SUBORDINATION RESULTS FOR FUNCTIONS OF COMPLEX ORDER DEFINED BY CONVOLUTION

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Abstract. In this paper, we drive several interesting subordination results for functions of complex order defined by convolution.

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1. Introduction

Let \( A \) denote the class of functions of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \phi \in A \) be given by

\[
\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k.
\]

Definition 1 (Hadamard product or convolution). Given two functions \( f \) and \( \phi \) in the class \( A \), where \( f(z) \) is given by (1.1) and \( \phi(z) \) is given by (1.2) the Hadamard product (or convolution) \( f * \phi \) of \( f \) and \( \phi \) is defined (as usual) by

\[
(f * \phi)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (\phi * f)(z).
\]

We also denote by \( K \) the class of functions \( f(z) \in A \) that are convex in \( U \).

A function \( f(z) \in A \) is said to be in the class of starlike functions of complex order \( b \), denoted by \( S(b) \) if

\[
\text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}; \ z \in U). \tag{1.4}
\]

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A function \( f(z) \in A \) is said to be in the class of convex functions of complex order \( b \), denoted by \( C(b) \) if
\[
\Re \left\{ 1 + \frac{1}{b} z f''(z) \right\} > 0 \quad (b \in \mathbb{C}^*; \ z \in \mathbb{U} ).
\] (1.5)

The class \( S(b) \) was introduced and studied by Nasr and Aouf [12] and the class \( C(b) \) was introduced and studied by Nasr and Aouf [11] and Waitrowski [16].

A function \( f(z) \in A \) is said to be in \( S(\eta) = S((1 - \eta) \cos \eta e^{-i\eta}) \), the class of \( \eta \)-spirallike functions of order \( \gamma \) if
\[
\Re \left\{ e^{i\eta} \frac{zf'(z)}{f(z)} \right\} > \gamma \cos \eta \quad (|\eta| < \frac{\pi}{2}; 0 \leq \gamma < 1).
\] (1.6)

A function \( f(z) \in A \) is said to be in \( C(\eta) = C((1 - \eta) \cos \eta e^{-i\eta}) \), the class of \( \eta \)-Robertson functions of order \( \gamma \) if
\[
\Re \left\{ e^{i\eta} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \gamma \cos \eta \quad (|\eta| < \frac{\pi}{2}; 0 \leq \gamma < 1).
\] (1.7)

It follows from (1.6) and (1.7) that
\[
f(z) \in C(\eta) \iff zf'(z) \in S(\eta).
\]

The class \( S(\eta) \) was introduced and studied by Libera [8] and the class \( C(\eta) \) was introduced and studied by Chichra [4].

For \( 0 \leq \lambda \leq 1, b \in \mathbb{C}^* \), we denote by \( M(f, g, b, \lambda) \) the subclass of \( A \) consisting of functions \( f(z) \) of the form (1.1), functions \( g(z) \) given by
\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k,
\] (1.8)
and satisfying the analytic criterion:
\[
\Re \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{(1 - \lambda)(f * g)(z) + \lambda z(f * g)'(z)} - 1 \right) \right\} > 0.
\] (1.9)

We note that for suitable choices of \( g, b \) and \( \lambda \), we obtain the following subclasses studied by various authors.

(i) \( M(f, \frac{z}{1 - z}, 1 - \alpha, 0) = S^*(\alpha) \) \((0 \leq \alpha \leq 1)\) (see Robertson [13]);
(ii) \( M(f, \frac{z}{(1 - z)^2}, 1 - \alpha, 0) = C(\alpha) \) \((0 \leq \alpha \leq 1)\) (see Robertson [13]);
(iii) \( M(f, \frac{z}{(1 - z)^2}, b, 0) = S(b) \) \((b \in \mathbb{C}^*)\) (see Nasr and Aouf [12]);
(iv) \( M(f, \frac{z}{1-z^2}, b, 0) = C(b) \) (\( b \in \mathbb{C}^* \)) (see Waitrowski [16], Nasr and Aouf [11]);

(v) \( M(f, \frac{z}{1-z}, (1-\gamma) \cos \eta e^{-i\eta}, 0) = S^\eta(\gamma) \) (\( |\eta| < \frac{\pi}{2}, 0 \leq \gamma < 1 \)) (see Libera [8]);

(vi) \( M(f, \frac{z}{1-z^2}, (1-\gamma) \cos \eta e^{-i\eta}, 0) = C^\eta(\gamma) \) (\( |\eta| < \frac{\pi}{2}, 0 \leq \gamma < 1 \)) (see Chichra [4]);

(vii) \( M(f, z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1)z^k, (1-\gamma) \cos \eta e^{-i\eta}, \lambda) = R^\eta_s(\eta, \gamma, \lambda) \) (\( |\eta| < \frac{\pi}{2}, 0 \leq \lambda \leq 1, 0 \leq \gamma < 1 \)) (see Murugusundaramoorthy and Magesh [10]), where

\[
\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \cdots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \cdots (\beta_s)_{k-1}(1)_{k-1}},
\]

(1.10)

for \( \alpha_i > 0, i = 1, \ldots, q; \beta_j > 0, j = 1, \ldots, s; q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), where \( \mathbb{N} = \{1, 2, \ldots\} \).

Also we note that:

(i) \( M(f, g, b, 0) = M(f, g, b) \)

\[
= \left\{ f \in A : Re \left[ 1 + \frac{1}{b} \frac{z(f * g)'(z)}{(f * g)(z)} - 1 \right] > 0, b \in \mathbb{C}^* \right\};
\]

(ii) \( M(f, z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1)z^k, b, \lambda) = M_{q,s}(\alpha_1, b, \lambda) \)

\[
= \left\{ f \in A : Re \left[ 1 + \frac{1}{b} \frac{z (H_{q,s}(\alpha_1, \beta_1)f(z))'}{(1-\lambda)H_{q,s}(\alpha_1, \beta_1)f(z) + \lambda z (H_{q,s}(\alpha_1, \beta_1)f(z))'} - 1 \right] > 0 \right\},
\]

(0 \leq \lambda \leq 1, b \in \mathbb{C}^*, z \in U \) and \( \Gamma_k(\alpha_1) \) is defined by (1.10),

and the operator \( H_{q,s}(\alpha_1, \beta_1) \) was introduced and studied by Dziok and Srivastava (see [5] and [6]), which is a generalization of many other linear operators considered earlier;

(iii) \( M(f, z + \sum_{k=2}^{\infty} \left[ \frac{\ell+1+\mu(k-1)}{\ell+1} \right]^m z^k, b, \lambda) = M(m, \mu, \ell, b, \lambda) \)

\[
= \left\{ f \in A : Re \left[ 1 + \frac{1}{b} \frac{z (I^m(\mu, \ell)f(z))'}{(1-\lambda)I^m(\mu, \ell)f(z) + \lambda z (I^m(\mu, \ell)f(z))'} - 1 \right] > 0 \right\},
\]

where 0 \leq \lambda \leq 1, b \in \mathbb{C}^*, m \in \mathbb{N}_0, \mu, \ell \geq 0, z \in U \) and the operator \( I^m(\mu, \ell) \) was defined by Càtás et al. [3], which is a generalization of many other linear operators considered earlier;
\( (iv) \) \( M(f, g, (1 - \gamma) \cos \eta e^{-i\eta}, \lambda) = M(f, g, \lambda, \eta) \)

\[ = \left\{ f \in A : \text{Re} \left[ e^{i\eta} \frac{z (f \ast g)'(z)}{(1 - \lambda) (f \ast g)(z) + \lambda z (f \ast g)'(z)} \right] > \gamma \cos \eta \right\}, \]

where \( |\eta| < \frac{\pi}{2}, 0 \leq \lambda \leq 1, 0 \leq \gamma < 1; \)

\( (v) \) \( M(f, z + \sum_{k=2}^{\infty} \left[ \frac{\mu + (k-1)}{\ell + 1} \right] m z^k, (1 - \gamma) \cos \eta e^{-i\eta}, \lambda) = M(m, \mu, \ell, \lambda, \gamma, \eta) \)

\[ = \left\{ f \in A : \text{Re} \left[ e^{i\eta} \frac{z (I_m (\mu, \ell) f(z))'}{(1 - \lambda) I_m (\mu, \ell) f(z) + \lambda z (I_m (\mu, \ell) f(z))'} \right] > \gamma \cos \eta \right\}. \]

where \( |\eta| < \frac{\pi}{2}, 0 \leq \lambda \leq 1, 0 \leq \gamma < 1. \)

**Definition 2 (Subordination principle).** For two functions \( f \) and \( \phi \), analytic in \( U \), we say that the function \( f(z) \) is subordinate to \( \phi(z) \) in \( U \), written \( f(z) \prec \phi(z) \), if there exists a Schwarz function \( w(z) \), which (by definition) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f(z) = \phi(w(z)) \). Indeed it is known that

\[ f(z) \prec \phi(z) \Rightarrow f(0) = \phi(0) \text{ and } f(U) \subset \phi(U). \]

Furthermore, if the function \( \phi \) is univalent in \( U \), then we have the following equivalence (see [2] and [9]):

\[ f(z) \prec \phi(z) \iff f(0) = \phi(0) \text{ and } f(U) \subset \phi(U). \quad (1.11) \]

**Definition 3 (Subordinating factor sequence) [17].** A sequence \( \{c_k\}_{k=1}^{\infty} \) of complex numbers is said to be a subordinating factor sequence if, whenever \( f \) of the form (1.1) is analytic, univalent and convex in \( U \), we have

\[ \sum_{k=2}^{\infty} a_k c_k z^k < f(z) \quad (a_1 = 1; z \in U). \quad (1.12) \]

2. **Main Result**

Unless otherwise mentioned, we assume throughout this section that \( |\eta| < \frac{\pi}{2}, 0 \leq \lambda \leq 1, 0 \leq \gamma < 1, b \in \mathbb{C}^*, z \in U \) and \( g(z) \) given by (1.8).

To prove our main result we need the following lemmas.
**Lemma 1** [17]. The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\text{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} c_k z^k \right\} > 0. \quad (2.1)$$

Now, we prove the following Lemma which gives a sufficient condition for functions belonging to the class $M(f, g, b, \lambda)$:

**Lemma 2.** A function $f(z)$ of the form (1.1) is said to be in the class $M(f, g, b, \lambda)$ if

$$\sum_{k=2}^{\infty} \left\{ (1 - \lambda) (k - 1) + |b| |1 + \lambda (k - 1)| \right\} b_k |a_k| \leq |b|, \quad (2.2)$$

where $b_{k+1} \geq b_k > 0 \ (k \geq 2)$.

**Proof.** Assume that, the inequality (2.2) holds true. Then it suffices to show that

$$\left| \frac{z (f \ast g)'(z)}{(1 - \lambda)(f \ast g)(z) + \lambda z (f \ast g)'(z)} - 1 \right| \leq |b|.$$

We have

$$\left| \frac{z (f \ast g)'(z)}{(1 - \lambda)(f \ast g)(z) + \lambda z (f \ast g)'(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} (1 - \lambda) (k - 1) b_k |a_k| |z^{k-1}|}{1 - \sum_{k=2}^{\infty} [1 + \lambda (k - 1)] b_k |a_k| |z^{k-1}|} \leq \frac{\sum_{k=2}^{\infty} (1 - \lambda) (k - 1) b_k |a_k|}{1 - \sum_{k=2}^{\infty} [1 + \lambda (k - 1)] b_k |a_k|} \leq |b|.$$

This completes the proof of Lemma 2

Let $M^*(f, g, b, \lambda)$ denote the class of $f(z) \in A$ whose coefficients satisfy the condition (2.2). We note that $M^*(f, g, b, \lambda) \subseteq M(f, g, b, \lambda)$.

Employing the technique used earlier by Attiya [1] and Srivastava and Attiya [15], we prove:
Thereom 1. Let $f(z) \in M^*(f,g,b,\lambda)$. Then
\[
\frac{[1 - \lambda + |b|(1 + \lambda)]_b}{2 \{|b| + [1 - \lambda + |b|(1 + \lambda)]_b\}} (f * h)(z) \prec h(z) \quad (2.3)
\]
for every function $h \in K$, and
\[
Re \{f(z)\} > -\frac{|b| + [1 - \lambda + |b|(1 + \lambda)]_b}{[1 - \lambda + |b|(1 + \lambda)]_b}. \quad (2.4)
\]
The constant factor \(\frac{[1 - \lambda + |b|(1 + \lambda)]_b}{2(|b| + [1 - \lambda + |b|(1 + \lambda)]_b)}\) in the subordination result (2.3) can not be replaced by a larger one.

Proof. Let $f(z) \in M^*(f,g,b,\lambda)$ and suppose that $h(z) = z + \sum_{k=2}^{\infty} c_k z^k$, then
\[
\frac{[1 - \lambda + |b|(1 + \lambda)]_b}{2 \{|b| + [1 - \lambda + |b|(1 + \lambda)]_b\}} (f * h)(z) = \frac{[1 - \lambda + |b|(1 + \lambda)]_b}{2 \{|b| + [1 - \lambda + |b|(1 + \lambda)]_b\}} \left( z + \sum_{k=2}^{\infty} c_k a_k z^k \right). \quad (2.5)
\]
Thus, by using Definition 3, the subordination result holds true if
\[
\left\{ \frac{[1 - \lambda + |b|(1 + \lambda)]_b}{2 \{|b| + [1 - \lambda + |b|(1 + \lambda)]_b\}} a_k \right\}_{k=1}^{\infty}
\]
is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:
\[
Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{[1 - \lambda + |b|(1 + \lambda)]_b}{\{|b| + [1 - \lambda + |b|(1 + \lambda)]_b\}} a_k z^k \right\} > 0. \quad (2.6)
\]
Now, since
\[
\Psi(k) = ((1 - \lambda) (k - 1) + |b| [1 + \lambda (k - 1)] b_k
\]
is an increasing function of $k (k \geq 2)$, we have
\[
Re \left\{ 1 + \frac{[1 - \lambda + |b|(1 + \lambda)]_b}{\{|b| + [1 - \lambda + |b|(1 + \lambda)]_b\}} \sum_{k=1}^{\infty} a_k z^k \right\}
\]
\[
\begin{align*}
&= Re \left\{ 1 + \frac{[1 - \lambda + |b|(1 + \lambda)] b_2}{\{|b| + [1 - \lambda + |b|(1 + \lambda)] b_2\}} z + \sum_{k=2}^{\infty} \frac{[1 - \lambda + |b|(1 + \lambda)] b_2 a_k z^k}{\{|b| + [1 - \lambda + |b|(1 + \lambda)] b_2\}} \right\} \\
&\geq 1 - \frac{[1 - \lambda + |b|(1 + \lambda)] b_2}{\{|b| + [1 - \lambda + |b|(1 + \lambda)] b_2\}} r \\
&\quad - \frac{1}{\{|b| + [1 - \lambda + |b|(1 + \lambda)] b_2\}} \sum_{k=2}^{\infty} \{(1 - \lambda)(k - 1) + |b| [1 + \lambda (k - 1)]\} b_k |a_k| r^k \\
&\geq 1 - \frac{[1 - \lambda + |b|(1 + \lambda)] b_2}{\{|b| + [1 - \lambda + |b|(1 + \lambda)] b_2\}} r - \frac{|b|}{\{|b| + [1 - \lambda + |b|(1 + \lambda)] b_2\}} > 0 (|z| = r < 1),
\end{align*}
\]

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.6) holds true in \( U \). This proves the inequality (2.3). The inequality (2.4) follows from (2.4) by taking the convex function

\[ h(z) = \frac{z}{1 - z} = z + \sum_{k=2}^{\infty} z^k \in K. \]  

(2.7)

To prove the sharpness of the constant

\[
\frac{[1 - \lambda + |b|(1 + \lambda)] b_2}{2 \{|b| + [1 - \lambda + |b|(1 + \lambda)] b_2\}},
\]

we consider the function \( f_0(z) \in M^*(f, g, b, \lambda) \) given by

\[ f_0(z) = z - \frac{|b|}{[1 - \lambda + |b|(1 + \lambda)] b_2} z^2. \]
Thus from (2.4), we have

\[
\frac{[1 - \lambda + |b| (1 + \lambda)] b_2}{2 \{|b| + [1 - \lambda + |b| (1 + \lambda)] b_2\}} f_0(z) \prec \frac{z}{1 - z}.
\]

It is easily verified that

\[
\min_{|z| \leq r} \left\{ \text{Re} \left( \frac{[1 - \lambda + |b| (1 + \lambda)] b_2}{2 \{|b| + [1 - \lambda + |b| (1 + \lambda)] b_2\}} f_0(z) \right) \right\} = -\frac{1}{2}.
\]

This shows that the constant

\[
\frac{[1 - \lambda + |b| (1 + \lambda)] b_2}{2 \{|b| + [1 - \lambda + |b| (1 + \lambda)] b_2\}}
\]

is the best possible. This completes the proof of Theorem 1.

Remark 1.

(i) Taking \(g(z) = \frac{z - z^1}{1 - z}, b = 1 - \alpha (0 \leq \alpha \leq 1)\) and \(\lambda = 0\) in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.3];

(ii) Taking \(g(z) = \frac{z - z^1}{1 - z}, b = 1\) and \(\lambda = 0\) in Theorem 1, we obtain the result obtained by Singh [14, Corollary 2.2];

(iii) Taking \(g(z) = \frac{z}{(1 - z)^2}, b = 1 - \alpha (0 \leq \alpha \leq 1)\) and \(\lambda = 0\) in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.6];

(iv) Taking \(g(z) = \frac{z}{(1 - z)^2}, b = 1\) and \(\lambda = 0\) in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.7];

(v) Taking \(g(z) = \frac{z}{1 - z}, b = \cos \eta e^{-i\eta} (|\eta| < \frac{\pi}{2})\) and \(\lambda = 0\) in Theorem 1, we obtain the result obtained by Singh [14];

(vi) Taking \(g(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k\), where \(\Gamma_k(\alpha_1)\) given by (1.10) and \(b = (1 - \gamma) \cos \eta e^{-i\eta} (|\eta| < \frac{\pi}{2}, 0 \leq \gamma < 1)\) in Theorem 1, we obtain the result obtained by Murugusundaramoorthy and Magesh [10].

Also, we establish subordination results for the associated subclasses, \(M^* (f, g, b), M^*_{m, \ell, a} (\alpha_1, b, \lambda), M^* (m, \mu, \ell, b, \lambda), M^* (f, g, \lambda, \gamma, \eta)\) and \(M^* (m, \mu, \ell, \lambda, \gamma, \eta)\), whose coefficients satisfy the condition (2.2) in the special cases as mentioned in the introduction.

By taking \(\lambda = 0\) in Lemma 2 and Theorem 1, we have:

**Corollary 1.** Let the function \(f(z)\) defined by (1.1) be in the class \(M^* (f, g, b)\) and satisfy the condition

\[
\sum_{k=2}^{\infty} (k - 1 + |b|) b_k |a_k| \leq |b|.
\]

Then for every function \(h \in K\), we have
\[
\frac{(1 + |b|) b_2}{2 [b + (1 + |b|) b_2]} (f * h) (z) < h(z), \quad (2.10)
\]

and

\[
\text{Re} \{f(z)\} > -\left\{ [b + (1 + |b|) b_2] \right\} \left(1 + |b| b_2 \right). \quad (2.11)
\]

The constant factor \( \frac{(1+|b|)b_2}{2[|b|+(1+|b|)b_2]} \) in (2.10) can not be replaced by a larger one.

By taking \( b_k = \Gamma_k(\alpha_1) \), where \( \Gamma_k(\alpha_1) \) defined by (1.10), in Lemma 2 and Theorem 1, we have:

**Corollary 2.** Let the function \( f(z) \) defined by (1.1) be in the class \( M^*_{q,s} (\alpha_1, b, \lambda) \) and satisfy the condition

\[
\sum_{k=2}^{\infty} \left\{ (1 - \lambda) (k - 1) + |b| [1 + \lambda (k - 1)] \right\} \Gamma_k(\alpha_1) |a_k| \leq |b|. \quad (2.12)
\]

Then for every function \( h \in K \), we have

\[
\frac{[1 - \lambda + |b| (1 + \lambda)] \Gamma_2(\alpha_1)}{2 \{ |b| + [1 - \lambda + |b| (1 + \lambda)] \Gamma_2(\alpha_1) \} (f * h) (z) < h(z), \quad (2.13)
\]

and

\[
\text{Re} \{f(z)\} > -\left\{ |b| + [1 - \lambda + |b| (1 + \lambda)] \Gamma_2(\alpha_1) \right\} \left[1 - \lambda + |b| (1 + \lambda) \right] \Gamma_2(\alpha_1). \quad (2.14)
\]

The constant factor \( \frac{[1-\lambda+|b|(1+\lambda)]\Gamma_2(\alpha_1)}{2\{|b|+[1-\lambda+|b|(1+\lambda)]\Gamma_2(\alpha_1)\}} \) in (2.13) can not be replaced by a larger one.

By taking \( b_k = \left( \frac{\ell+1+\mu(k-1)}{\ell+1} \right)^{m} \) \( (m \in \mathbb{N}, \mu, \ell \geq 0) \) in Lemma 2 and Theorem 1, we have:

**Corollary 3.** Let the function \( f(z) \) defined by (1.1) be in the class \( M^*(m, \mu, \ell, b, \lambda) \) and satisfy the condition

\[
\sum_{k=2}^{\infty} \left\{ (1 - \lambda) (k - 1) + |b| [1 + \lambda (k - 1)] \right\} \left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m |a_k| \leq |b|. \quad (2.15)
\]

Then for every function \( h \in K \), we have
\[
\frac{[1 - \lambda + |b| (1 + \lambda)] \ell + 1 + \mu}{2 \{ (1 + \lambda) |b| + [1 - \lambda + |b| (1 + \lambda)] \ell + 1 + \mu \}} \quad (f \ast h)(z) < h(z), \quad (2.16)
\]

and

\[
\Re \{ f(z) \} > \frac{\{ (1 + \lambda) |b| + [1 - \lambda + |b| (1 + \lambda)] \ell + 1 + \mu \}}{[1 - \lambda + |b| (1 + \lambda)] \ell + 1 + \mu \}}. \quad (2.17)
\]

The constant factor \( \frac{[1 - \lambda + |b| (1 + \lambda)] \ell + 1 + \mu}{2 \{ (1 + \lambda) |b| + [1 - \lambda + |b| (1 + \lambda)] \ell + 1 + \mu \}} \) in (2.16) can not be replaced by a larger one.

By taking \( b = (1 - \gamma) \cos \eta e^{-i\eta} \) (\( |\eta| < \frac{\pi}{2}, 0 \leq \gamma < 1 \)) in Lemma 2 and Theorem 1, we have:

**Corollary 4.** Let the function \( f(z) \) defined by (1.1) be in the class \( M^*(f, g, \lambda, \gamma, \eta) \) and satisfy the condition

\[
\sum_{k=2}^{\infty} \left\{ (1 - \lambda)(k - 1) \sec \eta + (1 - \gamma)[1 + \lambda(k - 1)] \right\} b_k |a_k| \leq 1 - \gamma. \quad (2.18)
\]

Then for every function \( h \in K \), we have

\[
\frac{[1 - \lambda \sec \eta + (1 - \gamma)(1 + \lambda)] b_2}{2 \{ 1 - \gamma + [(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] b_2 \}} (f \ast h)(z) < h(z), \quad (2.19)
\]

and

\[
\Re \{ f(z) \} > \frac{1 - \gamma + [(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] b_2}{[(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] b_2}. \quad (2.20)
\]

The constant factor \( \frac{[1 - \lambda \sec \eta + (1 - \gamma)(1 + \lambda)] b_2}{2 \{ 1 - \gamma + [(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] b_2 \}} \) in (2.19) can not be replaced by a larger one.

By taking \( b_k = \left( \frac{\ell + 1 + \mu(k - 1)}{\ell + 1} \right)^m \) (\( m \in \mathbb{N}_0, \mu, \ell \geq 0 \)) and \( b = (1 - \gamma) \cos \eta e^{-i\eta} \) (\( |\eta| < \frac{\pi}{2}, 0 \leq \gamma < 1 \)) in Lemma 2 and Theorem 1, we have:

**Corollary 5.** Let the function \( f(z) \) defined by (1.1) be in the class \( M^*(m, \mu, \ell, \lambda, \gamma, \eta) \) and satisfy the condition

\[
\sum_{k=2}^{\infty} \left\{ (1 - \lambda)(k - 1) \sec \eta + (1 - \gamma)[1 + \lambda(k - 1)] \right\} \left[ \frac{\ell + 1 + \mu(k - 1)}{\ell + 1} \right]^m |a_k| \leq 1 - \gamma \quad (2.21)
\]
Then for every function \( h \in K \), we have
\[
\frac{[(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] [\ell + 1 + \mu]^m}{2 \{(1 - \gamma)(\ell + 1)^m + [(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] [\ell + 1 + \mu]^m\}} \quad (f \ast h)(z) < h(z)
\]
(2.22)

and
\[
\text{Re}\{f(z)\} > \frac{[(1 - \gamma)(\ell + 1)^m + [(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] [\ell + 1 + \mu]^m]}{[(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] [\ell + 1 + \mu]^m}\,
\]
(2.23)

The constant factor
\[
\frac{[(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] [\ell + 1 + \mu]^m}{2 \{(1 - \gamma)(\ell + 1)^m + [(1 - \lambda) \sec \eta + (1 - \gamma)(1 + \lambda)] [\ell + 1 + \mu]^m\}}
\]
in (2.22) can not be replaced by a larger one.

References


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