A SUBCLASS OF MULTIVALENT UNIFORMLY CONVEX FUNCTIONS ASSOCIATED WITH GENERALIZED ŞALÅGEAN AND RUSCHEWEYH DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper a new subclass of Multivalent uniformly convex functions with negative coefficients defined by a linear combination of generalized Şalågean and Ruscheweyh differential operators is introduced. Several results concerning coefficient estimates, the result of modified Hadamard product and results for a family of class preserving integral operators are considered. Extreme points and other interesting properties for this class are also indicated.

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1. Introduction and Definitions

Let \( A_p \) denote the class of functions of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,
\]

which are analytic and \( p \)-valent in the unit disk \( U = \{ z : |z| < 1 \} \). Also denote by \( T_p \) the class of functions of the form

\[
f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad (a_k \geq 0, z \in U),
\]

which are analytic and \( p \)-valent in \( U \).

For functions

\[
f_j(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,j} z^k, \quad (a_{k,j} \geq 0), \quad (j = 1, 2)
\]

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Hadamard product \((f_1 * f_2)(z)\) of \(f_1(z)\) and \(f_2(z)\) is defined by
\[
(f_1 * f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k.
\] (1.4)

A function \(f(z) \in A_p\) is said to be \(\beta\)-uniformly starlike functions of order \(\alpha\) denoted by \(\beta - S_p(\alpha)\) if it satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \geq \beta \left| \frac{zf'(z)}{f(z)} - p \right|,
\] (1.5)
in the class \(T_p\), the modified for some \(\alpha(-p \leq \alpha < p), \beta \geq 0\), and is said to be \(\beta\)-uniformly convex of order \(\alpha\) denoted by \(\beta - K_p(\alpha)\) if it satisfies
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} \geq \beta \left| \frac{zf''(z)}{f'(z)} + 1 - p \right|,
\] (1.6)
for some \(\alpha(-p \leq \alpha < p), \beta \geq 0\) and all \((z \in U)\). The class \(0 - S_p(\alpha) = S_p(\alpha)\), and \(0 - K_p(\alpha) = K_p(\alpha)\), where \(S_p(\alpha)\) and \(K_p(\alpha)\) are respectively the well-known classes of starlike and convex functions of order \(\alpha\) \((0 \leq \alpha < p)\).

The classes \(S_p^*(\alpha)\) and \(K_p^*(\alpha)\) are introduced by Patil and Thakare \([7]\) while the classes \(S(\alpha)\) and \(K(\alpha)\) were first studied by Reboorton \([8]\), Schild \([12]\), Silverman \([13]\), and others. The classes \(\beta - S_p(\alpha)\) and \(\beta - K_p(\alpha)\) were introduced and studied by Goodman \([2]\), Rønning\([9]\), and Minda and Ma \([5]\).

Let
\[
\beta - S_p^*(\alpha) = [\beta - S_p(\alpha)] \cap T_p, \quad K_p^*(\alpha) = K_p(\alpha) \cap T_p,
\] (1.7)
\[
\beta - S_p^*(\alpha) = [\beta - S_p(\alpha)] \cap T_p, \quad \text{and} \quad \beta - K_p^*(\alpha) = [\beta - K_p(\alpha)] \cap T_p.
\]

The Sălăgean differential operator \([11]\) can be generalized for a function \(f(z) \in A_p\) as follows
\[
S_{\delta,p}^0 f(z) = f(z),
\]
\[
S_{\delta,p}^1 f(z) = (1 - \delta)f(z) + \delta \frac{zf'(z)}{p} = S_{\delta,p} f(z),
\] (1.8)
\[
S_{\delta,p}^n f(z) = S_{\delta,p}(S_{\delta,p}^{n-1} f(z)). \quad (n \in \mathbb{N}, \delta \geq 0, z \in U)
\]

The \(n\)th Ruscheweyh derivative \([1]\) for a function \(f(z) \in A_p\), is defined by
\[
R_{p}^n f(z) = \frac{z^p}{n!} \frac{d^n}{dz^n} (z^{n-p} f(z)) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in U)
\] (1.9)
It can be easily seen that the operators $S^n_{p}$ and $R^n_{p}$ on the function $f(z) \in A_p$ are given by

$$S^n_{p}f(z) = z^p + \sum_{k=p+1}^{\infty} \left(1 + \left(\frac{k}{p} - 1\right)\delta\right)^n a_k z^k, \quad (1.10)$$

and

$$R^n_{p}f(z) = z^p + \sum_{k=p+1}^{\infty} C^n_{n+k-p} a_k z^k. \quad (1.11)$$

where $C^n_{n+k-p} = \frac{(n+k-p)!}{n!(k-p)!}$.

**Definition 1.** Let $n \in N_0$ and $\lambda \geq 0$. Let $D^n_{\lambda,\delta,p}f$ denote the operator defined by

$$D^n_{\lambda,\delta,p}f(z) = (1 - \lambda)S^n_{p}f(z) + \lambda R^n_{p}f(z) \quad (z \in U). \quad (1.12)$$

Notice that $D^n_{\lambda,\delta,p}$ is a linear operator and for $f(z) \in A_p$ we have

$$D^n_{\lambda,\delta,p}f(z) = z^p + \sum_{k=p+1}^{\infty} \phi_k(n, \lambda, \delta, p)a_k z^k, \quad (1.13)$$

where

$$\phi_k(n, \lambda, \delta, p) = \left[(1 - \lambda) \left(1 + \left(\frac{k}{p} - 1\right)\delta\right)^n + \lambda C^n_{n+k-p}\right] \quad (1.14)$$

It is clear that $D^0_{\lambda,\delta,p}f(z) = f(z)$ and $D^1_{\lambda,1,p}f(z) = z p f'(z)$. When $p = 1$, we get the differential operator studied by Khairnar and More [3].

**Definition 2.** For $-p \leq \alpha < p$, $\beta \geq 0$, we let $S^n_{p}(\alpha, \beta, \lambda, \delta)$ be the subclass of $A_p$ consisting of functions $f(z)$ of the form (1.1) and satisfying the following condition

$$\text{Re} \left\{ \frac{z \left(D^n_{\lambda,\delta,p}f(z)\right)'}{D^n_{\lambda,\delta,p}f(z)} - \alpha \right\} \geq \beta \left| \frac{z \left(D^n_{\lambda,\delta,p}f(z)\right)'}{D^n_{\lambda,\delta,p}f(z)} - p \right|, \quad (1.15)$$

also let $T^n_{p}(\alpha, \beta, \lambda, \delta) = S^n_{p}(\alpha, \beta, \lambda, \delta) \cap T_p$.

It may be noted that the class $T^n_{p}(\alpha, \beta, \lambda, \delta)$ extends the classes of starlike, convex, $\beta$-uniformly starlike and $\beta$-uniformly convex for suitable choice of $\alpha, \beta, \lambda, \delta$ and $n$. For example
i) For \( n = 0, \lambda = \delta = 1 \) the class \( T^n_p(\alpha, \beta, \lambda, \delta) \) reduces to the class of \( \beta \)-uniformly starlike functions.

(ii) For \( n = 1, \lambda = \delta = 1 \) we obtain the class of \( \beta \)-uniformly convex function.

Several other classes studied by various research workers can be obtained from the class \( T^n_p(\alpha, \beta, \lambda, \delta) \).

2. COEFFICIENT ESTIMATES

**Theorem 1.** A function \( f(z) \) defined by (1.2) is in the class \( T^n_p(\alpha, \beta, \lambda, \delta) \), 

\[-p \leq \alpha < p, \beta \geq 0\]

if and only if

\[
\sum_{k=p+1}^{\infty} \{k(1+\beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p) a_k \leq (p - \alpha), \quad (2.1)
\]

where \( \phi_k(n, \lambda, \delta, p) \) is given by (1.14) and the result is sharp.

**Proof.** Let \( f(z) \in T^n_p(\alpha, \beta, \lambda, \delta) \) and \( z \) be real then by virtue of (1.13) we have

\[
\frac{p - \sum_{k=p+1}^{\infty} k \phi_k(n, \lambda, \delta, p) a_k z^{k-p}}{1 - \sum_{n=p+1}^{\infty} \phi_k(n, \lambda, \delta, p) a_k z^{k-p}} \geq \beta \left| \frac{\sum_{k=p+1}^{\infty} (k-p)\phi_k(n, \lambda, \delta, p) a_k z^k}{1 - \sum_{n=p+1}^{\infty} \phi_k(n, \lambda, \delta, p) a_n z^n} \right|.
\]

Letting \( z \to 1 \) along the real axis, we obtain the desire inequality (2.1).

Conversely, assuming that (2.1) holds, then we show that

\[
\beta \left| \frac{z \left(D^n_{\lambda,\delta,p}f(z)\right)'}{D^n_{\lambda,\delta,p}f(z)} \right| - Re \left\{ \frac{z \left(D^n_{\lambda,\delta,p}f(z)\right)'}{D^n_{\lambda,\delta,p}f(z)} \right\} \leq p - \alpha \quad (2.2)
\]

We have

\[
\beta \left| \frac{z \left(D^n_{\lambda,\delta,p}f(z)\right)'}{D^n_{\lambda,\delta,p}f(z)} \right| - Re \left\{ \frac{z \left(D^n_{\lambda,\delta,p}f(z)\right)'}{D^n_{\lambda,\delta,p}f(z)} \right\} \leq (1+\beta) \left| \frac{z \left(D^n_{\lambda,\delta,p}f(z)\right)'}{D^n_{\lambda,\delta,p}f(z)} \right|
\]

\[
\leq \frac{(1+\beta) \sum_{k=p+1}^{\infty} (k-p) \phi_k(n, \lambda, \delta, p) a_k}{1 - \sum_{n=p+1}^{\infty} \phi_k(n, \lambda, \delta, p) a_k}.
\]
This expression is bounded above by \((p - \alpha)\) if

\[
\sum_{k=p+1}^{\infty} \left\{ k(1 + \beta) - (\alpha + p\beta) \right\} \phi_k(n, \lambda, \delta, p) a_k \leq (p - \alpha) ,
\]

The equality in (2.1) is attained for the function

\[
f(z) = z^p - \frac{(p - \alpha)}{k(1 + \beta) - (\alpha + p\beta)} \phi_k(n, \lambda, \delta, p) z^k \cdot \quad k \geq p+1 \quad (2.3)
\]

This completes the proof of the theorem.

**Corollary 1.** Let the function \(f(z)\) defined by (1.2) be in the class \(T^n_p(\alpha, \beta, \lambda, \delta)\), \(-p \leq \alpha < p\), \(\beta \geq 0\), then

\[
a_k \leq \frac{(p - \alpha)}{k(1 + \beta) - (\alpha + p\beta)} \phi_k(n, \lambda, \delta, p) , \quad k \geq p+1.
\]

3. **Results Involving Modified Hadamard Product**

**Theorem 2.** For \(n \in \mathbb{N}_0, \lambda, \delta \geq 0, -p \leq \alpha < p\) and \(\beta \geq 0\) let \(f_1(z) \in T^n_p(\alpha, \beta, \lambda, \delta)\) and \(f_2(z) \in T^n_p(\gamma, \beta, \lambda, \delta)\). Then \(f_1 * f_2(z) \in T^n_p(\sigma, \beta, \lambda, \delta)\), where

\[
\sigma = p - \frac{(1 + \beta)(p - \alpha)(p - \gamma)}{(p + 1 + \beta - \alpha)(p + 1 + \beta - \gamma)} \left[ (1 - \lambda) \left( 1 + \frac{\beta}{p} \right)^n + \lambda(n + 1) \right] - (p - \alpha)(p - \gamma)
\]

and the result is sharp.

**Proof.** To prove the theorem it is sufficient to assert that

\[
\sum_{k=p+1}^{\infty} \frac{\left\{ k(1 + \beta) - (\sigma + p\beta) \right\}}{p - \sigma} \phi_k(n, \lambda, \delta, p) a_{k,1} a_{k,2} \leq 1 , \quad (3.2)
\]

where \(\phi_k(n, \lambda, \delta, p)\) is defined in (1.14) and \(\sigma\) is defined in (3.1). Now by virtue of Cauchy-Schwarz inequality and Theorem 1, it follows that

\[
\sum_{k=p+1}^{\infty} \frac{\left\{ k(1 + \beta) - (\alpha + p\beta) \right\}^{1/2} \left\{ k(1 + \beta) - (\gamma + p\beta) \right\}^{1/2}}{\sqrt{(p - \alpha)(p - \gamma)}} \phi_k(n, \lambda, \delta, p) \sqrt{a_{n,1} a_{n,2}} \leq 1 ,
\]
Hence (3.2) is true if
\[
\frac{(k+1) - (\sigma+p\beta)}{p-\sigma} \phi_k(n, \lambda, \delta, p) a_{n,1} a_{n,2}
\]
\[
\leq \frac{(k+1) - (\alpha+p\beta)}{p-\sigma} \frac{\phi_k(n, \lambda, \delta, p)}{\sqrt{(p-\alpha)(p-\gamma)}} \leq \frac{(k+1) - (\gamma+p\beta)}{p-\sigma} \phi_k(n, \lambda, \delta, p) \sqrt{a_{n,1}a_{n,2}}
\]
or equivalently
\[
\sqrt{a_{n,1}a_{n,2}} \leq \frac{(k+1) - (\alpha+p\beta)}{p-\sigma} \frac{\phi_k(n, \lambda, \delta, p)}{\sqrt{(p-\alpha)(p-\gamma)}} \leq \frac{(k+1) - (\gamma+p\beta)}{p-\sigma} \phi_k(n, \lambda, \delta, p)
\]
By virtue of (3.3), (3.2) is true if
\[
\sqrt{(p-\alpha)(p-\gamma)} \frac{(k+1) - (\alpha+p\beta)}{\phi_k(n, \lambda, \delta, p)} \frac{\phi_k(n, \lambda, \delta, p)}{\sqrt{(p-\alpha)(p-\gamma)}} \leq \frac{(k+1) - (\gamma+p\beta)}{p-\sigma} \phi_k(n, \lambda, \delta, p)
\]
which yields
\[
\sigma \leq p - \frac{(k+1)(p-\alpha)}{(k+1) - (\alpha+p\beta)} \frac{(p-\alpha)}{(p-\gamma)} \frac{\phi_k(n, \lambda, \delta, p)}{(p-\alpha)(p-\gamma)}.
\]
Under the stated conditions in the theorem, we observe that the function \(\phi_k(n, \lambda, \delta, p)\) is a decreasing for \(k \geq p+1\), and thus (3.5) is satisfied if \(\sigma\) is given by (3.1).
Finally the result is sharp for
\[
f_1(z) = z^p - \frac{(p-\alpha)}{(p+1+\beta-\alpha) \left( (1-\lambda) \left( 1 + \frac{\delta}{p} \right)^n + \lambda(n+1) \right)} z^{p+1},
\]
\[
f_2(z) = z^p - \frac{(p-\gamma)}{(p+1+\beta-\gamma) \left( (1-\lambda) \left( 1 + \frac{\delta}{p} \right)^n + \lambda(n+1) \right)} z^{p+1}.
\]
Theorem 3. Under the conditions stated in Theorem 2, let the functions \( f_j(z) \) \((j = 1, 2)\) defined by (1.3) be in the class \( T_p^n(\alpha, \beta, \lambda, \delta) \). Then \( f_1 \ast f_2(z) \in T_p^n(\sigma, \beta, \lambda, \delta) \), where

\[
\sigma = p - \frac{(1 + \beta)(p - \alpha)^2}{(p + 1 + \beta - \alpha)^2 \left[ (1 - \lambda) \left( 1 + \frac{\delta}{p} \right)^n + \lambda(n + 1) \right] - (p - \alpha)^2} .
\] (3.6)

Proof. The result follows by setting \( \alpha = \gamma \) in Theorem 3.

Theorem 4. Under the conditions stated in Theorem 2, let the functions \( f_j(z) \) \((j = 1, 2)\) defined by (1.3) be in the class \( T_p^n(\alpha, \beta, \lambda, \delta) \). Then

\[
h(z) = z^p - \sum_{k=p+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^p
\]

is in the class \( T_p^n(\sigma, \beta, \lambda, \delta) \), where

\[
\sigma = p - \frac{2(1 + \beta)(p - \alpha)^2}{(p + 1 + \beta - \alpha)^2 \left[ (1 - \lambda) \left( 1 + \frac{\delta}{p} \right)^n + \lambda(n + 1) \right] - 2(p - \alpha)^2} .
\] (3.8)

Proof. In view of Theorem 1, it is sufficient to prove that

\[
\sum_{k=p+1}^{\infty} \left\{ k(1 + \beta) - (\sigma + p\beta) \right\} \phi_k(n, \lambda, \delta, p) \left( a_{k,1}^2 + a_{k,2}^2 \right) \leq 1 ,
\]

where \( \phi_k(n, \lambda, \delta, p) \) is defined in (1.14) and \( \sigma \) is defined in (3.8). as \( f_j(z) \in T_p^n(\alpha, \beta, \lambda, \delta)(j = 1, 2) \), Theorem 1 yields

\[
\sum_{k=p+1}^{\infty} \left[ \frac{k(1 + \beta) - (\alpha + p\beta)}{(p - \alpha)} \phi_k(n, \lambda, \delta, p) \right]^2 a_{k,j}^2
\]

\[
\leq \sum_{k=p+1}^{\infty} \left[ \frac{k(1 + \beta) - (\alpha + p\beta)}{(p - \alpha)} \phi_k(n, \lambda, \delta, p) a_{k,j} \right]^2 \leq 1
\]

hence

\[
\sum_{k=p+1}^{\infty} \frac{1}{2} \left[ \frac{k(1 + \beta) - (\alpha + p\beta)}{(p - \alpha)} \phi_k(n, \lambda, \delta, p) \right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1
\] (3.10)
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(3.9) is true if 
\[
\frac{k(1 + \beta) - (\sigma + p\beta)}{p - \sigma} \phi_k(n, \lambda, \delta, p)(a_{n,1}^2 + a_{n,2}^2) 
\leq \frac{1}{2} \left[ \frac{k(1 + \beta) - (\alpha + p\beta)}{p - \alpha} \phi_k(n, \lambda, \delta, p) \right]^2 (a_{n,1}^2 + a_{n,2}^2) ,
\]
that is, if 
\[
\sigma \leq p - \frac{2(k - p)(1 + \beta)(p - \alpha)^2}{n(1 + \beta) - (\alpha + p\beta)^2} \phi_k(n, \lambda, \delta, p) - 2(p - \alpha)^2 .
\]

Under the stated conditions in the theorem, we observe that the function \(\phi_k(n, \lambda, \delta, p)\) is a decreasing for \(k \geq p + 1\), and thus (3.11) is satisfied if \(\sigma\) is given by (3.8).

4. FAMILY OF CLASS PRESERVING INTEGRAL OPERATORS

In this section, we discuss some class preserving integral operators. We recall here the Komatu operator [6] defined by 
\[
H(z) = P^d_{c,p}f(z) = \frac{(c + p)^d}{\Gamma(d) z^c} \int_0^z t^{d-1} \left( \log \frac{t}{z} \right)^d f(t) dt \tag{4.1}
\]
where \(d > 0, c > -p\) and \(z \in U\).

Also we recall the generalized Jung-Kim-Srivastava integral operator [4] defined by 
\[
I(z) = Q^d_{c,p}f(z) = \frac{\Gamma(d + c + p)}{\Gamma(c + p) \Gamma(d)} \frac{1}{z^p} \int_0^z t^{d-1} \left( 1 - \frac{t}{z} \right)^d f(t) dt . \tag{4.2}
\]

Theorem 5. If \(f(z) \in T^n_p(\alpha, \beta, \lambda, \delta)\), then \(H(z) \in T^n_p(\alpha, \beta, \lambda, \delta)\).

Proof. Let the function \(f(z) \in T^n_p(\delta, \beta, \lambda, \delta)\) be defined by (1.2). It can be easily verified that 
\[
H(z) = z^p - \sum_{k=p+1}^{\infty} \left( \frac{c + p}{c + k + p} \right)^d a_k z^k \quad (a_k \geq 0, p \in N) \tag{4.3}
\]

Now \(H(z) \in T^n_p(\alpha, \beta, \lambda, \delta)\) if 
\[
\sum_{k=p+1}^{\infty} \left[ \frac{k(1 + \beta) - (\alpha + p\beta)}{p - \alpha} \phi_k(n, \lambda, \delta, p) \right] \left( \frac{c + p}{c + k + p} \right)^d a_k \leq 1 \tag{4.4}
\]
Now as \( \frac{c+p}{c+k+p} \leq 1 \) for \( k \in N \), so it is clear that

\[
\sum_{k=p+1}^{\infty} \left\{ \frac{[k(1+\beta) - (\alpha + p\beta)] \phi_k(n, \lambda, \delta, p)}{(p-\alpha)} \right\} \left( \frac{c + p}{c + k + p} \right)^d a_k
\]

\[
\leq \sum_{k=p+1}^{\infty} \left\{ \frac{[k(1+\beta) - (\alpha + p\beta)] \phi_k(n, \lambda, \delta, p)}{(p-\alpha)} \right\} a_k \leq 1
\]

Therefore \( H(z) \in T_p^n(\alpha, \beta, \lambda, \delta) \).

**Theorem 6.** Let \( d > 0, c > -p \) and \( f(z) \in T_p^n(\alpha, \beta, \lambda, \delta) \). Then \( H(z) \) defined by (4.1) is \( p \)-valent in the disk \( |z| < R_1 \), where

\[
R_1 = \inf_k \left\{ \frac{p[k(1+\beta) - (\alpha + p\beta)](c + k + p)^d \phi_k(n, \lambda, \delta, p)}{k(c + p)^d(p-\alpha)} \right\}^\frac{1}{k}
\]

(4.5)

**Proof.** In order to prove the assertion, it is enough to show that

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq p
\]

(4.6)

Now, in view of (4.3), we get

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| = \left| - \sum_{k=p+1}^{\infty} k \left( \frac{c + p}{c + k + p} \right)^d a_k \right| \leq \sum_{k=p+1}^{\infty} k \left( \frac{c + p}{c + k + p} \right)^d a_k \left| z \right|^k
\]

This expression is bounded by \( p \) if

\[
\sum_{k=p+1}^{\infty} k \left( \frac{c + p}{c + k + p} \right)^d a_k \left| z \right|^k \leq 1
\]

(4.7)

Given that \( f(z) \in T_p^n(\alpha, \beta, \lambda, \delta) \), so in view of Theorem 1, we have

\[
\sum_{k=p+1}^{\infty} \left\{ \frac{[k(1+\beta) - (\alpha + p\beta)] \phi_k(n, \lambda, \delta, p)}{(p-\alpha)} \right\} a_k \leq 1
\]

Thus, (4.7) holds if

\[
k \left( \frac{c + p}{c + k + p} \right)^d a_k \left| z \right|^k \leq \frac{p[k(1+\beta) - (\alpha + p\beta)] \phi_k(n, \lambda, \delta, p)}{(p-\alpha)}
\]

that is

\[
\left| z \right| \leq \left\{ \frac{p[k(1+\beta) - (\alpha + p\beta)](c + k + p)^d \phi_k(n, \lambda, \delta, p)}{k(c + p)^d(p-\alpha)} \right\}^\frac{1}{k}
\]

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The result follows by setting $|z| = R_1$.

Following similar steps as in the proofs of Theorem 5 and Theorem 6, we can state the following two theorems concerning the generalized Jung-Kim-Srivastava integral operator $I(z)$.

**Theorem 7.** If $f(z) \in T^n_p(\alpha, \beta, \lambda, \delta)$, then $I(z) \in T^n_p(\alpha, \beta, \lambda, \delta)$.

**Theorem 8.** Let $d > 0, c > -p$ and $f(z) \in T^n_p(\alpha, \beta, \lambda, \delta)$. Then $I(z)$ defined by (4.2) is $p$-valent in the disk $|z| < R_2$, where

$$R_2 = \inf_k \left\{ \frac{p \{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p)(p + c + d)_k}{k(p - \alpha)(p + c)_k} \right\}^{\frac{1}{p}}. \quad (4.8)$$

5. **Extreme Points of the Class $T^n_p(\alpha, \beta, \lambda, \delta)$**

**Theorem 9.** Let $f_p(z) = z^p$ and

$$f_k(z) = z^p - \frac{(p - \alpha)}{\{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p)} z^k, \quad (k \geq p + 1). \quad (5.1)$$

Then $f(z) \in T^n_p(\alpha, \beta, \lambda, \delta)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z), \quad (5.2)$$

where $\lambda_k \geq 0$ and $\sum_{k=p}^{\infty} \lambda_k = 1$, and $\phi_k(n, \lambda, \delta, p)$ is given in (1.14).

**Proof.** Let (5.2) holds, then by (5.1) we have

$$f(z) = \lambda_p z^p - \sum_{k=p+1}^{\infty} \frac{(p - \alpha)}{\{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p)} \lambda_k z^k.$$

Now

$$\sum_{k=p+1}^{\infty} \{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p) a_k \quad \frac{(p - \alpha)}{\{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p)} \lambda_k$$

$$= \sum_{k=p+1}^{\infty} \{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p) \times \frac{(p - \alpha)}{\{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p)} \lambda_k$$

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= (p - \alpha) \sum_{k=p+1}^{\infty} \lambda_k \leq (p - \alpha) \sum_{k=p}^{\infty} \lambda_k \leq p - \alpha.

Hence by Theorem 1, \( f(z) \in T_p^n(\alpha, \beta, \lambda, \delta) \).

Conversely, suppose \( f(z) \in T_p^n(\alpha, \beta, \lambda, \delta) \). Since

\[ a_k \leq \frac{(p - \alpha)}{\{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p)} \leq \frac{(p - \alpha)}{\{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p)} \]

setting \( \lambda_k = \frac{(k(1 + \beta) - (\alpha + p\beta)) \phi_k(n, \lambda, \delta, p)}{(p - \alpha)} \) \( a_k \) and \( \lambda_p = 1 - \sum_{k=p+1}^{\infty} \lambda_k \), we get (5.2). This completes the proof of the theorem.

6. Closure Properties

**Theorem 10.** Let the functions \( f_j(z) \) defined by (1.3) be in the class \( T_p^n(\alpha, \beta, \lambda, \delta) \). Then the function \( h(z) \) defined by

\[ h(z) = z^p - \sum_{k=p+1}^{\infty} d_k z^k \]

belongs to \( T_p^n(\alpha, \beta, \lambda, \delta) \), where

\[ d_k = \frac{1}{m} \sum_{j=1}^{m} a_{k,j}, \quad (a_{k,j} \geq 0). \]

**Proof.** Since \( f_j(z) \in T_p^n(\delta, \alpha, \lambda, \delta) \), it follows from Theorem 1 that

\[ \sum_{k=p+1}^{\infty} \{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p) \leq (p - \alpha), \quad (6.1) \]

where \( \phi_k(n, \lambda, \delta, p) \) is given in (1.14). Therefore

\[ \sum_{k=p+1}^{\infty} \{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p) d_k \]

\[ = \sum_{k=p+1}^{\infty} \{k(1 + \beta) - (\alpha + p\beta)\} \phi_k(n, \lambda, \delta, p) \left( \frac{1}{m} \sum_{j=1}^{m} a_{kj} \right) \leq p - \alpha, \]

by (6.1), which yields that \( h(z) \in T_p^n(\delta, \beta, \lambda, \delta) \).
References


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