8(1+\beta)^2(\beta+2)} c_1^2 + \frac{(1-\beta)^2}{2(1+\beta)} c_2 + \frac{(1-\beta)^2}{4(1+\beta)^2} c_1^2,$$

$$= \frac{(1-\beta)}{2(\beta+\beta)} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} \left(1 - \frac{1-\beta}{2(2+\beta)}\right) c_1^2 + \frac{(1-\beta)^2}{8(1+\beta)^2(\beta+2)} c_1^2,$$

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)^2(1+3\beta)}{8(1+\beta)^2(\beta+2)} + \frac{(1-\beta)}{2(2+\beta)} \left(2 - \frac{|c_1|}{2} \right)$$

$$= \phi(x), \text{ with } x = |c_1|,$$

where we have used Lemma 1. and equations

$$a_2 = \frac{1-\beta}{2(1+\beta)} c_1,$$

and

$$a_3 = \frac{(1-\beta)^2(1+3\beta)}{8(1+\beta)^2(\beta+2)} c_1^2 + \frac{(1-\beta)}{2(2+\beta)} c_2.$$

Elementary calculation indicates that the function attains its maximum value at

$$x_o = 0$$

and thus establishing

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\[ |a_3 - \mu a_2^2| \leq \phi(x_0) = \frac{1 - \beta}{2 + \beta}. \]

Next, we have
\[
|a_3 - \mu a_2^2| \leq \frac{(1 - \beta)^2(1 + 3\beta) + 2(1 + \beta)^2(1 - \beta) - 2\mu(\beta + 2)(1 - \beta)^2}{8(1 + \beta)^2(\beta + 2)}|c_1|^2
+ \frac{1 - \beta}{2 + \beta} - \frac{1 - \beta}{4(2 + \beta)}|c_1|^2
= \frac{(1 - \beta)^2(1 + 3\beta) - 2\mu(\beta + 2)(1 - \beta)^2}{8(1 + \beta)^2(\beta + 2)}|c_1|^2 + \frac{1 - \beta}{2 + \beta}.
\]

Secondly, we consider the case \( \mu \geq \frac{1 + 3\beta}{2(\beta + 2)}. \)

Write
\[ a_3 - \mu a_2^2 = a_3 - \frac{1 + 3\beta}{2(\beta + 2)} a_2^2 + \left( \frac{1 + 3\beta}{2(\beta + 2)} - \mu \right) a_2^2. \]

From (9), we have
\[ |a_2| \leq \frac{1 - \beta}{2(\beta + 1)}, \]
and
\[ |a_3| \leq \frac{1 - \beta}{2(\beta + 2)}. \]

Then
\[ |a_3 - \mu a_2^2| \leq |a_3 - \frac{1 + 3\beta}{2(\beta + 2)} a_2^2| + \left( \frac{1 + 3\beta}{2(\beta + 2)} - \mu \right) |a_2|^2, \]
and hence \( |a_3 - \mu a_2^2| \leq \frac{1 - \beta}{2(\beta + 2)} - \mu \left( \frac{1 - \beta}{2(\beta + 2)} \right)^2. \)

The proof of Theorem 3.1 is now complete.

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References


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