INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION-II

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ABSTRACT. In the present paper we have studied invariant submanifolds of Kenmotsu manifolds admitting a semi-symmetric metric connection and obtained some interesting results.


Keywords: Invariant submanifolds, Kenmotsu manifold, Semi-symmetric metric connection.

1. Semi-symmetric metric connection

The study of the geometry of invariant submanifolds of Kenmotsu manifolds is carried out by C.S. Bagewadi and V.S. Prasad [9], S. Sular and C. Ozgur [12] and M. Kobayashi [8]. The author [8] has shown that the submanifold \( M \) of a Kenmotsu manifold \( \tilde{M} \) has parallel second fundamental form if and only if \( M \) is totally geodesic. The authors [8, 9, 12] have shown the equivalence of totally geodesicity of \( M \) with parallelism and semiparallelism of \( \sigma \). Also they have shown that invariant submanifold of Kenmotsu manifold carried Kenmotsu structure and \( K \leq \tilde{K} \), then \( M \) is totally geodesic. Further the author [12] have shown the equivalence of totally geodesicity of \( M \), if \( \sigma \) is recurrent, \( M \) has parallel third fundamental form and \( \sigma \) is generalized 2-recurrent. In this paper we extend the results to invariant submanifolds \( M \) of Kenmotsu Manifolds admitting Semi-symmetric metric connection.

We know that a connection \( \nabla \) on a manifold \( M \) is called a metric connection if there is a Riemannian metric \( g \) on \( M \) if \( \nabla g = 0 \) otherwise it is non-metric. Further it is said to be semi-symmetric if its torsion tensor \( T(X,Y) = 0 \) i.e., \( T(X,Y) = w(Y)X - w(X)Y \), where \( w \) is a 1-form. In 1924, A. Friedmann and J.A. Schouten [6] introduced the idea of semi-symmetric linear connection on differentiable manifold. In 1932, H.A. Hayden [7] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection
on a Riemannian manifold was published by K. Yano [13] in 1970. After that the properties of semi-symmetric metric connection have studied by many authors like K.S. Amur and S.S. Pujar [1], C.S. Bagewadi, D.G. Prakasha and Venkatesha [2, 3], A. Sharfuddin and S.I. Hussain [11], U.C. De and G. Pathak [5] etc. If $\nabla$ denotes semi-symmetric metric connection on a contact metric manifold, then it is given by

$$\nabla_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi,$$

where $\eta(Y) = g(Y,\xi).

2. Isometric Immersion

Let $f: (M,g) \rightarrow (\tilde{M},\tilde{g})$ be an isometric immersion from an $n$-dimensional Riemannian manifold $(M,g)$ into $(n+d)$-dimensional Riemannian manifold $(\tilde{M},\tilde{g})$, $n \geq 2$, $d \geq 1$. We denote by $\nabla$ and $\tilde{\nabla}$ as Levi-Civita connection of $M$ and $\tilde{M}$ respectively. Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),$$

$$(\tilde{\nabla}_X N) = -A_N X + \nabla^\perp_X N,$$

for any tangent vector fields $X,Y$ and the normal vector field $N$ on $M$, where $\sigma$, $A$ and $\nabla^\perp$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form $\sigma$ is identically zero, then the manifold is said to be totally geodesic. The second fundamental form $\sigma$ and $A_N$ are related by

$$\tilde{g}(\sigma(X,Y),N) = g(A_N X,Y),$$

for tangent vector fields $X,Y$. The first and second covariant derivatives of the second fundamental form $\sigma$ are given by

$$(\tilde{\nabla}_X \sigma)(Y,Z) = \nabla^\perp_X (\sigma(Y,Z)) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z),$$

$$(\tilde{\nabla}^2\sigma)(Z,W,X,Y) = (\tilde{\nabla}_X \tilde{\nabla}_Y \sigma)(Z,W),$$

$$= \nabla^\perp_X ((\tilde{\nabla}_Y \sigma)(Z,W)) - (\tilde{\nabla}_Y \sigma)(\nabla_X Z,W) - (\nabla_X \sigma)(Z,\nabla_Y W) - (\nabla_{\nabla_X Y \sigma})(Z,W)$$

respectively, where $\tilde{\nabla}$ is called the vander Waerden-Bortolotti connection of $M$ [4]. If $\tilde{\nabla}\sigma = 0$, then $M$ is said to have parallel second fundamental form [4]. We next define endomorphisms $R(X,Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y.$$
respectively, where \( X, Y, Z \in \chi(M) \) and \( B \) is a symmetric \((0, 2)\)-tensor.

Now, for a \((0, k)\)-tensor field \( T \), \( k \geq 1 \) and a \((0, 2)\)-tensor field \( B \) on \((M, g)\), we define the tensor \( Q(B, T) \) by

\[
Q(B, T)(X_1, \ldots, X_k; X, Y) = - (T(X \wedge_B Y)X_1, \ldots, X_k) \\
- \cdots - T(X_1, \ldots, X_{k-1})(X \wedge_B Y)X_k.
\]

Putting into the above formula \( T = \tilde{\nabla}_\sigma \) and \( B = g, S \), we obtain the tensors \( Q(g, \tilde{\nabla}_\sigma) \) and \( Q(S, \tilde{\nabla}_\sigma) \).

### 3. Kenmotsu Manifolds

Let \( \tilde{M} \) be a \( n \)-dimensional almost contact metric manifold with structure \((\phi, \xi, \eta, g)\), where \( \phi \) is a tensor field of type \((1, 1)\), \( \xi \) is a vector field, \( \eta \) is a 1-form and \( g \) is the Riemannian metric satisfying

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi \xi = 0,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),
\]

for all vector fields \( X, Y \) on \( M \). If

\[
(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,
\]

\[
\nabla_X \xi = X - \eta(X)\xi,
\]

where \( \nabla \) denotes the Riemannian connection of \( g \), then \((M, \phi, \xi, \eta, g)\) is called an almost Kenmotsu manifold.

Example of Kenmotsu manifold: Consider the 3-dimensional manifold \( M = \{(x, y, z) \in R^3 : z \neq 0\} \), where \((x, y, z)\) are the standard co-ordinates in \( R^3 \). Let \( \{E_1, E_2, E_3\} \) be linearly independent global frame field on \( M \) given by

\[
E_1 = -e^z \frac{\partial}{\partial x}, \quad E_2 = -e^z \frac{\partial}{\partial y}, \quad E_3 = -\frac{\partial}{\partial z}.
\]

Let \( g \) be the Riemannian metric defined by

\[
g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.
\]

The \((\phi, \xi, \eta)\) is given by

\[
\eta = -dz, \quad \xi = E_3 = \frac{\partial}{\partial z},
\]

\[
\phi E_1 = E_2, \quad \phi E_2 = -E_1, \quad \phi E_3 = 0.
\]
The linearity property of $\phi$ and $g$ yields that
\[
\eta(E_3) = 1, \quad \phi^2 U = -U + \eta(U)E_3,
\]
\[
g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W),
\]
for any vector fields $U, W$ on $M$. By definition of Lie bracket, we have
\[
[E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.
\]
The Levi-Civita connection with respect to above metric $g$ is given by Koszula formula
\[
2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]
Then we have,
\[
\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1,
\]
\[
\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = E_2,
\]
\[
\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.
\]
The tangent vectors $X$ and $Y$ to $M$ are expressed as linear combination of $E_1, E_2, E_3$, i.e., $X = a_1 E_1 + a_2 E_2 + a_3 E_3$ and $Y = b_1 E_1 + b_2 E_2 + b_3 E_3$, where $a_i$ and $b_j$ are scalars. Clearly $(\phi, \xi, \eta, g)$ and $X, Y$ satisfy equations (8), (9), (10) and (11). Thus $M$ is a Kenmotsu manifold.

In Kenmotsu manifolds the following relations hold:
\[
R(X, Y)Z = \{g(X, Z)Y - g(Y, Z)X\},
\]
\[
R(X, Y)\xi = \{\eta(X)Y - \eta(Y)X\},
\]
\[
R(\xi, X)Y = \{\eta(Y)X - g(X, Y)\xi\},
\]
\[
R(\xi, X)\xi = \{X - \eta(X)\xi\},
\]
\[
S(X, \xi) = -(n - 1)\eta(X),
\]
\[
Q \xi = -(n - 1)\xi.
\]

4. Invariant submanifolds of Kenmotsu manifolds admitting
Semi-symmetric metric connection

A submanifold $M$ of a Kenmotsu manifold $\tilde{M}$ with a semi-symmetric metric connection is called an invariant submanifold of $\tilde{M}$ with a semi-symmetric metric connection, if for each $x \in M$, $\phi(T_x M) \subset T_x M$. As a consequence, $\xi$ becomes tangent
to $M$. For an invariant submanifold of a Sasakian manifold with a semi-symmetric metric connection, we have
\[ \sigma(X, \xi) = 0, \quad (18) \]
for any vector $X$ tangent to $M$.

Let $M$ be a Kenmotsu manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$.

**Lemma 1.** Let $M$ be an invariant submanifold of contact metric manifold $\tilde{M}$ which admits semi-symmetric metric connection $\tilde{\nabla}$ and let $\sigma$ and $\tilde{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and semi-symmetric metric connection, then (1) $M$ admits semi-symmetric metric connection, (2) the second fundamental forms with respect to $\tilde{\nabla}$ and $\tilde{\nabla}$ are equal.

**Proof.** We know that the contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\tilde{M}$ induces $(\phi, \xi, \eta, g)$ on invariant submanifold. By virtue of (1), we get
\[ \tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi. \quad (19) \]
By using (2) in (19), we get
\[ \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) + \eta(Y)X - g(X,Y)\xi. \quad (20) \]
Now Gauss formula (2) with respect to semi-symmetric metric connection is given by
\[ \tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \tilde{\sigma}(X,Y). \quad (21) \]
Equating (20) and (21), we get (1) and
\[ \tilde{\sigma}(X,Y) = \sigma(X,Y). \quad (22) \]

We know that an invariant submanifold of Kenmotsu manifold is also Kenmotsu. Also by the above lemma an invariant submanifold which admits semi-symmetric metric connection. Further second fundamental forms are equal. Hence by (18) and (22),
\[ \tilde{\sigma}(X, \xi) = \sigma(X, \xi). \quad (23) \]

**Definition 1.** An immersion is said to be 2-semi, 2-pseudo and 2-Ricci-generalized pseudoparallel with respect to semi-symmetric metric connection, respectively, if the following conditions hold for all vector fields $X, Y$ tangent to $M$
\[ \tilde{\nabla} \cdot \sigma = 0, \quad (24) \]
\[ \tilde{\nabla} \cdot \sigma = L_1 Q(g, \tilde{\nabla}\sigma) \quad \text{and} \quad (25) \]
\[ \tilde{\nabla} \cdot \sigma = L_2 Q(S, \tilde{\nabla}\sigma), \quad (26) \]
where $\overline{R}$ denotes the curvature tensor with respect to connection $\overline{\nabla}$. Here $L_1$ and $L_2$ are functions depending on $\overline{\nabla}\sigma$.

**Lemma 2.** Let $M$ be an invariant submanifold of contact manifold $\widetilde{M}$ which admits semi-symmetric metric connection. Then Gauss and Weingarten formulae with respect to semi-symmetric metric connection are given by

$$\tan(\overline{R}(X, Y)Z) = R(X, Y)Z + \eta(\nabla_Y Z)X - g(X, \nabla_Y Z)\xi + \eta(Z)\nabla_X Y - \eta(Z)\nabla_X X - \eta(\nabla_X Z)Y + g(Y, \nabla_X Z)\xi - \eta(Z)\nabla_Y X$$

(27)

$$+ \eta(Z)\eta(Y)X - \eta(Z)g(X, Y)\xi - \eta(\nabla_X Z)Y + g(Y, \nabla_X Z)\xi - \eta(Z)\nabla_Y X - \eta(Z)\nabla(X, Y)Z + \eta(Z)\nabla(Y, X)Z + g([X, Y], Z)\xi - \eta(Z)[X, Y]$$

$$+ \tan \left\{ \overline{\nabla}_X \left\{ \sigma(Y, Z) \right\} - \overline{\nabla}_Y \left\{ \sigma(X, Z) \right\} - (\overline{\nabla}_Y \eta(Z))X - (\overline{\nabla}_X \eta(Z))Y - (\overline{\nabla}_X g(Y, Z))\xi + (\overline{\nabla}_Y g(X, Z))\xi \right\}.$$  

(28)

**Proof.** The Riemannian curvature tensor $\overline{R}$ on $\widetilde{M}$ with respect to semi-symmetric metric connection is given by

$$\overline{R}(X, Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X, Y]} Z.$$  

(29)

Using (1) and (2) in (29), we get

$$\overline{R}(X, Y)Z = R(X, Y)Z + \sigma(X, \nabla_Y Z) + \eta(\nabla_Y Z)X - g(X, \nabla_Y Z)\xi$$

(30)

$$+ \overline{\nabla}_X \left\{ \sigma(Y, Z) \right\} + (\overline{\nabla}_X \eta(Z))Y + \eta(Z)\nabla_X Y + \eta(Z)\sigma(X, Y) + \eta(Z)\eta(Y)X - \eta(Z)g(X, Y)\xi - (\overline{\nabla}_X g(Y, Z))\xi - \sigma(Y, \nabla_X Z) - \eta(\nabla_X Z)Y + g(Y, \nabla_X Z)\xi$$

$$- \overline{\nabla}_Y \left\{ \sigma(X, Z) \right\} - (\overline{\nabla}_Y \eta(Z))X - \eta(Z)\nabla_Y X - \eta(Z)\sigma(Y, X) - \eta(Z)\eta(X)Y + \eta(Z)g(Y, X)\xi + (\overline{\nabla}_Y g(X, Z))\xi - \sigma([X, Y], Z) - \eta(Z)[X, Y] + g([X, Y], Z)\xi.$$  

Comparing tangential and normal part of (30), we obtain Gauss and Weingarten formulae (27) and (28).

We obtain the condition in the following lemma for 2-semi, 2-pseudo, 2-Ricci-generalized pseudoparallelism for invariant submanifold $M$ of Kenmotsu manifold $\widetilde{M}$.
Lemma 3. Let $M$ be an invariant submanifold of contact manifold $\tilde{M}$ which admits semi-symmetric metric connection. Then
\begin{align}
(R(X,Y) \cdot \overline{\nabla}) (U,V,W) &= \overline{R}(X,Y) \{ \nabla^\perp_U \sigma(V,W) - \sigma(\nabla^\perp_U V,W) \\
- \sigma(V,\nabla^\perp_U W) \} - \overline{\nabla} \sigma(\overline{R}(X,Y) U,V,W) - \overline{\nabla} \sigma(U,\overline{R}(X,Y) V,W) \\
- \overline{\nabla} \sigma(U,V,\overline{R}(X,Y) W) - \eta(\nabla_Y U) \overline{\nabla} \sigma(X,V,W) + g(X,\nabla_Y U) \overline{\nabla} \sigma(\xi,V,W) \\
- \overline{\nabla} \sigma(\tan(\nabla_X \{ \sigma(Y,U) \}),V,W) - \overline{\nabla} \sigma(\eta(\nabla_X \eta(U))) Y,V,W) - \eta(U) \overline{\nabla} \sigma(\nabla_X Y,V,W) \\
- \eta(U) \eta(Y) \overline{\nabla} \sigma(X,V,W) + \eta(U) g(X,Y) \overline{\nabla} \sigma(\xi,V,W) + \overline{\nabla} \sigma((\nabla_x g(Y,U)) \xi,V,W) \\
+ \eta(\nabla_X U) \overline{\nabla} \sigma(Y,V,W) - g(Y,\nabla_U U) \overline{\nabla} \sigma(\xi,V,W) + \overline{\nabla} \sigma(\tan(\nabla_Y \{ \sigma(X,U) \}),V,W) \\
+ \overline{\nabla} \sigma((\nabla_\gamma \eta(U)) X,V,W) + \eta(U) \overline{\nabla} \sigma(\xi_\gamma X,V,W) + \eta(U) \eta(X) \overline{\nabla} \sigma(Y,V,W) \\
- \eta(U) g(Y,X) \overline{\nabla} \sigma(\xi,Y,V,W) - \overline{\nabla} \sigma((\nabla_x g(Y,U)) \xi,Y,V,W) + \eta(U) \overline{\nabla} \sigma([X,Y],V,W) \\
g([X,Y],U) \overline{\nabla} \sigma(\xi,Y,V,W) - \overline{\nabla} \sigma(\eta(\nabla_Y V) \overline{\nabla} \sigma(U,X,W) + g(X,\nabla_Y V) \overline{\nabla} \sigma(U,\xi,W) \\
- \overline{\nabla} \sigma(U,\nabla_t(\nabla_X \{ \sigma(Y,V) \}),V,W) - \overline{\nabla} \sigma(U,\nabla_t(\nabla_Y \{ \sigma(X,V) \}),W) \\
+ \eta(\nabla_X V) \overline{\nabla} \sigma(U,Y,W) - g(Y,\nabla_X V) \overline{\nabla} \sigma(U,\xi,W) + \overline{\nabla} \sigma(U,\nabla_t(\nabla_Y \{ \sigma(X,V) \}),W) \\
+ \overline{\nabla} \sigma(U,\nabla_t(\nabla_Y \eta(V)) X,W) + \eta(V) \overline{\nabla} \sigma(U,\xi_\gamma X,W) + \eta(V) \eta(X) \overline{\nabla} \sigma(U,Y,W) \\
- \eta(V) g(Y,X) \overline{\nabla} \sigma(U,\xi,W) - \overline{\nabla} \sigma(U,\nabla_t(Y g(X,U)) \xi,W) + \eta(V) \overline{\nabla} \sigma(U,[X,Y],W) \\
g([X,Y],V) \overline{\nabla} \sigma(\xi,Y,V,W) - \overline{\nabla} \sigma(\eta(\nabla_Y W) \overline{\nabla} \sigma(U,V,X) + g(X,\nabla_Y W) \overline{\nabla} \sigma(U,V,\xi) \\
- \overline{\nabla} \sigma(U,V,\nabla_t(\nabla_X \{ \sigma(Y,W) \})) \xi,V,W) - \overline{\nabla} \sigma(U,V,\nabla_t(W g(Y,U)) \xi,Y,V,W) \\
- \eta(W) \eta(Y) \overline{\nabla} \sigma(U,V,X) + \eta(W) g(X,Y) \overline{\nabla} \sigma(U,V,\xi) + \overline{\nabla} \sigma(U,V,\nabla_t(W g(Y,U)) \xi,X,V,W) \\
+ \eta(\nabla_X W) \overline{\nabla} \sigma(U,V,Y) - g(Y,\nabla_X W) \overline{\nabla} \sigma(U,V,\xi) + \overline{\nabla} \sigma(U,V,\nabla_t(\nabla_Y \{ \sigma(X,W) \})) \\
+ \overline{\nabla} \sigma(U,V,\nabla_t(\nabla_Y \eta(W)) X) + \eta(W) \overline{\nabla} \sigma(U,V,\nabla_Y X) + \eta(W) \eta(X) \overline{\nabla} \sigma(U,V,Y) \\
- \eta(W) g(Y,X) \overline{\nabla} \sigma(U,V,\xi) - \overline{\nabla} \sigma(U,V,\nabla_t(W g(Y,X)) \xi) + \eta(W) \overline{\nabla} \sigma(U,V,[X,Y]) \\
- g([X,Y],W) \overline{\nabla} \sigma(U,V,\xi),
\end{align}
for all vector fields $X,Y,U$ and $V$ tangent to $M$, where
\[
R^\perp(X,Y) = [\nabla^\perp_X, \nabla^\perp_Y] - \nabla^\perp_{[X,Y]}.
\]

Proof. We know, from tensor algebra, that
\begin{align}
(R(X,Y) \cdot \overline{\nabla}) (U,V,W) &= \overline{R}(X,Y) \overline{\nabla} \sigma(U,V,W) - \overline{\nabla} \sigma(\overline{R}(X,Y) U,V,W) \\
- \overline{\nabla} \sigma(U,\overline{R}(X,Y) Y,V,W) - \overline{\nabla} \sigma(U,V,\overline{R}(X,Y) W).
\end{align}
We write the equation (4) with respect to semi-symmetric metric connection and in the form, we have the following equalities:

$$\overline{\nabla} \sigma(U, V, W) = \nabla_{U}^{\perp} \sigma(V, W) - \sigma(\nabla_{U} V, W) - \sigma(V, \nabla_{U} W).$$  \hspace{1cm} (33)

By using (30) in $\overline{\nabla} \sigma(\overline{R}(X, Y)U, V, W), \overline{\nabla} \sigma(U, \overline{R}(X, Y) V, W)$ and $\overline{\nabla} \sigma(U, V, \overline{R}(X, Y) W)$ to get

$$\overline{\nabla} \sigma(\overline{R}(X, Y)U, V, W) = \overline{\nabla} \sigma(R(X, Y)U, V, W) + \eta(\nabla_{Y} \overline{\nabla} \sigma(U, V, W)) + \overline{\nabla} \sigma(U, \overline{\nabla} \sigma(\nabla_{X} Y, V, W) + \eta(U) \eta(Y) \overline{\nabla} \sigma(X, V, W))$$

$$- \eta(U) g(X, Y) \overline{\nabla} \sigma(U, V, W) - \eta(U) g(Y, X) \overline{\nabla} \sigma(U, V, W) + \eta(U) g(Y, X) \overline{\nabla} \sigma([X, Y], V, W)$$

and

$$\overline{\nabla} \sigma(U, \overline{R}(X, Y) V, W) = \overline{\nabla} \sigma(U, R(X, Y) V, W) + \eta(\nabla_{Y} \overline{\nabla} \sigma(U, X, W))$$

$$- g(X, \nabla_{Y} V) \overline{\nabla} \sigma(U, V, W) + \eta(\nabla_{Y} \overline{\nabla} \sigma(U, V, W)) + \eta(V) \eta(Y) \overline{\nabla} \sigma(U, X, W)$$

$$- \eta(U) g(X, Y) \overline{\nabla} \sigma(U, V, W) - \eta(U) g(Y, X) \overline{\nabla} \sigma(U, V, W) + \eta(U) g(Y, X) \overline{\nabla} \sigma([X, Y], V, W)$$

$$+ g([X, Y], V) \overline{\nabla} \sigma(U, X, W).$$

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Substituting (33) – (36) into (32), we get (31).

5. 2-Semiparallel, 2-Pseudoparallel and 2-Ricci-Generalized Pseudoparallel Invariant Submanifolds of Kenmotsu Manifolds Admitting Semi-symmetric Metric Connection

We consider invariant submanifolds of Kenmotsu manifolds admitting semi-symmetric metric connection satisfying the conditions \( \overline{R} \cdot \overline{\nabla} \sigma = 0, \overline{R} \cdot \overline{\nabla} \sigma = L_1 Q(g, \overline{\nabla} \sigma), \overline{R} \cdot \overline{\nabla} \sigma = L_2 Q(S, \overline{\nabla} \sigma) \). We write the equation (4) with respect to semi symmetric metric connection in the form

\[
(\overline{\nabla}_X \sigma)(Y, Z) = \overline{\nabla}_X(\sigma(Y, Z)) - \sigma(\overline{\nabla}_X Y, Z) - \sigma(Y, \overline{\nabla}_X Z), \quad (37)
\]

and prove the following theorems

**Theorem 4.** Let \( M \) be an invariant submanifold of a Kenmotsu manifold \( \overline{M} \) admitting a semi-symmetric metric connection. Then \( M \) is 2-semiparallel with respect to semi symmetric metric connection if and only if the derivative of the second fundamental form with respect to the characteristic vector \( \xi \) is zero \( \Rightarrow \sigma(U, Y) = \text{constant} \).

**Proof.** Let \( M \) be 2-semiparallel satisfying \( \overline{R} \cdot \overline{\nabla} \sigma = 0 \). Put \( X = V = \xi \) and use (8), (11) and (23) in (31) to get

\[
\begin{align*}
0 &= -\overline{R}(\xi, Y) \left\{ \sigma(\nabla_U \xi, W) + \sigma(\xi, \nabla_U W) \right\} - \overline{\nabla} \sigma(R(\xi, Y)U, \xi, W) \\
&- \overline{\nabla} \sigma(U, \nabla(\xi, Y)\xi, W) - \overline{\nabla} \sigma(U, \xi, R(\xi, Y)W) - \overline{\nabla} \sigma(tan(\nabla_\xi \{ \sigma(Y, U) \}), \xi, W) \\
&- \overline{\nabla} \sigma(\nabla_\xi (\nabla_\eta(U))), Y, \xi, W) - \eta(U)\overline{\nabla} \sigma(\nabla_\xi Y, \xi, W) + \overline{\nabla} \sigma(\nabla_\xi (g(Y, U))\xi, \xi, W) \\
&+ \eta(\nabla_\xi U)\overline{\nabla} \sigma(Y, \xi, W) - g(Y, \nabla_\xi U)\overline{\nabla} \sigma(\xi, \xi, W) + \overline{\nabla} \sigma(\nabla_\xi (\overline{\nabla} \eta(U)))\xi, \xi, W) \\
&+ \eta(U)\overline{\nabla} \sigma(\nabla_\eta \xi, \xi, W) + \eta(U)\overline{\nabla} \sigma(Y, \xi, W) - \eta(U)\eta(Y)\overline{\nabla} \sigma(\xi, \xi, W) \\
&- \overline{\nabla} \sigma(\nabla_\eta (\nabla_Y U)))\xi, \xi, W) + \eta(U)\overline{\nabla} \sigma([\xi, Y]), \xi, W) - g([\xi, Y], U)\overline{\nabla} \sigma(\xi, \xi, W) \\
&- \overline{\nabla} \sigma(U, \nabla_\xi Y, W) - \overline{\nabla} \sigma(U, \nabla_\xi Y, W) + \overline{\nabla} \sigma(U, \overline{\nabla}_\xi(\nabla_\eta(Y)))\xi, W) + \overline{\nabla} \sigma(U, \overline{\nabla}_\xi Y, W) \\
&+ \overline{\nabla} \sigma(U, \nabla_\xi \xi, W) + \overline{\nabla} \sigma(U, Y, W) - \eta(Y)\overline{\nabla} \sigma(U, \xi, Y) - \overline{\nabla} \sigma(U, \overline{\nabla}_\xi Y, W) \\
&+ \overline{\nabla} \sigma(U, [\xi, Y], W) - \eta([\xi, Y])\overline{\nabla} \sigma(U, \xi, W) - \overline{\nabla} \sigma(U, \xi, tan(\nabla_\xi \{ \sigma(Y, W) \})) \\
&- \overline{\nabla} \sigma(U, \xi, (\overline{\nabla}_\eta(\nabla(Y)))W) - \eta(W)\overline{\nabla} \sigma(U, \xi, \nabla_\xi Y) + \overline{\nabla} \sigma(U, \xi, (\overline{\nabla}_\xi g(Y, W))\xi)
\end{align*}
\]
+η(∇ξW)Wσ(U,ξ,Y)−g(Y,∇ξW)Wσ(U,ξ,ξ) +Wσ(U,ξ,ξ) +η(W)Wσ(U,ξ,Y)−η(W)η(Y)Wσ(U,ξ,ξ

In view of (1), (8), (11), (14), (15), (23) and (37), we have the following equalities:

\[ \nabla_\sigma (R(\xi,Y)U,\xi,W) = (\nabla_{R(\xi,Y)U} \sigma)(\xi,W), \quad (39) \]

\[ = \nabla^L_{R(\xi,Y)U} \sigma(\xi,W) - \sigma(\nabla_{R(\xi,Y)U} \xi, W) - \sigma(\xi, \nabla_{R(\xi,Y)U} W), \]

\[ = -2\eta(U)\sigma(Y,W), \]

\[ \nabla_\sigma (U,R(\xi,Y)\xi,W) = (\nabla_U \sigma)(R(\xi,Y)\xi,W), \quad (40) \]

\[ = \nabla^L_U \sigma(R(\xi,Y)\xi,W) - \sigma(\nabla_U R(\xi,Y)\xi, W) - \sigma(R(\xi,Y)\xi, \nabla_U W), \]

\[ = \nabla^L_U \sigma(Y - \eta(Y)\xi, W) - \sigma(\nabla_U \{Y - \eta(Y)\xi\}, W) \]

\[ - \sigma(Y, \nabla_U W) \]

and

\[ \nabla_\sigma (U,\xi,R(\xi,Y)W) = (\nabla_U \sigma)(\xi,R(\xi,Y)W), \quad (41) \]

\[ = \nabla^L_U \sigma(\xi,R(\xi,Y)W) - \sigma(\nabla_U \xi, R(\xi,Y)W) - \sigma(\xi, \nabla_U R(\xi,Y)W), \]

\[ = -2\eta(W)\sigma(U,Y). \]

Substituting (39 – 41) into (38) and W = ξ, using (1), (2), (8), (11), (23) and (37), we get

\[ \sigma(\nabla_\xi U,Y) = 0. \quad (42) \]

Interchanging Y and U in (42), we get

\[ \sigma(\nabla_\xi Y,U) = 0. \quad (43) \]

Adding (42) and (43), we get ξσ(U,Y) = 0 that is the derivative of the second fundamental form with respect to the characteristic vector ξ is zero ⇒ σ(U,Y) = constant.

**Theorem 5.** Let M be an invariant submanifold of a Kenmotsu manifold \( \tilde{M} \) admitting a semi-symmetric metric connection. Then M is 2-pseudoparallel with respect to semi-symmetric metric connection if and only if the derivative of the second fundamental form with respect to the characteristic vector ξ is zero ⇒ σ(U,Y) = constant.
Proof. Let $M$ be 2-pseudoparallel satisfying \( \widehat{R} \cdot \overline{\nabla} \sigma = L_1 Q(g, \overline{\nabla} \sigma) \). Put $X = V = \xi$ and use (8), (11) and (23) in (7), (31) to get

\begin{align*}
-\widehat{R}(\xi, Y) \{ \sigma(\overline{\nabla}_U \xi, W) + \sigma(\xi, \overline{\nabla}_U W) \} - \overline{\nabla} \sigma(R(\xi, Y)U, \xi, W) \\
- \overline{\nabla} \sigma(U, R(\xi, Y) \xi, W) - \overline{\nabla} \sigma(U, \xi, R(\xi, Y)W) - \overline{\nabla} \sigma(\tan(\overline{\xi} \{ \sigma(Y, U) \}), \xi, W) \\
- \overline{\nabla} \sigma((\overline{\xi} \eta(U))Y, \xi, W) - \eta(U)\overline{\nabla} \sigma(\overline{\xi} \xi, \xi, W) + \overline{\nabla} \sigma((\overline{\xi} g(Y, U)) \xi, \xi, W) \\
+ \eta(\overline{\nabla}_U \overline{\nabla} \sigma(Y, \xi, W) - g(Y, \overline{\nabla}_U \overline{\nabla} \sigma(\xi, \xi, W) + \overline{\nabla} \sigma((\overline{\xi} Y, \eta(U)) \xi, \xi, W) \\
+ \eta(\overline{\nabla}_U \overline{\nabla} \sigma(Y, \xi, W) + \eta(\overline{\nabla}_U \overline{\nabla} \sigma(Y, \xi, W) - \eta(U)\overline{\nabla} \sigma(Y, \xi, W) - \overline{\nabla} \sigma(U, \overline{\nabla}_Y \xi, W) \\
+ \overline{\nabla} \sigma(U, \overline{\nabla}_Y \xi, W) + \overline{\nabla} \sigma(U, Y, W) - \eta(\overline{\nabla} \sigma(U, \xi, W) - \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi, W) \\
+ \overline{\nabla} \sigma(U, \overline{\nabla}_Y \xi, W) - \eta([\xi, Y]) \overline{\nabla} \sigma(U, \xi, W) - \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi, W) \\
+ \overline{\nabla} \sigma(U, \overline{\nabla}_Y \xi, W) - \eta(W) \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) + \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi, W) \\
+ \eta(\overline{\nabla}_W \eta(U)) \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) + \eta(W) \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) - \eta(W)\overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) \\
- \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \eta(W)) \xi) + \eta(W) \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) - \eta(W)\overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) \\
- \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \eta(W)) \xi) + \eta(W) \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) - \eta(W)\overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) \\
- \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \eta(W)) \xi) + \eta(W) \overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi) - \eta(W)\overline{\nabla} \sigma(U, \xi, \overline{\nabla}_Y \xi)
\end{align*}

Substituting (39 – 41) into (44) and $W = \xi$, using (1), (2), (8), (11), (18) and (37), we get

\begin{equation}
\sigma(\overline{\nabla}_U Y, Y) = 0. \tag{45}
\end{equation}

Interchanging $Y$ and $U$ in (45), we get

\begin{equation}
\sigma(\overline{\nabla}_Y U, Y) = 0. \tag{46}
\end{equation}

Adding (45) and (46), we get $\xi \sigma(U, Y) = 0$ that is the derivative of the second fundamental form with respect to the characteristic vector $\xi$ is zero $\Rightarrow \sigma(U, Y) = \text{constant}$.

**Theorem 6.** Let $M$ be an invariant submanifold of a Kenmotsu manifold $\widehat{M}$ admitting a semi-symmetric metric connection. Then $M$ is 2-Ricci-generalized pseudoparallel with respect to semi-symmetric metric connection if and only if the derivative...
of the second fundamental form with respect to the characteristic vector $\xi$ is zero
$\Rightarrow \sigma(U, Y) = \text{constant.}$

Proof. Let $M$ be 2-Ricci-generalized pseudoparallel satisfying $\overline{R} \overline{\nabla} \sigma = L_2 Q(S, \overline{\nabla} \sigma)$. Put $X = V = \xi$ and use (8), (11), (16) and (23) in (7), (31) to get

\[
\begin{align*}
-\overline{R}(\xi, Y) \{ \sigma(\overline{\nabla} U, \xi, W) + \sigma(\xi, \overline{\nabla} U) W \} - \overline{\nabla} \sigma(R(\xi, Y) U, \xi, W) \quad (47) \\
-\overline{\nabla} \sigma(U, R(\xi, Y) W) - \overline{\nabla} \sigma(U, R(\xi, Y) W) - \overline{\nabla} \sigma(tan(\overline{\nabla} \xi, \{ \sigma(Y, U) \}), \xi, W) \\
-\overline{\nabla} \sigma((\overline{\nabla} \xi \eta(U)) Y, \xi, W) - \eta(U) \overline{\nabla} \sigma(\overline{\nabla} \xi Y, \xi, W) + \overline{\nabla} \sigma((\overline{\nabla} \xi \eta(U)) \xi, \xi, W) \\
+ \eta(\overline{\nabla} \sigma(\overline{\nabla} \xi Y, \xi, W) - g(Y, \overline{\nabla} U) \overline{\nabla} \sigma(\xi, \xi, W) + \overline{\nabla} \sigma((\overline{\nabla} \xi \eta(U)) \xi, \xi, W) \\
+ \eta(U) \overline{\nabla} \sigma(\overline{\nabla} \xi Y, \xi, W) + \eta(U) \overline{\nabla} \sigma(\xi, \xi, W) - \eta(U) \eta(Y) \overline{\nabla} \sigma(\xi, \xi, W) \\
- \overline{\nabla} \sigma((\overline{\nabla} \xi \eta(U)) \xi, \xi, W) + \eta(U) \overline{\nabla} \sigma([\xi, Y], \xi, W) - g([\xi, Y], U) \overline{\nabla} \sigma(\xi, \xi, W) \\
- \overline{\nabla} \sigma(U, \overline{\nabla} \xi Y, W) + \overline{\nabla} \sigma(U, \overline{\nabla} \xi Y, W) + \overline{\nabla} \sigma(U, (\overline{\nabla} \xi \eta(U)) \xi, W) + \overline{\nabla} \sigma(U, \overline{\nabla} \xi Y, W) \\
+ \overline{\nabla} \sigma(U, [\xi, Y], W) - \eta([\xi, Y]) \overline{\nabla} \sigma(U, \xi, W) - \overline{\nabla} \sigma(U, \xi, \xi, W) \\
- \overline{\nabla} \sigma(\xi, \xi, W) - \eta(W) \overline{\nabla} \sigma(U, \xi, \xi, W) + \overline{\nabla} \sigma(U, \xi, \xi, W) + \overline{\nabla} \sigma(U, (\overline{\nabla} \xi \eta(W)) \xi) \\
+ \eta([\xi, Y]) \overline{\nabla} \sigma(U, \xi, \xi, W) - \eta(W) \overline{\nabla} \sigma(U, \xi, \xi, W) + \overline{\nabla} \sigma(U, \xi, \xi, W) + \overline{\nabla} \sigma(U, (\overline{\nabla} \xi \eta(W)) \xi) \\
- \overline{\nabla} \sigma(U, [\xi, (\overline{\nabla} \xi \eta(W))] \xi) + \eta(W) \overline{\nabla} \sigma(U, \xi, \xi, W) - \eta(W) \eta(Y) \overline{\nabla} \sigma(U, \xi, \xi, W) \\
+ (n - 1) \eta(Y) \left( \nabla^\perp \xi \sigma(Y, U) - \sigma(\overline{\nabla} \xi Y, U) - \sigma(Y, \overline{\nabla} \xi U) \right) \\
+ \left( \nabla^\perp \xi \sigma(Y, U) - \sigma(\overline{\nabla} \xi Y, U) - \sigma(Y, \overline{\nabla} \xi U) \right) \right). \\
\end{align*}
\]

Substituting (39 – 41) into (47) and $W = \xi$, using (1), (2), (8), (11), (18) and (37), we get

\[
\sigma(\overline{\nabla} \xi U, Y) = 0. \quad (48)
\]

Interchanging $Y$ and $U$ in (48), we get

\[
\sigma(\overline{\nabla} \xi Y, U) = 0. \quad (49)
\]
Adding (48) and (49), we get $\xi \sigma(U,Y) = 0$ that is the derivative of the second fundamental form with respect to the characteristic vector $\xi$ is zero $\Rightarrow \sigma(U,Y) = \text{constant}$.

**Remark.** Let $M$ be an invariant submanifold of a Kenmotsu manifold which admits semi-symmetric metric connection. If $M$ is 2-semi, 2-pseudo and 2-Ricci-generalized pseudoparallel, then we have obtained conditions connecting $\xi$. These conditions need further investigation and are to be interpreted geometrically.

**References**


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