CERTAIN PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce a new class $UT_{q,s}([\alpha_1]; \alpha, \beta)$ of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ defined by Dziok-Srivastava operator. The object of the present paper is to determine coefficient estimates, extreme points, distortion theorems, the radii of close-to-convexity, starlikeness and convexity and a family of integral operators for functions belonging to the class $UT_{q,s}([\alpha_1]; \alpha, \beta)$. We also obtain several results for the modified Hadamard products of functions belonging to this class.

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1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of $A$ which are, respectively, convex and starlike functions of order $\alpha$, $0 \leq \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (see [16]).

The Hadamard product (or convolution) $(f * g)(z)$ of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$
For positive real parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^{-}$, $\mathbb{Z}_0^{-} = 0, -1, -2, ..., j = 1, 2, ..., s$), the generalized hypergeometric function $qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is defined by

$$qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_s)_n n!} z^n$$

($q \leq s + 1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, .......\}; z \in U$), where $(\theta)_n$ is the Pochhammer symbol defined in terms of the Gamma function $\Gamma$, by

$$(\theta)_n = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} = \begin{cases} 1 & (n = 0) \\ \theta(\theta + 1)....(\theta + n - 1) & (n \in \mathbb{N}) \end{cases}$$

For the function $h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$, the Dziok-Srivastava linear operator (see [5] and [6]) $H_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) : A \rightarrow A$, is defined by the Hadamard product as follows:

$$H_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \ast f(z) = z + \sum_{n=2}^{\infty} \Psi_n(a_1) a_n z^n \quad (z \in U),$$

where

$$\Psi_n(a_1) = \frac{(\alpha_1)_{n-1} ... (\alpha_q)_{n-1}}{(\beta_1)_{n-1} ... (\beta_s)_{n-1} (n - 1)!}.$$  

(1.3)

For brevity, we write

$$H_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) f(z) = H_{q,s}(\alpha_1) f(z).$$  

(1.4)

For $0 \leq \alpha < 1, \beta \geq 0$ and for all $z \in U$, let $US_{q,s}(\alpha_1; \alpha, \beta)$ denote the subclass of $A$ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\text{Re} \left\{ \frac{H_{q,s}(\alpha_1) f(z)}{z(H_{q,s}(\alpha_1) f(z))'} - \alpha \right\} > \beta \left| \frac{H_{q,s}(\alpha_1) f(z)}{z(H_{q,s}(\alpha_1) f(z))'} - 1 \right|.$$  

(1.5)

Denote by $T$ the subclass of $A$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$  

(1.6)
which are analytic in $U$. We define the class $UT_{q,s}([\alpha_1]; \alpha, \beta)$ by:

$$UT_{q,s}([\alpha_1]; \alpha, \beta) = US_{q,s}([\alpha_1]; \alpha, \beta) \cap T.$$  

(1.7)

We note that for suitable choices of $q, s, \alpha$ and $\beta$, we obtain the following subclasses studied by various authors.

(1) For $q = 2, s = 1$ and $\alpha_1 = \alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}(1; \alpha, \beta)$ reduces to the class $ST(\alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left\{ \frac{f(z)}{zf'(z)} - \alpha \right\} > \beta \left| \frac{f(z)}{zf'(z)} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, z \in U \right\}$$

and the class $ST(\alpha, 0) = ST(\alpha)$ is the class of functions $f(z) \in T$ which satisfy the following condition (see [7] and [17])

$$ST(\alpha) = \text{Re} \left\{ \frac{f(z)}{zf'(z)} \right\} > \alpha \ (0 \leq \alpha < 1);$$

(2) For $q = 2, s = 1, \alpha_1 = a(a > 0), \alpha_2 = 1$ and $\beta_1 = c(c > 0)$ in (1.5), the class $UT_{2,1}(a, 1; \alpha, \beta)$ reduces to the class $\mathcal{L}T(a, c; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left\{ \frac{L(a, c)f(z)}{z(L(a, c)f(z))} - \alpha \right\} > \beta \left| \frac{L(a, c)f(z)}{z(L(a, c)f(z))} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, a > 0, c > 0, z \in U \right\},$$

where $L(a, c)$ is the Carlson - Shaffer operator (see [2]);

(3) For $q = 2, s = 1, \alpha_1 = \lambda + 1(\lambda > -1)$ and $\alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}(\lambda + 1; \alpha, \beta)$ reduces to the class $W_\lambda(\alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left\{ \frac{D^\lambda f(z)}{z(D^\lambda f(z))} - \alpha \right\} > \beta \left| \frac{D^\lambda f(z)}{z(D^\lambda f(z))} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, \lambda > -1, z \in U \right\} \ (\text{see [10]}),$$

where $D^\lambda(\lambda > -1)$ is the Ruscheweyh derivative operator (see [14]);

(4) For $q = 2, s = 1, \alpha_1 = v + 1(v > -1), \alpha_2 = 1$ and $\beta_1 = v + 2$ in (1.5), the class $UT_{2,1}(v + 1; v + 2; \alpha, \beta)$ reduces to the class $\zeta T(v; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left\{ \frac{J_v f(z)}{z(J_v f(z))} - \alpha \right\} > \beta \left| \frac{J_v f(z)}{z(J_v f(z))} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, v > -1, z \in U \right\},$$
where $J_{v}f(z)$ is the generalized Bernardi - Libera - Livingston operator (see [1], [8] and [9]);

(5) For $q = 2, s = 1, \alpha_{1} = 2, \alpha_{2} = 1$ and $\beta_{1} = 2 - \mu (\mu \neq 2, 3, \ldots)$ in (1.5), the class $UT_{2,1}([2, 1; 2 - \mu]; \alpha, \beta)$ reduces to the class $\mathcal{F}(\mu; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left( \frac{\Omega_{\mu}^{\mu}f(z)}{z(\Omega_{\mu}^{\mu}f(z))'} - \alpha \right) > \beta \left| \frac{\Omega_{\mu}^{\mu}f(z)}{z(\Omega_{\mu}^{\mu}f(z))'} - 1 \right|, 0 \leq \alpha < 1, \beta \geq 0, \mu \neq 2, 3, \ldots, z \in U \right\},$$

where $\Omega_{\mu}^{\mu}f(z)$ is the Srivastava - Owa fractional derivative operator (see [12] and [13]);

(6) For $q = 2, s = 1, \alpha_{1} = \mu (\mu > 0), \alpha_{2} = 1$ and $\beta_{1} = \lambda + 1 (\lambda > -1)$ in (1.5), the class $UT_{2,1}([\mu, 1; \lambda + 1]; \alpha, \beta)$ reduces to the class $\mathcal{L}(\mu, \lambda; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left( \frac{I_{\lambda, \mu}f(z)}{z(I_{\lambda, \mu}f(z))'} - \alpha \right) > \beta \left| \frac{I_{\lambda, \mu}f(z)}{z(I_{\lambda, \mu}f(z))'} - 1 \right|, 0 \leq \alpha < 1, \beta \geq 0, \mu > 0, \lambda > -1, z \in U \right\},$$

where $I_{\lambda, \mu}f(z)$ is the Choi-Saigo-Srivastava operator (see [4]);

(7) For $q = 2, s = 1, \alpha_{1} = 2, \alpha_{2} = 1$ and $\beta_{1} = k + 1 (k > -1)$ in (1.5), the class $UT_{2,1}([2, 1; k + 1]; \alpha, \beta)$ reduces to the class $\mathcal{A}(k; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left( \frac{I_{k}f(z)}{z(I_{k}f(z))'} - \alpha \right) > \beta \left| \frac{I_{k}f(z)}{z(I_{k}f(z))'} - 1 \right|, 0 \leq \alpha < 1, \beta \geq 0, k > -1, z \in U \right\},$$

where $I_{k}f(z)$ is the Noor integral operator (see [11]);

(8) For $q = 2, s = 1, \alpha_{1} = c (c > 0), \alpha_{2} = \lambda + 1 (\lambda > -1) and \beta_{1} = a (a > 0)$ in (1.5), the class $UT_{2,1}([c, \lambda + 1; a]; \alpha, \beta)$ reduces to the class $\mathcal{F}(c, a, \lambda; \alpha, \beta)$

$$= \left\{ f \in T : \text{Re} \left( \frac{I_{\lambda}(a, c)f(z)}{z(I_{\lambda}(a, c)f(z))'} - \alpha \right) > \beta \left| \frac{I_{\lambda}(a, c)f(z)}{z(I_{\lambda}(a, c)f(z))'} - 1 \right|, 0 \leq \alpha < 1, \beta \geq 0, c > 0, \lambda > -1, a > 0, z \in U \right\},$$

where $I_{\lambda}(a, c)f(z)$ is the Cho-Kwon-Srivastava operator (see [3]).
2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters \( \alpha_1, \ldots, \alpha_q \) and \( \beta_1, \ldots, \beta_s \) are positive real numbers, \( 0 \leq \alpha < 1, \beta \geq 0, \ n \geq 2, \ z \in U \) and \( \Psi_n(\alpha_1) \) is defined by (1.3).

**Theorem 1.** A function \( f(z) \) of the form (1.6) is in the class \( UT_{q,s}(\alpha_1; \alpha, \beta) \) if

\[
\sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1) a_n \leq 1 - \alpha.
\]  

(2.1)

**Proof.** Suppose that (2.1) is true. Since

\[
\frac{[2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{1 - \alpha} - n \Psi_n(\alpha_1) = \frac{(n - 1)(1 + \beta) \Psi_n(\alpha_1)}{1 - \alpha} > 0,
\]

we deduce

\[
\sum_{n=2}^{\infty} n \Psi_n(\alpha_1) a_n < \sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{1 - \alpha} a_n \leq 1.
\]

It suffices to show that

\[
\beta \left| \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right| - \text{Re} \left( \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right) \leq 1 - \alpha,
\]

we have

\[
\beta \left| \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right| - \text{Re} \left( \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right) \leq (1 + \beta) \left| \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right|
\]

\[\leq \frac{(1 + \beta) \sum_{n=2}^{\infty} (n - 1) \Psi_n(\alpha_1) a_n}{1 - \sum_{n=2}^{\infty} n \Psi_n(\alpha_1) a_n},\]

which yields
\[
(1 - \alpha) - (1 + \beta) \left| \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right| > (1 - \alpha) - \frac{\sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1) a_n}{1 - \sum_{n=2}^{\infty} n\Psi_n(\alpha_1) a_n} \geq 0. \tag{2.2}
\]

This completes the proof of Theorem 1.

Unfortunately, the converse of the above Theorem 1 is not true. So we define the subclass \( T_{q,s}(\alpha, \beta) \) of \( UT_{q,s}(\alpha_1, \alpha, \beta) \) consisting of functions \( f(z) \) which satisfy (2.1).

**Remark 1.** Putting \( q = 2, s = 1, \beta = 0 \) and \( \alpha_1 = \alpha_2 = \beta_1 = 1 \), in Theorem 1 reduces to the result obtained by Yamakawa [17, Lemma 2.1, with \( n = p = 1 \)].

**Corollary 1.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_{q,s}(\alpha_1, \alpha, \beta) \), then
\[
a_n \leq \frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} \quad (n \geq 2). \tag{2.3}
\]
The result is sharp for the function
\[
f(z) = z - \frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} z^n \quad (n \geq 2). \tag{2.4}
\]

Putting \( q = 2, s = 1, \alpha_1 = \lambda + 1(\lambda > -1) \) and \( \alpha_2 = \beta_1 = 1 \) in Theorem 1, we obtain the following corollary.

**Corollary 2.** A function \( f(z) \) of the form (1.6) is in the class \( W_\lambda(\alpha, \beta) \) if
\[
\sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)]\frac{(\lambda + 1)_{n-1} a_n}{(n - 1)!} \leq 1 - \alpha.
\]

**Remark 2.** The result in Corollary 2 correct the result obtained by Najafzadeh and Kulkarni [10, Lemma 1.1].
3. Distortion theorems

**Theorem 2.** Let the function $f(z)$ defined by (1.6) belong to the class $T_{q,s}([\alpha_1]; \alpha, \beta)$. Then for $|z| = r < 1$, we have

$$r - \frac{(1 - \alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)} r^2 \leq |f(z)| \leq r + \frac{(1 - \alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)} r^2,$$

(3.1)

provided $\Psi_n(\alpha_1) \geq \Psi_2(\alpha_1)$ ($n \geq 2$). The result is sharp with equality for the function $f(z)$ defined by

$$f(z) = z - \frac{(1 - \alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)} z^2$$

(3.2)

at $z = r$ and $z = re^{i(2n+1)\pi}$ ($n \in \mathbb{N}$).

**Proof.** We have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + r^2 \sum_{n=2}^{\infty} a_n.$$

(3.3)

Since for $n \geq 2$, we have

$$(3 - 2\alpha + \beta)\Psi_2(\alpha_1) \leq [2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1),$$

then (2.1) yields

$$(3 - 2\alpha + \beta)\Psi_2(\alpha_1) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1) a_n \leq (1 - \alpha)$$

(3.4)

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1 - \alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)}.$$

(3.5)

From (3.5) and (3.3) we have

$$|f(z)| \leq r + \frac{(1 - \alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)} r^2$$

and similarly, we have

$$|f(z)| \geq r - \frac{(1 - \alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)} r^2.$$

This completes the proof of Theorem 2.
Theorem 3. Let the function $f(z)$ defined by (1.6) belong to the class $T_{q,s}([\alpha_1]; \alpha, \beta)$. Then for $|z| = r < 1$, we have

$$1 - \frac{2(1-\alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)} r,$$

provided $\Psi_n(\alpha_1) \geq \Psi_2(\alpha_1)$ ($n \geq 2$). The result is sharp for the function $f(z)$ given by (3.2) at $z = r$ and $z = re^{i(2n+1)\pi}$ ($n \in \mathbb{N}$).

Proof. For a function $f(z) \in UT_{q,s}([\alpha_1]; \alpha, \beta)$, it follows from (2.2) and (3.5) that

$$\sum_{n=2}^{\infty} na_n \leq \frac{2(1-\alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)}.$$

4. Extreme points

Theorem 4. The class $T_{q,s}([\alpha_1]; \alpha, \beta)$ is closed under convex linear combinations.

Proof. Let $f_j(z) \in T_{q,s}([\alpha_1]; \alpha, \beta)$ ($j = 1, 2$), where

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \ (a_{n,j} \geq 0; \ j = 1, 2).$$

(4.1)

Then it is sufficient to prove that the function $h(z)$ given by

$$h(z) = \mu f_1(z) + (1 - \mu)f_2(z) \ (0 \leq \mu \leq 1)$$

is also in the class $T_{q,s}([\alpha_1]; \alpha, \beta)$. For $0 \leq \mu \leq 1$

$$h(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1 - \mu)a_{n,2}] z^n$$

and with the aid of Theorem 1, we have

$$\sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1) \cdot [\mu a_{n,1} + (1 - \mu)a_{n,2}]$$

$$\leq \mu(1-\alpha) + (1 - \mu)(1-\alpha) = 1 - \alpha,$$

which implies that $h(z) \in T_{q,s}([\alpha_1]; \alpha, \beta)$. This completes the proof of Theorem 4.

As a consequence of Theorem 4, there exist extreme points of the class $T_{q,s}([\alpha_1]; \alpha, \beta)$, which are given by:

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Theorem 5. Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} z^n \quad (n \geq 2).$$

Then $f(z)$ is in the class $T_{q,s}([\alpha_1]; \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad (4.2)$$

where $\mu_n \geq 0 \ (n \geq 1)$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof. Assume that

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} \mu_n z^n.$$ 

Then it follows that

$$\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} \mu_n$$

$$= \sum_{n=2}^{\infty} \mu_n = (1 - \mu_1) \leq 1. \quad (4.3)$$

So, by Theorem 1, we have $f(z) \in T_{q,s}([\alpha_1]; \alpha, \beta)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_{q,s}([\alpha_1]; \alpha, \beta)$. Then $a_n$ are given by (2.3). Setting

$$\mu_n = \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} a_n \quad (4.4)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n,$$

we can see that $f(z)$ can be expressed in the form (4.2). This completes the proof of Theorem 5.
Corollary 3. The extreme points of the class $T_{q,s}([\alpha_1]; \alpha, \beta)$ are the functions $f_1(z) = z$ and

$$f_n(z) = z - \frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} z^n (n \geq 2).$$

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 6. Let the function $f(z)$ defined by (1.6) be in the class $T_{q,s}([\alpha_1]; \alpha, \beta)$. Then $f(z)$ is close-to-convex of order $\rho$ ($0 \leq \rho < 1$) in $|z| < r_1$, where

$$r_1 = \inf_{n \geq 2} \left\{ \frac{(1 - \rho)[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{n(1 - \alpha)} \right\} \frac{1}{n - 1}. \quad (5.1)$$

The result is sharp, the extremal function being given by (2.4).

Proof. We must show that

$$\left| f'(z) - 1 \right| \leq 1 - \rho \quad \text{for } |z| < r_1,$$

where $r_1$ is given by (5.1). Indeed we find from the definition (1.6) that

$$\left| f'(z) - 1 \right| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus

$$\left| f'(z) - 1 \right| \leq 1 - \rho,$$

if

$$\sum_{n=2}^{\infty} \left( \frac{n}{1 - \rho} \right) a_n |z|^{n-1} \leq 1. \quad (5.2)$$

But, by Theorem 1, (5.2) will be true if

$$\left( \frac{n}{1 - \rho} \right) |z|^{n-1} \leq \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)},$$

that is, if

$$|z| \leq \left\{ \frac{(1 - \rho)[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{n(1 - \alpha)} \right\} \frac{1}{n - 1} (n \geq 2). \quad (5.3)$$
Theorem 6 follows easily from (5.3).

**Theorem 7.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_{q,s}(\alpha_1; \alpha, \beta) \). Then \( f(z) \) is starlike of order \( \rho \) \((0 \leq \rho < 1)\) in \(|z| < r_2\), where

\[
r_2 = \inf_{n \geq 2} \left\{ \frac{(1 - \rho) [2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{(n - \rho)(1 - \alpha)} \right\}^{1/n - 1}. \tag{5.4}
\]

The result is sharp, with the extremal function \( f(z) \) given by (2.4).

**Proof.** It is sufficient to show that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad (|z| < r_2),
\]

where \( r_2 \) is given by (5.4). Indeed we find, again from the definition (1.6) that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.
\]

Thus

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho,
\]

if

\[
\sum_{n=2}^{\infty} \left( \frac{n - \rho}{1 - \rho} \right) a_n |z|^{n-1} \leq 1. \tag{5.5}
\]

But, by Theorem 1, (5.5) will be true if

\[
\left( \frac{n - \rho}{1 - \rho} \right) |z|^{n-1} \leq \frac{[2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{(1 - \alpha)}
\]

that is, if

\[
|z| \leq \left\{ \frac{(1 - \rho) [2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{(n - \rho)(1 - \alpha)} \right\}^{1/n - 1} (n \geq 2). \tag{5.6}
\]

Theorem 7 follows easily from (5.6).
Similarly, we can prove the following theorem.

**Theorem 8.** Let the functions \( f(z) \) defined by (1.6) be in the class \( T_{q,s}([\alpha_1]; \alpha, \beta) \). Then \( f(z) \) is convex of order \( \rho \) \((0 \leq \rho < 1)\) in \(|z| < r_3\), where

\[
r_3 = \inf_{n \geq 2} \left\{ \frac{(1 - \rho)[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{n(n - \rho)(1 - \alpha)} \right\} \frac{1}{n-1}.
\]

(5.7)

The result is sharp, with the extremal function \( f(z) \) given by (2.4).

6. A family of integral operators

**Theorem 9.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_{q,s}([\alpha_1]; \alpha, \beta) \) and let \( c \) be a real number such that \( c > -1 \). Then the function \( F(z) \) defined by

\[
F(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) \, dt \quad (c > -1)
\]

(6.1)

also belongs to the class \( T_{q,s}([\alpha_1]; \alpha, \beta) \).

**Proof.** Let the function \( f(z) \) be defined by (1.6). Then from the representation (6.1) of \( F(z) \), it follows that

\[
F(z) = z - \sum_{n=2}^{\infty} d_n z^n,
\]

where

\[
d_n = \left( \frac{c + 1}{c + n} \right) a_n.
\]

Therefore, we have

\[
\sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1) d_n
\]

\[
= \sum_{k=2}^{\infty} [2k - k(\alpha - \beta) - (\beta + 1)]\Psi_k(\alpha_1) \left( \frac{c + 1}{c + n} \right) a_n
\]

\[
\leq \sum_{n=2}^{\infty} [2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1) a_n \leq (1 - \alpha)
\]

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since $f(z) \in T_{q,s}([\alpha_1]; \alpha, \beta)$. Hence, by Theorem 1, $F(z) \in T_{q,s}([\alpha_1]; \alpha, \beta)$.

This completes the proof of Theorem 9.

**Theorem 10.** Let the function $F(z) = z - \sum_{n=2}^{\infty} a_nz^n (a_n \geq 0)$ be in the class $T_{q,s}([\alpha_1]; \alpha, \beta)$ and let $c$ be a real number such that $c > -1$. Then the function $f(z)$ given by (6.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_{n \geq 2} \left\{ \frac{(c + 1) [2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{n(c + n)(1 - \alpha)} \right\}^{\frac{1}{n-1}}. \quad (6.2)$$

The result is sharp.

**Proof.** From (6.1) we have

$$f(z) = \frac{z^{1-c}}{c+1} (z^c F(z))' = z - \sum_{k=2}^{\infty} \left( \frac{c+k}{c+1} \right) a_k z^k.$$

To prove the assertion of the theorem, it suffices to show that

$$|f'(z) - 1| < 1$$

for $|z| < R^*$, where $R^*$ is defined by (6.2). Now

$$|f'(z) - 1| = \left| - \sum_{n=2}^{\infty} n \left( \frac{c+n}{c+1} \right) a_n z^{n-1} \right|$$

$$\leq \sum_{n=2}^{\infty} n \left( \frac{c+n}{c+1} \right) a_n |z|^{n-1}.$$

Thus

$$|f'(z) - 1| < 1 \text{ if } \sum_{n=2}^{\infty} n \left( \frac{c+n}{c+1} \right) a_n |z|^{n-1} < 1.$$

But Theorem 1 confirms that

$$\sum_{n=2}^{\infty} \left[ 2n - n(\alpha - \beta) - (\beta + 1) \right] \Psi_n(\alpha_1) \frac{a_n}{(1 - \alpha)} \leq 1. \quad (6.3)$$

Thus (6.3) will be satisfied if

$$n \left( \frac{c+n}{c+1} \right) |z|^{n-1} \leq \sum_{n=2}^{\infty} \left[ 2n - n(\alpha - \beta) - (\beta + 1) \right] \Psi_n(\alpha_1) \frac{a_n}{(1 - \alpha)} (n \geq 2),$$

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that is, if
\[
|z| \leq \left\{ \frac{(c + 1) [2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{n (c + n) (1 - \alpha)} \right\}^{\frac{1}{n-1}} (n \geq 2). \tag{6.4}
\]

The required result follows now from (6.4).

Finally, the result is sharp for the function \(f(z)\) given by
\[
f(z) = z - \frac{(c + 1) (1 - \alpha)}{(c + n) [2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)} z^n (n \geq 2; c > -1). \tag{6.5}
\]

7. Modified Hadamard products

Let the functions \(f_j(z) (j = 1, 2)\) be defined by (4.1). The modified Hadamard product of \(f_1(z)\) and \(f_2(z)\) is defined by
\[
(f_1 \ast f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 \ast f_1)(z). \tag{7.1}
\]

**Theorem 11.** Let each of the functions \(f_j(z) (j = 1, 2)\) defined by (4.1) be in the class \(T_{q,s}([\alpha_1]; \alpha, \beta)\). If the sequence \(\{\delta_n(\alpha, \beta)\} (n \geq 2)\), where
\[
\delta_n(\alpha, \beta) = \{[2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)\}
\]

is non-decreasing, then \((f_1 \ast f_2)(z) \in T_{q,s}([\alpha_1]; \eta(q, s, \Psi_2(\alpha_1), \alpha, \beta), \beta)\) where \(\eta\) is given by
\[
\eta(q, s, \Psi_2(\alpha_1), \alpha, \beta) = 1 - \frac{(1 + \beta)(1 - \alpha)^2}{(3 - 2\alpha + \beta)^2 \Psi_2(\alpha_1) - 2(1 - \alpha)^2}. \tag{7.3}
\]

The result is sharp.

**Proof.** Employing the technique used earlier by Schild and Sliveman [15], we need to fined the largest \(\eta = \eta(q, s, \Psi_2(\alpha_1), \alpha, \beta)\) such that
\[
\sum_{n=2}^{\infty} \frac{[2n - n(\eta - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{(1 - \eta)} a_{n,1} a_{n,2} \leq 1. \tag{7.4}
\]

Since
\[
\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)] \Psi_n(\alpha_1)}{(1 - \alpha)} a_{n,1} \leq 1, \tag{7.5}
\]
and
\[
\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} a_{n,2} \leq 1,
\] (7.6)
by the Cauchy-Schwarz inequality, we have
\[
\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \sqrt{a_{n,1}a_{n,2}} \leq 1.
\] (7.7)
Thus it is sufficient to show that
\[
\sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \sqrt{a_{n,1}a_{n,2}} \leq 1.
\]
(7.7)

Thus it is sufficient to show that
\[
\sum_{n=2}^{\infty} \frac{[2n - n(\eta - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \eta)} a_{n,1}a_{n,2} \leq \sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \sqrt{a_{n,1}a_{n,2}},
\]
that is, that
\[
\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1 - \eta)[2n - n(\alpha - \beta) - (\beta + 1)]}{(1 - \alpha)[2n - n(\eta - \beta) - (\beta + 1)]}.
\]
Note that
\[
\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}.
\]
Consequently, we need only to prove that
\[
\frac{(1 - \alpha)}{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)} \leq \frac{(1 - \eta)[2n - n(\alpha - \beta) - (\beta + 1)]}{(1 - \alpha)[2n - n(\eta - \beta) - (\beta + 1)]},
\]
or, equivalently, that
\[
\eta \leq 1 - \frac{(n - 1)(1 + \beta)(1 - \alpha)^2}{[2n - n(\alpha - \beta) - (\beta + 1)]^2\Psi_n(\alpha_1) - n(1 - \alpha)^2}.
\] (7.8)
Since
\[
\varphi(n) = 1 - \frac{(n - 1)(1 + \beta)(1 - \alpha)^2}{[2n - n(\alpha - \beta) - (\beta + 1)]^2\Psi_n(\alpha_1) - n(1 - \alpha)^2}
\]
is an increasing function of \( n \) \((n \geq 2)\), letting \( n = 2 \) in (7.8), we obtain
\[
\eta \leq \varphi(2) = 1 - \frac{(1 + \beta)(1 - \alpha)^2}{(3 - 2\alpha + \beta)^2\Psi_2(\alpha_1) - 2(1 - \alpha)^2},
\] (7.9)
which proves the main assertion of Theorem 11.
Finally, by taking the functions \( f_j(z)(j = 1, 2) \) given by
\[
f_j(z) = z - \frac{(1 - \alpha)}{(3 - 2\alpha + \beta)\Psi_2(\alpha_1)}z^2 \quad (j = 1, 2),
\] (7.10)
we can see that the result is sharp.

**Remark 3.** Putting $q = 2, s = 1, \beta = 0$ and $\alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 11, we will obtain the result obtained by Kang et al. [7, Corollary 1, with $n = p = 1$ and $m = 2$].

**Theorem 12.** Let the functions $f_j(z)(j = 1, 2)$ defined by (4.1) be in the class $T_{q,s}([\alpha_1]; \alpha, \beta)$ if the sequence $\{\delta_n(\alpha, \beta)\}$ $(n \geq 2)$ defined by (7.2) is non-decreasing, then the function

$$g(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2)z^n$$

(7.11)

belongs to the class $T_{q,s}([\alpha_1]; \xi(q, s, \Psi_2(\alpha_1), \alpha, \beta, \beta)$, where

$$\xi(q, s, \Psi_2(\alpha_1), \alpha, \beta) = 1 - \frac{2(1 + \beta)(1 - \alpha)^2}{(3 - 2\alpha + \beta)^2\Psi_2(\alpha_1) - 4(1 - \alpha)^2}.$$  

(7.12)

The result is sharp for the functions $f_j(z)$ given by (7.10).

**Proof.** By virtue of Theorem 1, we obtain

$$\sum_{n=2}^{\infty} \left\{ \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \right\}^2 a_{n,1}^2$$

$$\leq \left\{ \sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} a_{n,1} \right\}^2 \leq 1,$$  

(7.13)

and

$$\sum_{n=2}^{\infty} \left\{ \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \right\}^2 a_{n,2}^2$$

$$\leq \left\{ \sum_{n=2}^{\infty} \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} a_{n,2} \right\}^2 \leq 1,$$  

(7.14)

it follows from (7.13) and (7.14) that

$$\sum_{n=2}^{\infty} \frac{1}{2} \left\{ \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \right\}^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1.$$

Therefore, we need to find the largest $\xi(q, s, \Psi_2(\alpha_1), \alpha, \beta)$ such that

$$\frac{[2n - n(\zeta - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \zeta)} \leq \frac{1}{2} \left\{ \frac{[2n - n(\alpha - \beta) - (\beta + 1)]\Psi_n(\alpha_1)}{(1 - \alpha)} \right\}^2 (n \geq 2),$$

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that is,
\[
\zeta \leq 1 - \frac{2(n-1)(1+\beta)(1-\alpha)^2}{[2n-n(\alpha-\beta)-((\beta+1)\Psi_n(\alpha_1) - 2n(1-\alpha)^2)(n \geq 2)}.
\]

Since
\[
B(n) = 1 - \frac{2(1-n)(1+\beta)(1-\alpha)^2}{[(1+\beta)-n(\alpha-\beta)]2\Psi_n(\alpha_1) - 2n(1-\alpha)^2}
\]
is an increasing function of \(n(n \geq 2)\), we readily have
\[
\zeta \leq B(2) = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(3-2\alpha+\beta)\Psi_2(\alpha_1) - 4(1-\alpha)^2}.
\]
This completes the proof of Theorem 12.

**Remark 4.** Specializing the parameters \(q, s, \alpha_1, ..., \alpha_q\) and \(\beta_1, ..., \beta_s\), in the above results, we obtain the corresponding results for the corresponding classes defined in the introduction.

**References**


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