THE ORDER OF CONVEXITY OF SOME INTEGRAL OPERATORS

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Abstract. In this paper we consider the classes of starlike functions of order \( \alpha \), convex functions of order \( \alpha \) and we study the convexity and \( \alpha \)-order convexity for some general integral operators. Several corollaries of the main results are also considered.

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1. Introduction

We consider the unit open disk of the complex plane denoted by \( U \), \( U = \{ z : |z| < 1 \} \) and let \( A \) be the class of holomorphic functions in \( U \) of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in \( U \). We denote by \( S \) the class of univalent functions in the unit disk.

A function \( f(z) \in S \) is a starlike of order \( \alpha \) if it satisfies

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad (z \in U)
\]

for some \( \alpha \) (0 \leq \alpha < 1). We denote by \( S^*(\alpha) \) the subclass of \( A \) consisting of the functions which are starlike of order \( \alpha \) in \( U \). For \( \alpha = 0 \) we obtain the class of starlike functions, denoted by \( S^* \).
A function \( f(z) \in S \) is convex of order \( \alpha \) if it satisfies
\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad (z \in U)
\]
for some \( \alpha \) \((0 \leq \alpha < 1)\). We denote by \( K(\alpha) \) the subclass of \( A \) consisting of the functions which are convex of order \( \alpha \) in \( U \). For \( \alpha = 0 \) we obtain the class of convex functions, denoted by \( K \).

A function \( f \in A \) is in the class \( R(\alpha) \) if \( \text{Re}(f'(z)) > \alpha \), \((z \in U)\).

Recently, Frasin and Jahangiri in [3] define the family \( B(\mu, \alpha) \), \( \mu \geq 0 \), \( 0 \leq \alpha < 1 \) so that it consists of functions \( f \in A \) satisfying the condition
\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^\mu - 1 \right| < 1 - \alpha, \quad (z \in U).
\]

In this paper we will obtain the order of convexity of the following integral operators:
\[
G_\gamma(z) = \int_0^z \left( te^{f(t)} \right)^{\frac{1}{\gamma}} dt,
\]
\[
G_{n,\gamma}(z) = \int_0^z \prod_{i=1}^n \left( te^{f_i(t)} \right)^{\frac{1}{\gamma}} dt,
\]
\[
H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^n \left( te^{f_i(t)} \right)^{\frac{1}{\gamma_i}} dt,
\]
and
\[
H_n(z) = \int_0^z \prod_{i=1}^n \left( te^{f_i(t)} \right)^{\gamma_i} dt,
\]
where the functions \( f_i \) for all \( i = 1, 2, ..., n \) and \( f \) are in \( B(\mu, \alpha) \).

Lemma 1. (General Schwarz Lemma).[5] Let the function \( f \) be regular in the disk \( U_R = \{ z \in \mathbb{C} : |z| < R \} \), with \( |f(z)| < M \) for fixed \( M \). If \( f \) has one zero with multiplicity order bigger than \( m \) for \( z = 0 \), then
\[
|f(z)| \leq \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R).
\]
The equality can hold only if
\[
f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m,
\]
where \( \theta \) is constant.
Theorem 1. [4]. Let $f \in A$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. If $|f(z)| \leq M$ ($M \geq 1$, $z \in U$) then the integral operator

$$G(z) = \int_{0}^{z} \left( te^{f(t)} \right)^{\gamma} \, dt$$

(9)

is in $K(\delta)$, where

$$\delta = 1 - |\gamma| \left[ (2 - \alpha)M^{\mu} + 1 \right]$$

(10)

and $|\gamma| < \frac{1}{(2 - \alpha)M^{\mu} + 1}$, $\gamma \in \mathbb{C}$.

2. Main results

Theorem 2. Let $f \in A$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. If $|f(z)| \leq M$ ($M \geq 1$, $z \in U$) then the integral operator

$$G_\gamma(z) = \int_{0}^{z} \left( te^{f(t)} \right)^{\frac{1}{\gamma}} \, dt$$

(11)

is in $K(\delta)$, where

$$\delta = 1 - \frac{1}{|\gamma|} \left[ (2 - \alpha)M^{\mu} + 1 \right]$$

(12)

and $\frac{1}{|\gamma|} < \frac{1}{(2 - \alpha)M^{\mu} + 1}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Let $f \in A$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. It follows from (11) that

$$G'_\gamma(z) = \left( ze^{f(z)} \right)^{\frac{1}{\gamma}}$$

and

$$G''_\gamma(z) = \frac{1}{\gamma} \left( ze^{f(z)} \right)^{\frac{1}{\gamma} - 1} \left( e^{f(z)} + ze^{f(z)} f'(z) \right).$$

Then $G''_\gamma(z) / G'_\gamma(z) = \frac{1}{\gamma} \left( \frac{1}{z} + f'(z) \right)$ and, hence

$$\left| \frac{zG''_\gamma(z)}{G'_\gamma(z)} \right| = \frac{1}{|\gamma|} \left( |1 + zf'(z)| \right) \leq \frac{1}{|\gamma|} \left( 1 + |f'(z)| \left( \frac{z}{f(z)} \right)^{\mu} \cdot \left( \frac{f(z)}{z} \right)^{\nu} \cdot |z| \right).$$

(13)

Applying the General Schwarz lemma, we have $\left| \frac{f(z)}{z} \right| \leq M$, ($z \in U$). Therefore, from (13), we obtain

$$\left| \frac{zG''_\gamma(z)}{G'_\gamma(z)} \right| \leq \frac{1}{|\gamma|} \left( 1 + |f'(z)| \left( \frac{z}{f(z)} \right)^{\mu} \cdot M^{\mu} \right), \quad z \in U.$$  

(14)
From (4) and (14), we see that
\[ \left| \frac{zG''(z)}{G'(z)} \right| \leq \frac{1}{|\gamma|} [(2 - \alpha)M^{\mu} + 1] = 1 - \delta. \]

Letting \( \mu = 0 \) in Theorem 2, we have \( B(0, \alpha) \equiv R(\alpha) \) and we obtain next corollary.

**Corollary 1.** Let \( f \in \mathcal{A} \) be in the class \( R(\alpha) \), \( 0 \leq \alpha < 1 \). Then the integral operator
\[ \int_0^z \left( te^{f(t)} \right)^{\frac{1}{2}} dt \in K(\delta), \]
where
\[ \delta = 1 - \frac{1}{|\gamma|} (3 - \alpha) \tag{15} \]
and \( \frac{1}{|\gamma|} < \frac{1}{3 - \alpha}, \gamma \in \mathbb{C} \setminus \{0\}. \)

Letting \( \mu = 1 \) in Theorem 2, we have \( B(1, \alpha) \equiv S^\ast(\alpha) \) and we obtain next corollary.

**Corollary 2.** Let \( f \in \mathcal{A} \) be in the class \( S^\ast(\alpha) \), \( 0 \leq \alpha < 1 \). If \( |f(z)| \leq M \) \((M \geq 1, z \in U)\) then the integral operator
\[ \int_0^z \left( te^{f(t)} \right)^{\frac{1}{2}} dt \in K(\delta), \]
where
\[ \delta = 1 - \frac{1}{|\gamma|} [(2 - \alpha)M + 1] \tag{16} \]
and \( \frac{1}{|\gamma|} < \frac{1}{(2 - \alpha)M + 1}, \gamma \in \mathbb{C} \setminus \{0\}. \)

Letting \( \alpha = \delta = 0 \) in Corollary 2, we have

**Corollary 3.** Let \( f \in \mathcal{A} \) be a starlike function in \( U \). If \( |f(z)| \leq M \) \((M \geq 1, z \in U)\) then the integral operator \( \int_0^z \left( te^{f(t)} \right)^{\frac{1}{2}} dt \) is convex in \( U \), where \( \frac{1}{|\gamma|} = \frac{1}{2M + 1}, \gamma \in \mathbb{C} \setminus \{0\}. \)
Theorem 3. Let \( f_i(z) \in A \) be in the class \( B(\mu, \alpha) \), \( \mu \geq 0 \), \( 0 \leq \alpha < 1 \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M_i \) \( (M_i \geq 1, \ z \in U) \) for all \( i = 1, 2, \ldots, n \), then the integral operator

\[
G_{n, \gamma}(z) = \int_0^z \prod_{i=1}^n \left(te^{t_i(t)}\right)^\gamma dt
\]

is in \( K(\delta) \), where

\[
\delta = 1 - |\gamma| \left[n + (2 - \alpha) \sum_{i=1}^n M_i^\mu\right]
\]

(17)

and \( |\gamma| < 1 + (2 - \alpha) \sum_{i=1}^n M_i^\mu \), \( \gamma \in \mathbb{C} \).

Proof. Let \( f_i \in A \) be in the class \( B(\mu, \alpha) \), \( \mu \geq 0 \), \( 0 \leq \alpha < 1 \). It follows from (6) that

\[
G_{n, \gamma}(z) = \int_0^z t^n e^{\sum_{i=1}^n f_i(t)} dt \quad \text{and} \quad G'_{n, \gamma}(z) = z^n e^{\sum_{i=1}^n f_i(z)}.
\]

Also

\[
G''_{n, \gamma}(z) = \gamma \left(\frac{z^n}{e^{z}} \sum_{i=1}^n f_i(z)\right)^{\gamma-1} \cdot \sum_{i=1}^n e^{f_i(z)} \left(n + z \sum_{i=1}^n f_i'(z)\right)
\]

Then

\[
\frac{G''_{n, \gamma}(z)}{G'_{n, \gamma}(z)} = \gamma \left(\frac{n}{z} + \sum_{i=1}^n f_i'(z)\right)
\]

and, hence

\[
\left|\frac{zG''_{n, \gamma}(z)}{G'_{n, \gamma}(z)}\right| = |\gamma| \left|n + z \sum_{i=1}^n f_i'(z)\right| \leq |\gamma| \sum_{i=1}^n \left|1 + zf_i'(z)\right|
\]

\[
\leq |\gamma| \sum_{i=1}^n \left[1 + \left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^\mu\right| \cdot \left|\left(\frac{M_i^\mu}{z}\right)\right| \cdot |z|\right].
\]

(18)

Applying the General Schwarz lemma, we have \( \left|\frac{f_i(z)}{z}\right| \leq M_i \), for all \( i = 1, 2, \ldots, n \).

Therefore, from (18), we obtain

\[
\left|\frac{zG''_{n, \gamma}(z)}{G'_{n, \gamma}(z)}\right| \leq |\gamma| \sum_{i=1}^n \left[1 + \left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^\mu\right| \cdot M_i^\mu\right], \ (z \in U).
\]

(19)
From (4) and (19), we see that
\[
\left| \frac{zG''_{n,\gamma}(z)}{G'_{n,\gamma}(z)} \right| \leq |\gamma| \left[ n + (2 - \alpha) \sum_{i=1}^{n} M_i^\mu \right] = 1 - \delta.
\]

This completes the proof.

For \( M_1 = M_2 = \ldots = M_n = M \) we have

**Corollary 4.** Let \( f_i(z) \in A \) be in the class \( B(\mu, \alpha) \), \( \mu \geq 0 \), \( 0 \leq \alpha < 1 \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M \) \( (M \geq 1, \ z \in U) \) for all \( i = 1, 2, \ldots, n \), then the integral operator
\[
G_{n,\gamma}(z) = \int_0^z \prod_{i=1}^{n} \left( te^{f_i(t)} \right)^\gamma dt
\]
is in \( K(\delta) \), where
\[
\delta = 1 - |\gamma| \left[ n(1 + (2 - \alpha)M^\mu) \right]
\]
and \( |\gamma| < \frac{1}{n[1 + (2 - \alpha)M^\mu]} \), \( \gamma \in \mathbb{C} \).

Letting \( \mu = 0 \) in Corollary 4, we have

**Corollary 5.** Let \( f_i(z) \in A \) be in the class \( R(\alpha) \), \( 0 \leq \alpha < 1 \) for all \( i = 1, 2, \ldots, n \). Then the integral operator defined in (6) is in \( K(\delta) \), where
\[
\delta = 1 - |\gamma|n(3 - \alpha)
\]
and \( |\gamma| < \frac{1}{n(3 - \alpha)} \), \( \gamma \in \mathbb{C} \).

Letting \( \mu = 1 \) in Corollary 4, we have

**Corollary 6.** Let \( f_i \in A \) be in the class \( S^*(\alpha) \), \( 0 \leq \alpha < 1 \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M \) \( (M \geq 1, \ z \in U) \) for all \( i = 1, 2, \ldots, n \), then the integral operator defined in (6) is in \( K(\delta) \), where
\[
\delta = 1 - |\gamma|[n(1 + (2 - \alpha)M)]
\]
and \( |\gamma| < \frac{1}{n[1 + (2 - \alpha)M]} \), \( \gamma \in \mathbb{C} \).

Letting \( \alpha = \delta = 0 \) in Corollary 6, we have
Corollary 7. Let \( f_i \in A \) be starlike functions in \( U \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M \) (\( M \geq 1, \ z \in U \)) for all \( i = 1, 2, \ldots, n \) then the integral operator defined in (6) is convex in \( U \), where \(|\gamma| = \frac{1}{n(2M + 1)}\), \( \gamma \in \mathbb{C} \).

Letting \( n = 1 \) in Corollary 4, we obtain Theorem 1 from paper [4].

Theorem 4. Let \( f_i(z) \in A \) be in the class \( B(\mu, \alpha) \), \( \mu \geq 0, \ 0 \leq \alpha < 1 \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M_i \) (\( M_i \geq 1, \ z \in U \)) for all \( i = 1, 2, \ldots, n \), then the integral operator

\[
H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^n \left( t e^{f_i(t)} \right)^{\frac{1}{\gamma}} \, dt
\]

is in \( K(\delta) \), where

\[
\delta = 1 - \frac{1}{|\gamma|} \left[ n + (2 - \alpha) \sum_{i=1}^n M_i^{\mu} \right]
\]

and \( \frac{1}{|\gamma|} < \frac{1}{n + (2 - \alpha) \sum_{i=1}^n M_i^{\mu}} \), \( \gamma \in \mathbb{C} \setminus \{0\} \).

Proof. Let \( f_i \in A \) be in the class \( B(\mu, \alpha) \), \( \mu \geq 0, \ 0 \leq \alpha < 1 \). We have from (7) that

\[
H_{n,\gamma}(z) = \int_0^z t^{\frac{n}{\gamma}} e^{\sum_{i=1}^n f_i(t)} \, dt \quad \text{and} \quad H'_{n,\gamma}(z) = z^{\frac{n}{\gamma}} e^{\sum_{i=1}^n f_i(z)}.
\]

Also

\[
H''_{n,\gamma}(z) = \frac{1}{\gamma} \left( \frac{1}{z^n} e^{\sum_{i=1}^n f_i(z)} \right)^{\frac{1}{\gamma} - 1} \cdot z^{n-1} \cdot e^{\sum_{i=1}^n f_i(z)} \left( n + z \sum_{i=1}^n f_i'(z) \right)
\]

Then

\[
\frac{H''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} = \frac{1}{\gamma} \left( \frac{n}{z} + \sum_{i=1}^n f_i'(z) \right)
\]

and, hence

\[
\left| \frac{z H''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} \right| = \frac{1}{|\gamma|} \left| n + z \sum_{i=1}^n f_i'(z) \right| \leq \frac{1}{|\gamma|} \left( \sum_{i=1}^n \left| 1 + z f_i'(z) \right| \right)
\]

\[
\leq \frac{1}{|\gamma|} \sum_{i=1}^n \left[ 1 + \left| f_i'(z) \left( \frac{z}{f_i(z)} \right) \right| \mu \cdot \left( \frac{f_i(z)}{z} \right)^{\mu} \cdot |z| \right]
\]

(24)
Applying the General Schwarz lemma, we have
$$\left| \frac{f_i(z)}{z} \right| \leq M_i, \text{ for all } i = 1, 2, \ldots, n.$$ Therefore, from (24), we obtain
$$\left| z \frac{H''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} \right| \leq \frac{1}{|\gamma|} \sum_{i=1}^{n} \left[ 1 + \left| f'_i(z) \left( \frac{z}{f_i(z)} \right)^\mu \cdot M_i^\mu \right| \right], \quad (z \in U). \quad (25)$$

From (4) and (25), we see that
$$\left| z \frac{H''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} \right| \leq \frac{1}{|\gamma|} \left[ n + (2 - \alpha) \sum_{i=1}^{n} M_i^\mu \right] = 1 - \delta.$$ 

For $M_1 = M_2 = \ldots = M_n = M$ we have

**Corollary 8.** Let $f_i(z) \in A$ be in the class $B(\mu, \alpha), \mu \geq 0, \ 0 \leq \alpha < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M$ $(M \geq 1, \ z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator
$$H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^{n} \left( te^{K(t)} \right)^{\frac{1}{\gamma}} dt$$
is in $K(\delta)$, where
$$\delta = 1 - \frac{n}{|\gamma|} \left[ (2 - \alpha)M^\mu + 1 \right] \quad (26)$$
and
$$\frac{1}{|\gamma|} < \frac{1}{n[(2 - \alpha)M^\mu + 1]^\gamma}, \gamma \in \mathbb{C} \setminus \{0\}.$$ 

Letting $\mu = 0$ in Corollary 8, we have

**Corollary 9.** Let $f_i(z) \in A$ be in the class $R(\alpha), \ 0 \leq \alpha < 1$ for all $i = 1, 2, \ldots, n$. Then the integral operator defined in (7) is in $K(\delta)$, where
$$\delta = 1 - \frac{n}{|\gamma|} (3 - \alpha) \quad (27)$$
and
$$\frac{1}{|\gamma|} < \frac{1}{n(3 - \alpha)}, \gamma \in \mathbb{C} \setminus \{0\}.$$ 

Letting $\mu = 1$ in Corollary 8, we have
Corollary 10. Let \( f_i \in A \) be in the class \( S^*(\alpha) \), \( 0 \leq \alpha < 1 \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M \) (\( M \geq 1, z \in U \)) for all \( i = 1, 2, \ldots, n \), then the integral operator defined in (7) is in \( K(\delta) \), where
\[
\delta = 1 - \frac{n}{\gamma} [1 + (2 - \alpha)M] \quad (28)
\]
and \( \frac{1}{|\gamma|} < \frac{1}{n[1 + (2 - \alpha)M]} \), \( \gamma \in \mathbb{C} \setminus \{0\} \).

Letting \( \alpha = \delta = 0 \) in Corollary 10, we have

Corollary 11. Let \( f_i(z) \in A \) be starlike functions in \( U \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M \) (\( M \geq 1, z \in U \)) for all \( i = 1, 2, \ldots, n \), then the integral operator defined in (7) is convex in \( U \), where
\[
\frac{1}{|\gamma|} = \frac{1}{n(2M + 1)}, \quad \gamma \in \mathbb{C} \setminus \{0\}.
\]

Letting \( n = 1 \) in Corollary 8, we obtain Theorem 2.

Theorem 5. Let \( f_i(z) \in A \) be in the class \( B(\mu, \alpha) \), \( \mu \geq 0 \), \( 0 \leq \alpha < 1 \) for all \( i = 1, 2, \ldots, n \). If \( |f_i(z)| \leq M_i \) (\( M_i \geq 1, z \in U \)) for all \( i = 1, 2, \ldots, n \), then the integral operator
\[
H_n(z) = \int_0^z \prod_{i=1}^n \left(t \gamma f_i(t)\right)^{\gamma_i} \, dt
\]
is in \( K(\delta) \), where
\[
\delta = 1 - \sum_{i=1}^n \frac{|\gamma_i|}{\gamma} [1 + (2 - \alpha)M_i]\mu \quad (29)
\]
and \( \sum_{i=1}^n |\gamma_i| [1 + (2 - \alpha)M_i]\mu < 1 \), \( \gamma_i \in \mathbb{C} \) for all \( i = 1, 2, \ldots, n \).

Proof. Let \( f_i \in A \) be in the class \( B(\mu, \alpha) \), \( \mu \geq 0 \), \( 0 \leq \alpha < 1 \). It follows from (8) that
\[
H_n(z) = \int_0^z \sum_{i=1}^n \gamma_i \prod_{i=1}^n \gamma_i f_i(t) \, dt \quad \text{and} \quad H_n'(z) = z \sum_{i=1}^n \gamma_i \prod_{i=1}^n \gamma_i f_i'(z).
\]

Also
\[
H_n''(z) = z^{\sum_{i=1}^n \gamma_i - 1} \prod_{i=1}^n \gamma_i f_i(z) \left[ \sum_{i=1}^n \gamma_i + z \sum_{i=1}^n \gamma_i f_i'(z) \right]
\]

Then
\[
\frac{H_n''(z)}{H_n'(z)} = \frac{\sum_{i=1}^n \gamma_i + z \sum_{i=1}^n \gamma_i f_i'(z)}{z}
\]
and, hence

\[
\left| \frac{zH_n''(z)}{H_n'(z)} \right| = \left| \sum_{i=1}^{n} \gamma_i + z \sum_{i=1}^{n} \gamma_i f'_i(z) \right| \leq \sum_{i=1}^{n} |\gamma_i| + |z| \sum_{i=1}^{n} |\gamma_i| \cdot |f'_i(z)| \\
\leq \sum_{i=1}^{n} |\gamma_i| + |z| \sum_{i=1}^{n} |\gamma_i| \cdot f'_i(z) \left( \frac{z}{f_i(z)} \right)^{\mu} \cdot \left( \frac{f_i(z)}{z} \right)^{\mu} 
\]

(30)

Applying the General Schwarz lemma, we have

\[
\left| \frac{f_i(z)}{z} \right| \leq M_i, \text{ for all } i = 1, 2, \ldots, n.
\]

Therefore, from (30), we obtain

\[
\left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \sum_{i=1}^{n} |\gamma_i| + |z| \sum_{i=1}^{n} |\gamma_i| \cdot f'_i(z) \left( \frac{z}{f_i(z)} \right)^{\mu} \cdot M_i^{\mu}, \text{ (z } \in U). 
\]

(31)

From (4) and (31), we see that

\[
\left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \sum_{i=1}^{n} |\gamma_i| \cdot \left[ 1 + (2 - \alpha)M_i^{\mu} \right] = 1 - \delta.
\]

This completes the proof. \(\square\)

For \(M_1 = M_2 = \ldots = M_n = M\) we have

**Corollary 12.** Let \(f_i(z) \in A\) be in the class \(B(\mu, \alpha), \mu \geq 0, 0 \leq \alpha < 1\) for all \(i = 1, 2, \ldots, n\). If \(|f_i(z)| \leq M \ (M \geq 1, \ z \in U)\) for all \(i = 1, 2, \ldots, n\), then the integral operator

\[
H_n(z) = \int_0^z \prod_{i=1}^{n} \left( te^{f_i(t)} \right)^{\gamma_i} dt
\]

is in \(K(\delta)\), where

\[
\delta = 1 - \sum_{i=1}^{n} |\gamma_i| \cdot [(2 - \alpha)M^{\mu} + 1] 
\]

(32)

and \(\sum_{i=1}^{n} |\gamma_i| < \frac{1}{(2 - \alpha)M^{\mu} + 1}, \gamma_i \in \mathbb{C}\) for all \(i = 1, 2, \ldots n\).

Letting \(\mu = 0\) in Corollary 12, we have

**Corollary 13.** Let \(f_i(z) \in A\) be in the class \(R(\alpha), 0 \leq \alpha < 1\) for all \(i = 1, 2, \ldots, n\). Then the integral operator defined in (8) is in \(K(\delta)\), where

\[
\delta = 1 - \sum_{i=1}^{n} |\gamma_i| (3 - \alpha)
\]

(33)

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\[ \sum_{i=1}^{n} |\gamma_i| < \frac{1}{3 - \alpha}, \; \gamma_i \in \mathbb{C} \text{ for all } i = 1, 2, ..., n. \]

Letting \( \mu = 1 \) in Corollary 12, we have

**Corollary 14.** Let \( f_i \in \mathcal{A} \) be in the class \( S^*(\alpha) \), \( 0 \leq \alpha < 1 \) for all \( i = 1, 2, ..., n \). If \( |f_i(z)| \leq M \) (\( M \geq 1, \; z \in U \)) for all \( i = 1, 2, ..., n \), then the integral operator defined in (8) is in \( K(\delta) \), where

\[ \delta = 1 - \sum_{i=1}^{n} |\gamma_i|[1 + (2 - \alpha)M] \quad (34) \]

and \( \sum_{i=1}^{n} |\gamma_i| < \frac{1}{1 + (2 - \alpha)M}, \; \gamma_i \in \mathbb{C} \text{ for all } i = 1, 2, ..., n. \)

Letting \( \alpha = \delta = 0 \) in Corollary 14, we have

**Corollary 15.** Let \( f_i \in \mathcal{A} \) be starlike functions in \( U \) for all \( i = 1, 2, ..., n \). If \( |f(z)| \leq M \) (\( M \geq 1, \; z \in U \)) for all \( i = 1, 2, ..., n \) then the integral operator defined in (8) is convex in \( U \), where \( \sum_{i=1}^{n} |\gamma_i| = \frac{1}{2M + 1}, \; \gamma_i \in \mathbb{C} \text{ for all } i = 1, 2, ..., n. \)

Letting \( n = 1 \) in Corollary 12, we obtain Theorem 1.

**References**


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