GENERALIZED SEQUENCE SPACES ON SEMINORMED SPACES

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Abstract. In this paper we define the sequence space $\ell M(u, p, q, s)$ on a semi-normed complex linear space by using Orlicz function and we give various properties and some inclusion relations on this space. This study generalized some results of Bektaş and Altın [1].

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1. Introduction

Let $\omega$ be the set of all sequences $x = (x_k)$ with complex terms.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the sequence space $\ell M = \{x \in \omega : \sum_{k=1}^{\infty} M(|x_k|/\rho) < \infty \text{ for some } \rho > 0\}$. The space $\ell M$ with the norm $\|x\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(|x_k|/\rho) \leq 1\}$ becomes a Banach space which is called an Orlicz sequence space. The space $\ell M$ is closely related to the space $\ell p$ which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark 1.1. If $M$ is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 \leq \lambda \leq 1$. 

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Let $X$ be a complex linear space with zero element $\theta$ and $(X, q)$ be a seminormed space with the seminorm $q$. By $S(X)$ we denote the linear space of all sequences $x = (x_k)$ with $(x_k) \in X$ and the usual coordinatewise operations:

$$\alpha x = (\alpha x_k) \text{ and } x + y = (x_k + y_k)$$

for each $\alpha \in C$ where $C$ denotes the set of all complex numbers. If $\lambda = (\lambda_k)$ is a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_k x_k)$. Let $U$ be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ and complex for all $k = 1, 2, \ldots$. Let $p = (p_k)$ be a sequence of positive real numbers and $M$ be an Orlicz function. Given $u \in U$. Let $s \geq 0$. Then we define the sequence space

$$\ell_M(u, p, q, s) = \{x \in S(X) : \sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k))]^{p_k} < \infty, \text{ for some } \rho > 0\}.$$

The following inequality and $p = (p_k)$ sequence will be used frequently throughout this paper.

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\},$$

where $a_k, b_k \in C$, $0 < p_k \leq \sup_k p_k = G$, $D = \max(1, 2^{G-1})[4]$.

A sequence space $E$ is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$.

2. Main Results

**Theorem 2.1.** The sequence space $\ell_M(u, p, q, s)$ is a linear space over the field $C$ complex numbers.

**Proof.** Let $x, y \in \ell_M(u, p, q, s)$ and $\alpha, \beta \in C$. Then there exist some positive numbers $\rho_1$ and $\rho_2$ such that

$$\sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k))]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^{-s}[M(q(u_k y_k))]^{p_k} < \infty.$$ 

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M$ is non-decreasing and convex, and since $q$ is a seminorm, we have

$$\sum_{k=1}^{\infty} k^{-s}[M(q(u_k(\alpha x_k + \beta y_k)))]^{p_k} \leq \sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k) + q(u_k y_k))]^{p_k}$$

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This proves that \( \ell_M(u, p, q, s) \) is a linear space.

**Theorem 2.2.** The space \( \ell_M(u, p, q, s) \) is paranormed (not necessarily totally paranormed) with

\[
g_u(x) = \inf \{ \rho^{p_k/H} : \left( \sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k))^{p_k}]^{1/\rho} \leq 1, \quad n = 1, 2, 3, \ldots \} \]

where \( H = \max(1, \sup_k p_k) \).

**Proof.** Clearly \( g_u(x) = g_u(-x) \). The subadditivity of \( g_u \) follows from (1'), on taking \( \alpha = 1 \) and \( \beta = 1 \). Since \( q(\theta) = 0 \) and \( M(0) = 0 \), we get \( \inf \{ \rho^{p_k/H} \} = 0 \) for \( x = \theta \).

Finally, we prove that the scalar multiplication is continuous. Let \( \lambda \) be any number. By definition,

\[
g_u(\lambda x) = \inf \{ \rho^{p_k/H} : \left( \sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k))^{p_k}]^{1/\rho} \leq 1, \quad n = 1, 2, 3, \ldots \} \]

Then

\[
g_u(\lambda x) = \inf \{ (\lambda r)^{p_k/H} : \left( \sum_{k=1}^{\infty} k^{-s}[M(q((\lambda x_k) x_k))^{p_k}]^{1/\rho} \leq 1, \quad n = 1, 2, 3, \ldots \} \]

where \( r = \rho/\lambda \). Since \( |\lambda|^{p_k} \leq \max(1, |\lambda|^H) \), then \( |\lambda|^{p_k/H} \leq (\max(1, |\lambda|^H))^{1/H} \).

Hence

\[
g_u(\lambda x) \leq (\max(1, |\lambda|^H))^{1/H} \inf \{ (r)^{p_k/H} : \left( \sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k))^{p_k}]^{1/\rho} \leq 1, \quad n = 1, 2, 3, \ldots \} \]

and therefore \( g_u(\lambda x) \) converges to zero when \( g_u(x) \) converges to zero in \( \ell_M(u, p, q, s) \).

Now suppose that \( \lambda_n \to 0 \) and \( x \) is in \( \ell_M(u, p, q, s) \). For arbitrary \( \varepsilon > 0 \), let \( N \) be a positive integer such that

\[
\sum_{k=N+1}^{\infty} k^{-s}[M(q(u_k x_k))^{p_k}]^{1/\rho} < \varepsilon^H
\]
for some $\rho > 0$. This implies that
\[
\left( \sum_{k=N+1}^{\infty} k^{-s} [M(q(\frac{u_k x_k}{\rho}))]^{p_k} \right)^{1/H} \leq \frac{\varepsilon}{2}.
\]

Let $0 < |\lambda| < 1$, then using Remark 1.1 we get
\[
\sum_{k=N+1}^{\infty} k^{-s} [M(q(\frac{\lambda u_k x_k}{\rho}))]^{p_k} < \sum_{k=N+1}^{\infty} k^{-s} [\lambda^s M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \left( \frac{\varepsilon}{2} \right)^H.
\]

Since $M$ is continuous everywhere in $[0, \infty)$, then
\[
f(t) = \sum_{k=1}^{N} k^{-s} \left[ M(q(\frac{t u_k x_k}{\rho})) \right]^{p_k}
\]
is continuous at 0. So there is $1 > \delta > 0$ such that $|f(t)| < \frac{\varepsilon}{2}$ for $0 < t < \delta$. Let $K$ be such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$ we have
\[
\left( \sum_{k=1}^{K} k^{-s} [M(q(\frac{\lambda_n u_k x_k}{\rho}))]^{p_k} \right)^{1/H} < \frac{\varepsilon}{2}.
\]

Since $0 < \varepsilon < 1$ we have
\[
\left( \sum_{k=1}^{\infty} k^{-s} [M(q(\frac{\lambda_n u_k x_k}{\rho}))]^{p_k} \right)^{1/H} < 1, \quad \text{for} \quad n > K.
\]

If we take limit on $\inf \{ \rho^{p_n/H} \}$ we get $g_u(\lambda x) \to 0$.

3. Some Particular Cases

We get the following sequence spaces from $\ell_M(u, p, q, s)$ on giving particular values to $p$ and $s$. Taking $p_k = 1$ for all $k \in N$, we have
\[
\ell_M(u, q, s) = \{ x \in S(X) : \sum_{k=1}^{\infty} k^{-s} [M(q(\frac{u_k x_k}{\rho}))] < \infty, \text{ for some } \rho > 0 \}.
\]

If we take $s = 0$, then we have
\[
\ell_M(u, p, q) = \{ x \in S(X) : \sum_{k=1}^{\infty} [M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty, \text{ for some } \rho > 0 \}.
\]
If we take \( p_k = 1 \) for all \( k \in N \) and \( s = 0 \), then we have

\[
\ell_M(u, q) = \{ x \in S(X) : \sum_{k=1}^{\infty} [M(q(\frac{u_k x_k}{\rho}))] < \infty, \text{ for some } \rho > 0 \}.
\]

If we take \( s = 0 \), \( q(x) = |x| \) and \( X = C \), then we have

\[
\ell_M(u, p) = \{ x \in S(X) : \sum_{k=1}^{\infty} [M(\frac{|u_k x_k|}{\rho})]^{p_k} < \infty, \text{ for some } \rho > 0 \}.
\]

In addition to the above sequence spaces, we write \( \ell_M(u, p, q, s) = \ell_M(p) \) due to Parashar and Choudhary [5], on taking \( u_k = 1 \) for all \( k \in N \), \( s = 0 \), \( q(x) = |x| \) and \( X = C \). If we take \( u_k = 1 \) for all \( k \in N \), we have \( \ell_M(u, p, q, s) = \ell_M(p, q, s) [1] \).

**Theorem 3.1.** (i) Let \( 0 < p_k \leq t_k < \infty \) for each \( k \in N \). Then \( \ell_M(u, p, q) \subseteq \ell_M(u, t, q) \).

(ii) \( \ell_M(u, q) \subseteq \ell_M(u, q, s) \).

(iii) \( \ell_M(u, p, q) \subseteq \ell_M(u, p, q, s) \).

**Proof.** (i) Let \( x \in \ell_M(u, p, q) \). Then there exists some \( \rho > 0 \) such that

\[
\sum_{k=1}^{\infty} [M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty.
\]

This implies that \( M(q(\frac{u_k x_k}{\rho})) \leq 1 \) for sufficiently large values of \( i \), say \( i \geq k_0 \) for some fixed \( k_0 \in N \).

Since \( M \) is non-decreasing, we get

\[
\sum_{k=1}^{\infty} [M(q(\frac{u_k x_k}{\rho}))]^{t_k} < \infty,
\]

since

\[
\sum_{k=k_0}^{\infty} [M(q(\frac{u_k x_k}{\rho}))]^{t_k} \leq \sum_{k=k_0}^{\infty} [M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty.
\]

Hence \( x \in \ell_M(u, t, q) \).

The proof of (ii) and (iii) is trivial.

**Theorem 3.2.** Let \( 0 < p_k \leq t_k < \infty \) for each \( k \). Then \( \ell_M(u, p) \subseteq \ell_M(u, t) \).

**Proof.** Proof can be proved by the same way as Theorem 3.1(i).

**Theorem 3.3.** (i) If \( 0 < p_k \leq 1 \) for all \( k \in N \), then \( \ell_M(u, p, q) \subseteq \ell_M(u, q) \).

(ii) If \( p_k \geq 1 \) for all \( k \in N \), then \( \ell_M(u, q) \subseteq \ell_M(u, p, q) \).

**Proof.** (i) If we take \( t_k = 1 \) for all \( k \in N \), in Theorem 3.1(i), then \( \ell_M(u, p, q) \subseteq \ell_M(u, q) \).
(ii) If we take $p_k = 1$ for all $k \in N$, in Theorem 3.1(i), then $\ell_M(u, q) \subseteq \ell_M(u, p, q)$.

Proposition 3.4 For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and any two seminorms $q_1$ and $q_2$ we have $\ell_M(u, p, q_1, r) \cap \ell_M(u, t, q_2, s) \neq \emptyset$ for $r, s > 0$.

Proof. Since the zero element belongs to $\ell_M(u, p, q_1, r)$ and $\ell_M(u, t, q_2, s)$, thus the intersection is nonempty.

Theorem 3.5. The sequence space $\ell_M(u, p, q, s)$ is solid.

Proof. Let $(x_k) \in \ell_M(u, p, q, s)$, i.e,

$$\sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k))/p_k] < \infty.$$ 

Let $(\alpha_k)$ be sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then we have

$$\sum_{k=1}^{\infty} k^{-s}[M(q(\alpha_k u_k x_k)/p_k)]p_k \leq \sum_{k=1}^{\infty} k^{-s}[M(q(u_k x_k)/p_k)]p_k < \infty.$$ 

Hence $(\alpha_k x_k) \in \ell_M(u, p, q, s)$ for all sequences of scalars $(\alpha_k)$ with $|\alpha_k| \leq 1$ for all $k \in N$, whenever $(x_k) \in \ell_M(u, p, q, s)$.

Therefore the space $\ell_M(u, p, q, s)$ is a solid sequence space.

Corollary 3.6. (i) Let $|u_k| \leq 1$ for all $k \in N$. Then $\ell_M(p, q, s) \subseteq \ell_M(u, p, q, s)$.

(ii) Let $|u_k| \geq 1$ for all $k \in N$. Then $\ell_M(u, p, q, s) \subseteq \ell_M(p, q, s)$.

Proof. Proof is trivial.

References


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