ON GROWTH PROPERTIES OF ITERATED ENTIRE FUNCTIONS

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ABSTRACT. In this paper we study growth properties of iterated entire functions which improves some earlier results.

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1. Introduction, definitions and notations

Let $f$ and $g$ be non constant entire functions defined in the open complex plane $\mathbb{C}$ and $M(r,f) = \max\{|f(z)| : |z| = r\}$. The order and lower order of $f$ are defined in the following way.

**Definition 1.** The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ are defined as follows:

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^2 M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 M(r,f)}{\log r}
\]

where we use the following notation (cf. [7]):

\[
\log^k x = \log(\log^{k-1} x) \quad \text{for} \quad k = 1, 2, 3 \ldots \quad \text{and} \quad \log^0 x = x.
\]

**Definition 2.** The type $\sigma_f$ of an entire function $f$ is defined as:

\[
\sigma_f = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
\]

In line of Lahiri and Banerjee [5] we define the iteration of $f(z)$ with respect to $g(z)$ as follows:

\[
\begin{align*}
    f_1(z) &= f(z) \\
    f_2(z) &= f(g(z)) = f(g_1(z)) \\
    f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))
\end{align*}
\]
\[ f_n(z) = \cdots (f(g(z)) or g(f(z)) \cdots) \]

according as \( n \) is odd or even,

and so

\[
\begin{align*}
g_1(z) &= g(z) \\
g_2(z) &= g(f(z)) = g(f_1(z)) \\
g_3(z) &= g(f_2(z)) = g(f(g(z))) \\
& \quad \vdots \\
g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{align*}
\]

Clearly all \( f_n(z) \) and \( g_n(z) \) are entire functions.

It is well known that

\[
\lim_{r \to \infty} \frac{M(r, fog)}{M(r, f)} = \lim_{r \to \infty} \frac{M(r, fog)}{M(r, g)} = \infty.
\]

Clunie [1] discussed on the behaviour of

\[
\frac{\log M(r, fog)}{\log M(r, f)} \quad \text{and} \quad \frac{\log M(r, fog)}{\log M(r, g)} \quad \text{as} \quad r \to \infty.
\]

Song and Yang [10] worked on

\[
\frac{\log[2] M(r, fog)}{\log[2] M(r, f)} \quad \text{and} \quad \frac{\log[2] M(r, fog)}{\log[2] M(r, g)} \quad \text{as} \quad r \to \infty.
\]

Replacing maximum modulus functions by Nevanlinna’s characteristic functions

Clunie [1] proved for any two transcendental entire functions defined in the open complex plane \( \mathbb{C} \),

\[
\lim_{r \to \infty} \frac{T(r, fog)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r \to \infty} \frac{T(r, fog)}{T(r, g)} = \infty.
\]

Singh [8] proved some comparative growth properties of \( \log T(r, fog) \) and \( T(r, f) \).

Singh [8] also raised the problem of investigating the comparative growth of \( \log T(r, fog) \) and \( T(r, g) \) and some results on the comparative growth of \( \log T(r, fog) \) and \( T(r, g) \) are proved in Lahiri [3].
Since $M(r, f)$ and $M(r, g)$ are increasing functions of $r$, Singh and Baloria \cite{9} asked whether for any two entire functions $f, g$ and for sufficiently large $R = R(r)$,
\[
\limsup_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, f)} < \infty \quad \text{and} \quad \limsup_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, g)} < \infty.
\]
Singh and Baloria \cite{9}, Lahiri and Sharma \cite{4}, Liao and Yang \cite{6} worked on this question.

In this paper we study growth properties of iterated entire functions which improves some results of Liao and Yang \cite{6}.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1.** Let $f(z)$ and $g(z)$ be two entire functions with non zero finite orders $\rho_f$ and $\rho_g$ respectively. Then for any $\varepsilon > 0$ and for all sufficiently large values of $r$
\[
\log^{[n]} M(r, f_n) \leq \begin{cases} 
(\rho_f + \varepsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even} \\
(\rho_g + \varepsilon) \log M(r, f) + O(1) & \text{when } n \text{ is odd}.
\end{cases}
\]

Lemma 1 follows from Lemma 2.4 of Dutta \cite{2} on putting $p = 1$.

**Lemma 2.** Let $f(z)$ and $g(z)$ be two entire functions with non zero finite lower orders $\lambda_f$ and $\lambda_g$ respectively. Then for any $0 < \varepsilon < \min\{\rho_f, \rho_g\}$ and for all sufficiently large values of $r$
\[
\log^{[n]} M(r, f_n) \geq \begin{cases} 
(\lambda_f - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even} \\
(\lambda_g - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd}.
\end{cases}
\]

Lemma 2 follows from Lemma 2.5 of Dutta \cite{2} on putting $p = 1$. 
3. Theorems

In this section we present the main results of our paper.

**Theorem 3.** Let \( f(z) \) and \( g(z) \) be two entire functions such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \lambda_g \leq \rho_g < \infty \). Then for each \( \alpha \in [0, \infty) \) and \( 0 < p < \min \{(1 + \alpha) \rho_g, (1 + \alpha) \rho_f\} \)

(i) \( \limsup_{r \to \infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(rp), f)} \geq \frac{(\lambda_f - \varepsilon)(1 + \alpha) \left( \frac{r}{2^n - 1} \right) (\rho_g - \varepsilon)(1 + \alpha) + O(1)}{(\rho_f + \varepsilon) \cdot \rho_g^{1 + \alpha}} \)

(ii) \( \limsup_{r \to \infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(rp), g)} = \infty. \)

**Proof.** (i) Case 1: When \( n \) is even.

From Lemma 2 we get for a sequence of values of \( r \) tending to infinity that

\[
\log^{[n]} M(r, f_n) \geq (\lambda_f - \varepsilon) \log M \left( \frac{r}{2^n - 1}, g \right) + O(1) > (\lambda_f - \varepsilon) \left( \frac{r}{2^n - 1} \right)^{\rho_g - \varepsilon} + O(1). \tag{1}
\]

Now from the definition of order we obtain for all sufficiently large values of \( r \) that

\[
\log^{[2]} M(\exp(rp), f) \leq (\rho_f + \varepsilon) \log \exp(rp) \]

i.e., \( \log^{[2]} M(\exp(rp), f) \leq (\rho_f + \varepsilon) \cdot r^p. \tag{2} \)

From (1) and (2) we obtain for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(rp), f)} \geq (\lambda_f - \varepsilon)^{(1 + \alpha)} \left( \frac{r}{2^n - 1} \right)^{(\rho_g - \varepsilon)(1 + \alpha)} + O(1) \]

As \( 0 < p < (1 + \alpha) \rho_g \) and \( \varepsilon > 0 \) is arbitrary, for each \( \alpha \in [0, \infty) \) we can choose \( \varepsilon > 0 \) such that \( p < (\rho_g - \varepsilon)(1 + \alpha) \) and therefore from (3) we obtain

\[
\limsup_{r \to \infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(rp), f)} = \infty.
\]

Case 2: When \( n \) is odd.

From Lemma 2 we get for a sequence of values of \( r \) tending to infinity that

\[
\log^{[n]} M(r, f_n) \geq (\lambda_g - \varepsilon) \log M \left( \frac{r}{2^n - 1}, f \right) + O(1)
\]
\[
> (\lambda_g - \varepsilon) \left( \frac{r}{2^{n-1}} \right)^{\rho_f - \varepsilon} + O(1). \tag{4}
\]

From (4) and (2) we obtain for a sequence of values of \( r \) tending to infinity that
\[
\left\{ \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^p), f)} \right\}^{1+\alpha} > \frac{(\lambda_g - \varepsilon)^{(1+\alpha)} \left( \frac{r}{2^{n-1}} \right)^{(\rho_f - \varepsilon)(1+\alpha)} + O(1)}{(\rho_f + \varepsilon) r^p}. \tag{5}
\]

As \( 0 < p < (1 + \alpha) \rho_f \) and \( \varepsilon > 0 \) is arbitrary, for each \( \alpha \in [0, \infty) \) we can choose \( \varepsilon > 0 \) such that \( p < (\rho_f - \varepsilon) (1 + \alpha) \) and therefore from (5) we obtain
\[
\limsup_{r \to \infty} \left\{ \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^p), f)} \right\}^{1+\alpha} = \infty.
\]

Hence in both the cases we get
\[
\limsup_{r \to \infty} \left\{ \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^p), f)} \right\}^{1+\alpha} = \infty.
\]

(ii) replacing \( f \) by \( g \) in (2) we get
\[
\limsup_{r \to \infty} \left\{ \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^p), g)} \right\}^{1+\alpha} = \infty.
\]

Remark 1. Theorem 3 improves Theorem 4 of Liao and Yang [6].

Remark 2. If we take \( 0 < p < \min \{ (1 + \alpha) \lambda_g, (1 + \alpha) \lambda_f \} \) instead of \( 0 < p < \min \{ (1 + \alpha) \rho_g, (1 + \alpha) \rho_f \} \) then Theorem 3 remains valid as we see in the following theorem.

Theorem 4. Let \( f(z) \) and \( g(z) \) be two entire functions such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \lambda_g \leq \rho_g < \infty \). Then for each \( \alpha \in [0, \infty) \) and \( 0 < p < \min \{ (1 + \alpha) \lambda_g, (1 + \alpha) \lambda_f \} \)
\[
(i) \lim_{r \to \infty} \left\{ \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^p), f)} \right\}^{1+\alpha} = \infty
\]
\[
(ii) \lim_{r \to \infty} \left\{ \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^p), g)} \right\}^{1+\alpha} = \infty.
\]
Proof of Theorem 4 is similar to Theorem 3 and so is omitted.

**Theorem 5.** Let \( f(z) \) and \( g(z) \) be two entire functions of finite order such that \( 0 < \lambda_f \). If \( \sigma_g < \infty \) then

\[
\begin{align*}
(i) \limsup_{r \to \infty} \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^{\rho_g}), f)} &< \infty \text{ if } n \text{ is even.} \\
(ii) \limsup_{r \to \infty} \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^{\rho_g}), f)} &= 0.
\end{align*}
\]

**Proof.** (i) As \( n \) is even from Lemma 1, we get for all sufficiently large values of \( r \) that

\[
\begin{align*}
\limsup_{r \to \infty} \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^{\rho_g}), f)} &\leq \limsup_{r \to \infty} \frac{(\rho_f + \epsilon) \log M(r, g) + O(1)}{\log^2 M(\exp(r^{\rho_g}), f)} \\
&\leq \limsup_{r \to \infty} \frac{\log M(r, g)}{r^{\rho_g}} \limsup_{r \to \infty} \frac{r^{\rho_g}}{\log^2 M(\exp(r^{\rho_g}), f)} \\
&\leq (\rho_f + \epsilon) \frac{\sigma_g}{\lambda_f} \frac{1}{\lambda_f}. \tag{6}
\end{align*}
\]

As \( \epsilon > 0 \) is arbitrary from (6), we obtain that

\[
\limsup_{r \to \infty} \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^{\rho_g}), f)} \leq \rho_f \frac{\sigma_g}{\lambda_f} < \infty.
\]

(ii) As \( n \) is odd from Lemma 1, we get for all sufficiently large values of \( r \) that

\[
\begin{align*}
\frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^{\rho_g}), f)} &\leq \frac{(\rho_g + \epsilon) \log M(r, f) + O(1)}{\log^2 M(\exp(r^{\rho_g}), f)} \\
&= \frac{(\rho_g + \epsilon) \log M(r, f) + O(1)}{r^{\rho_f}} \frac{r^{\rho_g}}{\log^2 M(\exp(r^{\rho_g}), f)} \tag{7}
\end{align*}
\]

As \( \epsilon > 0 \) is arbitrary and \( \rho_g > \rho_f \) from (7), we get that

\[
\limsup_{r \to \infty} \frac{\log^n M(r, f_n)}{\log^2 M(\exp(r^{\rho_g}), f)} = 0.
\]
Remark 3. If we replace $f$ by $g$ in the denominator of Theorem 5 then Theorem 5 is still valid which is evident from the following theorem.

Theorem 6. Let $f(z)$ and $g(z)$ be two entire functions of finite order such that $0 < \lambda_f$. If $\sigma_g < \infty$ then

(i) \[ \limsup_{r \to \infty} \frac{\log[n] M (r, f_n)}{\log[2] M (\exp (r^{\rho_g}), g)} < \infty \] if $n$ is even.

(ii) Further if $\rho_g > \rho_f$, $\sigma_f < \infty$ and $n$ is odd then

\[ \limsup_{r \to \infty} \frac{\log[n] M (r, f_n)}{\log[2] M (\exp (r^{\rho_g}), g)} = 0. \]

Proof of Theorem 6 is similar to Theorem 5 and so is omitted.

Theorem 7. Let $f(z)$ and $g(z)$ be two entire functions of finite order such that $0 < \lambda_f$. If $\sigma_g = \infty$ then

(i) \[ \limsup_{r \to \infty} \frac{\log[n] M (r, f_n)}{\log[2] M (\exp (r^{\rho_g}), f)} = \infty \] if $n$ is even.

(ii) Further if $\rho_g < \rho_f$, $\sigma_f = \infty$ and $n$ is odd then

\[ \limsup_{r \to \infty} \frac{\log[n] M (r, f_n)}{\log[2] M (\exp (r^{\rho_g}), f)} = \infty. \]

Proof. (i) As $n$ is even from Lemma 2 we get for all sufficiently large values of $r$ that

\[ \frac{\log[n] M (r, f_n)}{\log[2] M (\exp (r^{\rho_g}), f)} \geq \frac{(\lambda_f - \varepsilon) \log M \left( \frac{r}{2^{n-1}}, g \right) + O(1)}{\log[2] M (\exp (r^{\rho_g}), f)} \]

\[ = \frac{(\lambda_f - \varepsilon) \log M \left( \frac{r}{2^{n-1}}, g \right) + O(1)}{\log[2] M (\exp (r^{\rho_g}), f)} \frac{r^{\rho_g}}{(2^{n-1})^{\rho_g}} \]

As $\varepsilon > 0$ is arbitrary and $\sigma_g = \infty$ from (8) we obtain that

\[ \limsup_{r \to \infty} \frac{\log[n] M (r, f_n)}{\log[2] M (\exp (r^{\rho_g}), f)} = \infty. \]

(ii) As $n$ is odd from Lemma 2 we get for all sufficiently large values of $r$ that

\[ \frac{\log[n] M (r, f_n)}{\log[2] M (\exp (r^{\rho_g}), f)} \geq \frac{(\lambda_g - \varepsilon) \log M \left( \frac{r}{2^{n-1}}, f \right) + O(1)}{\log[2] M (\exp (r^{\rho_g}), f)} \]
\[
= (\lambda_g - \varepsilon) \log M \left( \frac{r}{2^{n^2}}, f \right) + O(1) \frac{r^{\rho_g}}{\log^{[2]} M(\exp(r^{\rho_g}) \cdot f) \left(2^{n-1}\right)^{\rho_f}}. \tag{9}
\]
As \(\varepsilon > 0\) is arbitrary and \(\rho_g > \rho_f, \sigma_f = \infty\) from (9) we get that
\[
\limsup_{r \to \infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}) \cdot f)} = 0.
\]

**Remark 4.** Replacing \(f\) by \(g\) in the denominator of Theorem 7 we obtain the following theorem.

**Theorem 8.** Let \(f(z)\) and \(g(z)\) be two entire functions of finite order such that \(0 < \lambda_f\). If \(\sigma_g = \infty\) then

(i) \(\limsup_{r \to \infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}) \cdot g)} = \infty\) if \(n\) is even.

(ii) Further if \(\rho_g < \rho_f\), \(\sigma_f = \infty\) and \(n\) is odd then

\[
\limsup_{r \to \infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}) \cdot g)} = \infty.
\]

**Remark 5.** Theorem 5, Theorem 6, Theorem 7 and Theorem 8 improves Theorem 5 of Liao and Yang [6].

Now we study the comparative growth properties of iteration of two set of entire functions.

**Theorem 9.** Let \(f, g, h\) and \(k\) be four entire functions with finite order such that \(\rho_g < \rho_k\) and \(\rho_f < \rho_h\). Then

\[
\limsup_{r \to \infty} \frac{\log^{[n]} M(r, h_n)}{\log^{[n]} M(r, f_n)} = \infty.
\]

**Proof.** Case 1: When \(n\) is even.

As \(n\) is even from Lemma 2 we get for all sufficiently large values of \(r\) that

\[
\log^{[n]} M(r, h_n) \geq (\lambda_h - \varepsilon) \log M \left( \frac{r}{2^{n^2}}, k \right) + O(1). \tag{10}
\]

Now for a sequence of values of \(r\) tending to infinity we obtain from (10) that

\[
\log^{[n]} M(r, h_n) \geq (\lambda_h - \varepsilon) \left( \frac{r}{2^{n^2}} \right)^{\rho_k - \varepsilon} + O(1). \tag{11}
\]
Now from Lemma [1] we get for all sufficiently large values of $r$ that
\[
\log^{[n]} M (r, f_n) \leq (\rho_f + \varepsilon) \log M (r, g) + O (1)
\]
\[
< (\rho_f + \varepsilon) r^{\rho_f + \varepsilon}.
\]
(12)
From (11) and (12) we obtain for a sequence of values of $r$ tending to infinity that
\[
\frac{\log^{[n]} M (r, h_n)}{\log^{[n]} M (r, f_n)} \geq \frac{(\lambda_h - \varepsilon) \left( \frac{r}{2n-1} \right)^{\rho_k - \varepsilon} + O (1)}{(\rho_f + \varepsilon) r^{\rho_k + \varepsilon}}.
\]
(13)
As $\varepsilon > 0$ is arbitrary and $\rho_g < \rho_k$ from (13) we get that
\[
\limsup_{r \to \infty} \frac{\log^{[n]} M (r, h_n)}{\log^{[n]} M (r, f_n)} = \infty.
\]
Case 2: When $n$ is odd.
From Lemma [2] we get for all sufficiently large values of $r$ that
\[
\log^{[n]} M (r, h_n) \geq (\lambda_k - \varepsilon) \log M \left( \frac{r}{2n-1}, h \right) + O (1).
\]
(14)
Now for a sequence of values of $r$ tending to infinity we obtain from (14) that
\[
\log^{[n]} M (r, h_n) \geq (\lambda_k - \varepsilon) \left( \frac{r}{2n-1} \right)^{\rho_h - \varepsilon} + O (1).
\]
(15)
Now from Lemma [1] we get for all sufficiently large values of $r$ that
\[
\log^{[n]} M (r, f_n) \leq (\rho_g + \varepsilon) \log M (r, f) + O (1)
\]
\[
< (\rho_g + \varepsilon) r^{\rho_f + \varepsilon}.
\]
(16)
From (15) and (16) we obtain for a sequence of values of $r$ tending to infinity that
\[
\frac{\log^{[n]} M (r, h_n)}{\log^{[n]} M (r, f_n)} \geq \frac{(\lambda_k - \varepsilon) \left( \frac{r}{2n-1} \right)^{\rho_h - \varepsilon} + O (1)}{(\rho_g + \varepsilon) r^{\rho_f + \varepsilon}}.
\]
(17)
As $\varepsilon > 0$ is arbitrary and $\rho_f < \rho_h$ from (17) we get that
\[
\limsup_{r \to \infty} \frac{\log^{[n]} M (r, h_n)}{\log^{[n]} M (r, f_n)} = \infty.
\]
Hence in both the cases
\[
\limsup_{r \to \infty} \frac{\log^{[n]} M (r, h_n)}{\log^{[n]} M (r, f_n)} = \infty.
\]
References


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