ON IMBALANCE SEQUENCES OF ORIENTED GRAPHS

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Abstract. A necessary and sufficient condition for a sequence of integers to be an irreducible imbalance sequence is obtained. We found bounds for imbalance \( b_i \) of a vertex \( v_i \) of oriented graphs. Some properties of imbalance sequence of oriented graphs, arranged in lexicographic order, are investigated. In the last we report a result on an imbalance sequence for a self-converse tournament and conjecture that it is true for oriented graphs.

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1. Introduction

An oriented graph is a digraph with no symmetric pair of directed arcs with no loops. The imbalance \( b(v_i) \) (or simply \( b_i \)) of a vertex \( v_i \) in a digraph is defined as \( d_i^+ - d_i^- \), where \( d_i^+ \) and \( d_i^- \) are out-degree and in-degree of vertex \( v_i \) respectively.

An oriented graph \( D \) is reducible if it is possible to partition its vertices into two nonempty sets \( V_1 \) and \( V_2 \) in such a way that every vertex of \( V_2 \) is adjacent to all vertices of \( V_1 \). Let \( D_1 \) and \( D_2 \) be induced digraphs having vertex sets \( V_1 \) and \( V_2 \) respectively. Then \( D \) consists of all the arcs of \( D_1, D_2 \) and every vertex of \( D_2 \) is adjacent to all vertices of \( D_1 \). We write \( D = [D_1, D_2] \). If this is not possible, then the oriented graph \( D \) is irreducible. Let \( D_1, D_2, \ldots, D_k \) be irreducible oriented graphs with disjoint vertex sets. \( D = [D_1, D_2, \ldots, D_k] \) denotes the oriented graph having all arcs of \( D_m, 1 \leq m \leq k \), and every vertex of \( D_j \) is adjacent to all vertices of \( D_i \) with \( 1 \leq i < j \leq k \). \( D_1, D_2, \ldots, D_k \) are called irreducible components of \( D \). Such decomposition is known as irreducible component decomposition of \( D \) and is unique.

An imbalance sequence \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is said to be irreducible if all the oriented graphs with the imbalance sequence \( B \) are irreducible.
2. NECESSARY AND SUFFICIENT CONDITION

A sequence of integers \( A = (a_1, a_2, \ldots, a_n) \) with \( a_1 \geq a_2 \geq \ldots \geq a_n \) is feasible if it has sum zero and satisfies

\[
\sum_{i=1}^{k} a_i \leq k(n-k) \text{ for } 1 \leq k < n.
\]

The following result gives a condition for a sequence of integers to be the imbalance sequence of a simple directed graph.

**Theorem 1.** [10] A sequence is realizable as an imbalance sequence if and only if it is feasible.

The above result is equivalent to saying that a sequence of integers \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \geq b_2 \geq \ldots \geq b_n \) imbalance sequence of a simple directed graph if and only if

\[
\sum_{i=1}^{k} b_i \leq k(n-k) \text{ for } 1 \leq k < n
\]

(1)

with equality when \( k = n \).

On arranging the imbalance sequence in nondecreasing order, we obtain the following Corollary 2.

**Corollary 2.** A sequence of integers \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is an imbalance sequence of a simple directed graph (without repeated arcs) if and only if

\[
\sum_{i=1}^{k} b_i \geq k(k-n) \text{ for } 1 \leq k < n
\]

(2)

with equality when \( k = n \).

**Proof.** Let \( \bar{b}_i = b_{n-i+1} \). Then the sequence \( \bar{B} = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n) \) satisfies condition...
(1) We have

\[
\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} \bar{b}_{n-i+1}
\]

\[
= \sum_{i=1}^{n} \bar{b}_{n-i+1} - \sum_{i=k+1}^{n} \bar{b}_{n-i+1}
\]

\[
= 0 - (\bar{b}_{n-k} + \bar{b}_{n-k+1} + \cdots + \bar{b}_1)
\]

\[
= -\sum_{j=1}^{n-k} \bar{b}_j
\]

\[
\geq -(n-k)\{n-(n-k)\} \quad \text{(from Condition 1)}
\]

\[
= k(k-n),
\]

where \(1 \leq k \leq n-1\) and equality holds when \(k = n\).

3. Construction of an oriented graph with a given imbalance sequence

A sequence of integers is graphic if it is a degree sequence of a simple undirected graph. For characterization of graphic sequences we refer to [2, 3, 6]. Klietman and Wang [7] observed that Havel and Hakimi [3, 6] argument works with the deletion of the any element \(d_k\) of the degree sequence \((d_1, d_2, \ldots, d_n)\) with \(d_1 \leq d_2 \leq \cdots \leq d_n\), subtracting 1 from the \(d_k\) largest other elements.

The analogous statement about imbalance sequence is false. Dhruv et al. [10] considered the imbalance sequence \((3, 1, -1, -3)\) of a transitive tournament. Deleting the element 1 and adding 1 to the smallest imbalance gives \((3, -1, -2)\), which has no realization by a simple digraph.

Theorem 1 provides us an algorithm to construct an oriented graph from a given imbalance sequence. At each stage we form \(\hat{B} = (\hat{b}_2, \ldots, \hat{b}_n)\) from \(B = (b_1, b_2, \ldots, b_n)\) by deleting the largest imbalance \(b_1\) and adding 1 to \(b_1\) smallest elements of \(B\). Arcs of an oriented graph are defined by \(v_1 \to v\) if and only if \(\hat{b}_v \neq b_v\). If this procedure applied recursively, then

\((i)\) it tests whether \(B\) is an imbalance sequence and if \(B\) is an imbalance sequence, then

\((ii)\) an oriented graph \(D_B\) with imbalance sequence \(B\) is constructed.

Example of algorithm, \(n = 5\), \(B = (2, 0, 0, 0, -2)\).
Stage  
1.  
2.  
3.  

$B$
\begin{align*}
1. & \quad (2,0,0,0,-2) \\
2. & \quad (-,1,0,0,-1) \\
3. & \quad (-,-,0,0,0) \\
\end{align*}

Aecs of $D_B$
\begin{align*}
& v_1 \rightarrow v_2, v_5 \\
& v_2 \rightarrow v_5 \\
\end{align*}

4. **Irreducible Imbalance Sequences of Oriented Graphs**

In case of tournaments, the score sequence $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \ldots \leq s_n$ used to decide whether a tournament $T$ having the score sequence $S$ is strong or not [4]. This is not true in case of oriented graphs. For example oriented graphs $D_1$ and $D_2$ both have imbalance sequence $(0,0,0)$, but $D_1$ is strong and $D_2$ is not.

The following Theorem characterizes irreducible imbalance sequences.

**Theorem 3.** Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ be an imbalance sequence of oriented graph. Then $B$ is irreducible if and only if

\begin{align*}
\sum_{i=1}^{k} b_i & > k(k-n), \text{ for } 1 \leq k \leq n-1 \quad (3) \\
\text{and} \quad \sum_{i=1}^{n} b_i & = 0. \quad (4)
\end{align*}

**Proof.** Suppose $D$ is an oriented graph with vertex set $V$, having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$. Equality condition (4) is obvious. To prove inequalities (3.4), let $U$ be the set of $k$ vertices with the smallest imbalances, the arcs within $U$ contribute nothing to $\sum_{i=1}^{k} b_i$, and the ordered pairs $(V\setminus U) \times U$ contributes atmost $-1$ to each $v \in U$, so

\begin{align*}
\sum_{i=1}^{k} b_i & \geq -k(n-k) \\
& = k(k-n), \text{ for } 1 \leq k \leq n-1.
\end{align*}

Since $D$ is irreducible, there must exist at least one arc from a vertex of $U$ to a vertex of $V\setminus U$. 

90
So condition (5) becomes,

\[ \sum_{i=1}^{k} b_i = k(k - n) + 2 \]
\[ = k(k - n), \text{ for } 1 \leq k \leq n - 1. \]

For the converse, suppose that conditions (3) and (4) hold. Hence from Corollary 2 there exist an oriented graph \( D \) having imbalance sequence \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \).

Suppose that such an oriented graph is reducible. Then there exist a vertex set \( W \) with \( k \) vertices \( (k < n) \), such that every vertex of \( V \setminus W \) is adjacent to all the vertices of \( W \). Hence

\[ \sum_{i=1}^{k} b_i = k(k - n), \]

a contradiction, proving the converse part.

**Corollary 4.** Let \( D \) be an oriented graph having imbalance sequence \( B = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n) \) with \( \tilde{b}_1 \geq \tilde{b}_2 \geq \ldots \geq \tilde{b}_n \). Then \( D \) is irreducible if and only if

\[ \sum_{i=1}^{k} \tilde{b}_i < k(n - k) \text{ for } 1 \leq k \leq n \]
\[ \text{and } \sum_{i=1}^{n} \tilde{b}_i = 0 \]

The next result is an extension of Theorem 3.

**Theorem 5.** Let \( D \) be an oriented graph having imbalance sequence \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \). Suppose that

\[ \sum_{i=1}^{p} b_i = p(p - n), \]
\[ \sum_{i=1}^{q} b_i = q(q - n) \]
\[ \text{and } \sum_{i=1}^{k} b_i > k(k - n), \text{ for } p + 1 \leq k \leq q - 1, \text{ where } 0 \leq p < q \leq n. \]
Then subdigraph induced by the vertices \( \{v_{p+1}, v_{p+2}, \ldots, v_q\} \) is an irreducible component of \( D \) with imbalance sequence
\[
(b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q).
\]

Proof. Suppose imbalance of vertex \( v_i \) in oriented graph \( D \) is \( b_i, 1 \leq i \leq n \). Since \( \sum_{i=1}^{q} b_i = q(n - q) \), so clearly each vertex of \( W = \{v_{q+1}, v_{q+2}, \ldots, v_n\} \) dominates all vertices of \( \{v_1, v_2, \ldots, v_q\} \). Thus the vertices within \( W \) contributes \(-(n - q)\) to imbalance of every vertex of \( \{v_1, v_2, \ldots, v_q\} \). Also \( \sum_{i=1}^{p} b_i = p(p - n) \), so each vertex of \( V = \{v_{p+1}, v_{p+2}, \ldots, v_q\} \) dominates all vertices of \( U = \{v_1, v_2, \ldots, v_p\} \). So vertices within \( U \) contribute \( p \) to imbalance of every vertex of \( V \). Hence the imbalance sequence of subdigraph induced by vertices \( \{v_{p+1}, v_{p+2}, \ldots, v_q\} \) is
\[
(b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q)
\]
i.e.,
\[
(b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q).
\]

Now we have to show that above imbalance sequence is irreducible. We have
\[
\sum_{i=1}^{k} b_i > k(k - n)
\]
\[
\Rightarrow \sum_{i=1}^{p} b_i + \sum_{i=p+1}^{k} b_i > k(k - n)
\]
\[
\Rightarrow p(p - n) + \sum_{i=p+1}^{k} b_i + (k - p)(n - p - q) > k(k - n) + (k - p)(n - p - q)
\]
\[
\Rightarrow \sum_{i=p+1}^{k} (b_i + n - p - q) > k(k - n) + (k - p)(n - p - q) - p(p - n)
\]
\[
= k^2 - kp - kq + pq
\]
\[
= (k - p)(k - q).
\]
Thus \( \sum_{i=p+1}^{k} (b_i + n - p - q) > (k - p)[(k - p) - (q - p)] \), and
\[
\sum_{i=p+1}^{q} (b_i + n - p - q) = \sum_{i=p+1}^{q} b_i + (q-p)(n-p-q)
\]
\[
= \sum_{i=1}^{k} b_i - \sum_{i=p+1}^{k} b_i + (q-p)(n-p-q)
\]
\[
= q(q-n) - p(p-n) + (q-p)(n-p-q)
\]
\[
= 0.
\]

Hence by Theorem 3 the imbalance sequence is irreducible.

Theorem 5 shows that the irreducible components of \(B\) are determined by the successive values of \(k\) for which
\[
\sum_{i=1}^{k} b_i = k(k-n) \text{ for } 1 \leq k \leq n.
\] (6)

Taking \(B = (-6, -5, -4, 1, 1, 6, 6)\), equation (6) is satisfied for \(k = 3, 6\) and 8. So the irreducible components of \(B\) are \((-1, 0, 1)\), \((0, 0, 0)\) and \((0, 0)\).

5. The bounds of imbalances

The converse of an oriented graph \(D\) is an oriented graph \(D'\), obtained by reversing orientation of all arcs of \(D\). Let \(B = (b_1, b_2, \ldots, b_n)\) with \(b_1 \leq b_2 \leq \ldots \leq b_n\) be imbalance sequence of an oriented graph \(D\). Then
\[
B' = (-b_n, -b_{n-1}, \ldots, b_1).
\]

Next result gives lower and upper bounds for the imbalance \(b_i\) of a vertex \(v_i\) of an oriented graph \(D\).

**Theorem 6.** If \(B = (b_1, b_2, \ldots, b_n)\) with \(b_1 \leq b_2 \leq \ldots \leq b_n\) is an imbalance sequence of an oriented graph \(D\), then for each \(i\),
\[
i - n \leq b_i \leq i - 1.
\]

**Proof.** First, we prove that
\[
b_i \geq i - n.
\]

Suppose that \(b_i < i - n\) then, for every \(k < i\)
\[
b_k \leq b_i < i - n.
\]
So that,
\[ \sum_{k=1}^{i} b_k < \sum_{k=1}^{i} (i - n) \]
\[ \Rightarrow \sum_{k=1}^{i} b_k < i(i - n). \]

As \( B = (b_1, b_2, \ldots, b_n) \) is an imbalance sequence so, by Corollary 2,
\[ \sum_{k=1}^{i} b_k \geq i(i - n). \]

This is a contradiction. Hence
\[ (i - n) \leq b_i. \] (7)

The second inequality is dual to the first. In the converse oriented graph \( D' \) with imbalance sequence \( B' = (b'_1, b'_2, \ldots, b'_n) \). We have
\[ b'_{n-i+1} \geq (n - i + 1) - n = 1 - i \] (using condition 7)
but \( b_i = -b'_{n-i+1} \) so,
\[ b_i \leq -(1 - i) = i - 1. \]

Proving the result.

6. Lexicographic enumeration of imbalance sequences

Let \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) and \( C = (c_1, c_2, \ldots, c_n) \) with \( c_1 \leq c_2 \leq \ldots \leq c_n \) be sequences of integers of order \( n \). Then \( B \) precedes \( C \) if there exist a positive integer \( k \leq n \) such that \( b_i = c_i \) for each \( 1 \leq i \leq k - 1 \) and \( b_k < c_k \) \((B = C \) if \( b_i = c_i \) for \( 1 \leq i \leq n \)).

We write \( B \prec C \) if \( B \) precedes \( C \), and we say that \( C \) is a successor of \( B \). If \( B \preceq C \) and \( C \preceq D \), then \( B \preceq D \), where \( D = (d_1, d_2, \ldots, d_n) \) with \( d_1 \leq d_2 \leq \ldots \leq d_n \). We say that \( C \) is an immediate successor of \( B \) if there is no \( D \) such that \( B \preceq D \preceq C \). An enumeration of all sequences of a given order with the property that the immediate successor of any sequence follows it in the list is called a lexicographic enumeration.

Let \( B = (b_1, b_2, \ldots, b_m) \) with \( b_1 \leq b_2 \leq \ldots \leq b_m \) and \( C = (c_1, c_2, \ldots, c_n) \) with \( c_1 \leq c_2 \leq \ldots \leq c_n \) are two imbalance sequences of order \( m \) and \( n \) respectively. Then we define
\[ B + C = (b_1 - n, b_2 - n, \ldots, b_m - n, c_1 + m, c_2 + m, \ldots, c_n + m). \]
The plus operation defined above is not commutative but it is associative.

Now we establish some results dealing with imbalance sequences that are tournament analogue to Merajuddin [9].

**Theorem 7.** Let \( B_1 = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) and \( B_2 = (-n, b_1 + 1, b_2 + 1, \ldots, b_n + 1) \). Then \( B_1 \) is \( m \text{th} \) imbalance sequence of order \( n \) if and only if \( B_2 \) is the \( m \text{th} \) imbalance sequence of order \( (n + 1) \).

**Proof.** Suppose \( D_1 \) be a realization of \( B_1 \). Then \( D_2 = [K, D_1] \), where \( K \) is an oriented graph of order 1, is a realization of \( B_2 \). This shows that \( B_2 \) is an imbalance sequence when \( B_1 \) is an imbalance sequence. For converse, suppose \( D \) be a realization of \( B_2 \). We can write \( D = [U, W] \), where \( U \) is an oriented graph of order 1, Clearly \( W \) is a realization of \( B_1 \). This shows that \( B_1 \) is an imbalance sequence when \( B_2 \) is an imbalance sequence. The unique correspondence shows that both are occupying the same position.

Let \( b_k(n) \) denotes the number of imbalance sequences of order \( n \), in nondecreasing order, having imbalance \( k \) atleast once, for \( 1 - n \leq k \leq n - 1 \). Then we have the following results.

**Theorem 8.**

\[
\begin{align*}
(i) & \quad b_k(n) = b_{-k}(n) \\
(ii) & \quad b_{1-n}(n) = b(n-1) \\
(iii) & \quad b_{n-1} = b(n-1).
\end{align*}
\]

**Proof.** (i) This is equivalent to proving that whenever \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is an imbalance sequence, then \( B' = (-b_n, -b_{n-1}, \ldots, b_1) \) is also an imbalance sequence. This always happens, since \( B \) is an imbalance sequence of an oriented graph \( D \) if and only if \( B' \) is an imbalance sequence of oriented graph \( D' \), the converse of \( D \).

(ii) Let \( B_1 = (b_1, b_2, \ldots, b_{n-1}) \) be the last imbalance, i.e., \( b(n-1) \text{th} \) imbalance sequence of order \( n - 1 \). By Theorem 7 \( B_2 = (-n, b_1 + 1, b_2 + 1, \ldots, b_{n-1} + 1) \) is the \( b(n-1) \text{th} \) imbalance sequence of order \( n \). Now we show that there does not exist any imbalance sequence \( B_3 = (t_1, t_2, \ldots, t_n) \), \( B_3 \neq B_2 \) such that \( t_1 = -(n - 1) \) and \( B_2 \preceq B_3 \).

Suppose that there exists one such \( B_3 \). Then by Theorem 7, \( B_4 = (t_2-1, \ldots, t_n-1) \) is an imbalance sequence of order \( n - 1 \) and \( B_1 \preceq B_4 \), a contradiction as \( B_1 \) is the last imbalance sequence of order \( (n - 1) \). Thus \( B_2 \) is the last imbalance sequence of order \( n \) in which the first entry is \( -(n - 1) \).

Hence \( b_{1-n}(n) = b(n-1) \).
Putting \( k = n - 1 \) in Theorem 8(i), we get
\[ b_{n-1} = b_{1-n} \]
and from Theorem 8(ii),
\[ b_{1-n} = b(n - 1) \]
Hence \( b_{n-1} = b(n - 1) \).

7. Self-converse imbalance sequences

A score sequence \( S = (s_1, s_2, \ldots, s_n) \) is said to be self-converse if all the tournaments \( T \), having the score sequence \( S \) are self-converse, i.e., \( T \cong T' \). If \( S = (s_1, s_2, \ldots, s_n) \) is a score sequence of a tournament \( T \), then \( S' \) score sequence of \( T' \), is given by
\[ S' = (n-1-s_1, n-1-s_2, \ldots, n-1-s_n). \]

In 1979, Eplett[1] characterized the self-converse score sequences.

**Theorem 9.**[1] A score sequence \( S = (s_1, s_2, \ldots, s_n) \) is self-converse if and only if
\[ s_i + s_{n+1-i} = n - 1, \text{ for } 1 \leq i \leq n. \] (8)

Let \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) be an imbalance sequence of an oriented graph \( D \). Then the imbalance sequence \( B' \) of the oriented graph \( D' \), the converse of \( D \), is given by \((-b_n, -b_{n-1}, \ldots, -b_1)\). An oriented graph \( D \) is said to be self-converse if \( D \cong D' \). An imbalance sequence \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is self-converse if all the oriented graph having imbalance sequence \( B \) are self-converse.

Next result characterizes self-converse imbalance sequences of tournaments.

**Theorem 10.** A sequence of integers \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is self-converse if and only if
\[ b_i + b_{n-i+1} = 0, \text{ for } 1 \leq i \leq n. \]

**Proof.** Consider a tournament \( T \) having \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) and \( S = (s_1, s_2, \ldots, s_n) \) with \( s_1 \leq s_2 \leq \ldots \leq s_n \) as its imbalance and score sequences. As tournament \( T \) is self-converse so by Theorem 9[1], we have
\[ s_i + s_{n-i+1} = n - 1 \]
\[ \Rightarrow \quad d_i^+ + d_{n-i+1}^+ = n - 1 \]
\[ \Rightarrow \quad 2d_i^+ + 2d_{n-i+1}^+ = 2(n - 1) \]
\[ \Rightarrow \quad d_i^+ - (n - 1 - d_i^+) + d_{n-i+1}^+ - (n - 1 - d_{n-i+1}^+) = 0 \]
\[ \Rightarrow \quad (d_i^+ - d_i^-) + (d_{n-i+1}^+ - d_{n-i+1}^-) = 0 \]
\[ \Rightarrow \quad b_i + b_{n-i+1} = 0. \]
This proves the necessity of Theorem. Converse also follows from Theorem 9.

Now we state a conjecture:

**Conjecture.** The above result is also true for oriented graphs.

Below we obtain following results on self-converse imbalance sequences of tournaments.

**Theorem 11.** If \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is an imbalance sequence of a tournament, then \( B + B' \) is a self-converse imbalance sequence.

**Proof.** Here \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is an imbalance sequence. By the definition of converse of imbalance sequence, \( B' = (-b_n, -b_{n-1}, \ldots, -b_1) \).

So, by the definition, we have

\[
B + B' = (b_1 - n, b_2 - n, \ldots, b_n - n, -b_n + n, \ldots, -b_1 + n) = (t_1, t_2, \ldots, t_{2n}), \text{ say}
\]

where

\[
t_i = \begin{cases} 
  b_i - n, & \text{for } 1 \leq i \leq n; \\
  -b_{n-i+1} + n, & \text{for } n + 1 \leq i \leq 2n.
\end{cases}
\]

Clearly \( t_i + t_{2n-i+1} = 0 \), for \( 1 \leq i \leq 2n \). Hence \( B + B' \) is self-converse.

**Theorem 12.** Let \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) be a self-converse imbalance sequence and \( C \) be any other imbalance sequence in nondecreasing order. Then \( C + B + C' \) is a self-converse imbalance sequence.

**Proof.** Suppose that \( C = (c_1, c_2, \ldots, c_m) \) with \( c_1 \leq c_2 \leq \ldots \leq c_m \) is an imbalance sequence of order \( m \). Then by definition of converse

\[
C' = (-c_m, -c_{m-1}, \ldots, -c_1)
\]

and by definition

\[
C + B + C' = (c_1 - m - n, c_2 - m - n, \ldots, c_m - m - n, b_1, b_2, \ldots, b_n, -c_m + m + n, -c_{m-1} + m + n, \ldots, -c_1 + m + n) = (r_1, r_2, \ldots, r_{2m+n}), \text{ say}
\]
where

\[ r_i = \begin{cases} 
  c_i - m - n, & 1 \leq i \leq m; \\
  b_i - m, & m + 1 \leq i \leq m + n; \\
  -c_{2m+m-i+1} + m + n, & m + n + 1 \leq i \leq 2m + n.
\end{cases} \]

**Case (i).** For \( 1 \leq j \leq m \),

\[ r_j + r_{2m+n-j+1} = c_j - m - n - c_j + m + n \]

\{ when \( 1 \leq j \leq m \) then \( m + n + 1 \leq 2m + n - j + 1 \leq 2m + 1 \} \]

\[ \Rightarrow r_j + r_{2m+n-j+1} = 0. \]

**Case (ii).** For \( m + 1 \leq j \leq m + n \),

\[ r_j + r_{2m+n-j+1} = b_j - m + b_{m+n-j+1} \]

\[ = b_k + b_{n-k+1} \text{ for } k = j - m \text{ and } 1 \leq k \leq n \]

\[ = 0. \]

As \( B = (b_1, b_2, \ldots, b_n) \) is a self-converse imbalance sequence, so \( b_i + b_{n-i+1} = 0 \), for \( 1 \leq i \leq n \). From above we have, \( r_j + r_{2m+n-j+1} = 0 \), for \( 1 \leq j \leq 2m + n \). Hence \( C + B + C' \) is a self-converse imbalance sequence.

**References**


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