ON CERTAIN SUBCLASSES OF HOLOMORPHIC FUNCTIONS DEFINED ON THE UNIT DISK

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Abstract. We present several results for certain subclasses of the uniformly \( \alpha \) spirallike functions. These include distortion and covering theorems, extreme points, radii of close-to-convexity, starlikeness and convexity for these classes. We also obtain integral means inequalities with the extremal functions for these classes.

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1. Introduction, definition and preliminaries

Let \( A \) denote the class of all analytic functions

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are regular in the unit disk \( \Delta = \{ z : |z| < 1 \} \) and normalized by \( f(0) = 0 \), \( f'(0) = 1 \). The function \( f \in A \) is spirallike if \( \Re \{ e^{-i\alpha} z f'(z) / f(z) \} > 0 \) for all \( z \in \Delta \) and for some \( \alpha \) with \( |\alpha| < \pi/2 \). Also \( f(z) \) is convex spirallike if \( z f'(z) \) is spirallike.

The class of uniformly convex functions was introduced and studied by various authors as in [1, 2, 4, 5, 6].

Let \( T \) denote the class consisting of functions \( f \) of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \text{ where } a_n \text{ is a non-negative real number.}
\]

Silverman [9] introduced and investigated many subclasses of \( T \).

We now defined \( UCSPT(\alpha, \beta) \) and \( SPPT(\alpha, \beta) \).
Definition 1. [7] Let $UCSPT(\alpha, \beta)$ be the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ which satisfy the condition

$$\Re e^{-i\alpha} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \left| \frac{zf''(z)}{f'(z)} \right| + \beta,$$

$|\alpha| < \pi/2$, $0 \leq \beta < 1$.

Definition 2. [7] Let $SP_T(\alpha, \beta)$ be the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ which satisfy the condition

$$\Re e^{-i\alpha} \frac{zf'(z)}{f(z)} \geq \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta,$$

$|\alpha| < \pi/2$, $0 \leq \beta < 1$.

In this paper we discuss several results for the classes $UCSPT(\alpha, \beta)$ and $SP_T(\alpha, \beta)$ like distortion bounds, extreme points, radii of close-to-convexity, starlikeness and convexity. We also obtain integral means inequality for the functions belonging to this class.

For proving our results we require the following lemmas.

Lemma 1. [7] Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. Then

$$\sum_{n=2}^{\infty} (2n - \cos \alpha - \beta)a_n \leq \cos \alpha - \beta,$$

if and only if $f(z)$ is in $UCSPT(\alpha, \beta)$.

Lemma 2. [7] $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ is in $SP_T(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} (2n - \cos \alpha - \beta)a_n \leq \cos \alpha - \beta.$$

2. Distortion and covering theorems

Theorem 3. If $f(z) \in UCSPT(\alpha, \beta)$ then

$$r - \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} r^2 \leq |f(z)| \leq r + \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} r^2$$
and
\[ 1 - \frac{\cos \alpha - \beta}{4 - \cos \alpha - \beta} r \leq |f'(z)| \leq 1 + \frac{\cos \alpha - \beta}{4 - \cos \alpha - \beta} r \]
and the extreme points are
\[ f_1(z) = z, \quad f_n(z) = z - \frac{\cos \alpha - \beta}{n(2n - \cos \alpha - \beta)} z^n, \quad n = 2, 3, \ldots \]
The result is sharp for \( f(z) = z - \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} z^2, \quad z = \pm r. \)

**Proof.** \( f(z) \in UCSPT(\alpha, \beta). \) Hence by Lemma 1
\[ \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) na_n \leq \cos \alpha - \beta. \]
\[ \therefore \sum_{n=2}^{\infty} a_n \leq \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} \]
From \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \) with \( |z| = r \) \( (r < 1) \) we have
\[ |f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \]
\[ \leq r + \sum_{n=2}^{\infty} a_n r^2 \]
\[ \leq r + \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} r^2. \]

**Theorem 4.** If \( f(z) \in SP_{PT}(\alpha, \beta) \) then
\[ r - \frac{\cos \alpha - \beta}{4 - \cos \alpha - \beta} r^2 \leq |f(z)| \leq r + \frac{\cos \alpha - \beta}{4 - \cos \alpha - \beta} r^2. \]
The result is sharp for \( f(z) = z - \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} z^2, \quad z = \pm r. \)

**Proof.** From Lemma 2,
\[ \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) a_n \leq \cos \alpha - \beta. \]
∴ \sum_{n=2}^{\infty} a_n \leq \frac{\cos \alpha - \beta}{4 - \cos \alpha - \beta}

From \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \) with \( |z| = r \ (r < 1) \) we have

\[
|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \\
\leq r + \sum_{n=2}^{\infty} a_n r^2 \\
\leq r + \frac{\cos \alpha - \beta}{4 - \cos \alpha - \beta} r^2.
\]

Also

\[
1 - \frac{2(\cos \alpha - \beta)}{4 - \cos \alpha - \beta} r \leq |f'(z)| \leq 1 + \frac{2(\cos \alpha - \beta)}{4 - \cos \alpha - \beta} r
\]

and the extreme points are

\[
f_1(z) = z, \quad f_n(z) = z - \frac{\cos \alpha - \beta}{2n - \cos \alpha - \beta} z^n, \quad n = 2, 3, \ldots
\]

3. Integral means inequalities

In [9], Silverman found that the function \( f_2(z) = z - \frac{z^2}{2} \) is often extremal over the family \( T \). He applied this function to resolve his integral means inequality conjectured in [10] and settled in [11], that

\[
\int_0^{2\pi} |f( r e^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_2( r e^{i\theta})|^\eta d\theta, \text{ for all } f \in T, \eta > 0 \text{ and } 0 < r < 1.
\]

In [11], he also proved his conjecture for some subclasses of \( T \).

Now, we prove Silverman’s conjecture for the class of functions \( UCSPT(\alpha, \beta) \). An analogous result is also obtained for the family of functions \( SPT(\alpha, \beta) \).

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [3].

Two given functions \( f \) and \( g \), which are analytic in \( \Delta \), the function \( f \) is said to be subordinate to \( g \) in \( \Delta \) if there exists a function \( w \) analytic in \( \Delta \) with \( w(0) = 0 \), \( |w(z)| < 1 \ (z \in \Delta) \), such that \( f(z) = g(w(z)) \ (z \in \Delta) \). We denote this subordination by \( f(z) \prec g(z) \).
Lemma 5. If the functions \( f \) and \( g \) are analytic in \( D \) with \( f(z) \prec g(z) \) then for \( \eta > 0 \) and \( z = re^{i\theta} \) \((0 < r < 1)\)

\[
\int_{0}^{2\pi} |g(re^{i\theta})|^\eta \, d\theta \leq \int_{0}^{2\pi} |f(re^{i\theta})|^\eta \, d\theta.
\]

Now we discuss the integral means inequalities for \( UCSPT(\alpha, \beta) \).

Theorem 6. Let \( f \in UCSPT(\alpha, \beta), |\alpha| < \pi/2, 0 \leq \beta < 1 \) and \( f_2(z) \) be defined by

\[
f_2(z) = z - \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} z^2.
\]

Then for \( z = re^{i\theta}, 0 < r < 1 \), we have

\[
\int_{0}^{2\pi} |f(z)|^\eta \, d\theta \leq \int_{0}^{2\pi} |f_2(z)|^\eta \, d\theta \quad (2)
\]

Proof. For \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), (2) is equivalent to

\[
\int_{0}^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\eta \, d\theta \leq \int_{0}^{2\pi} \left| 1 - \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} z \right|^\eta \, d\theta.
\]

By Lemma 2 it is enough to prove that

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} z.
\]

Assuming

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{\cos \alpha - \beta}{2(4 - \cos \alpha - \beta)} w(z)
\]

and using \( \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) a_n \leq \cos \alpha - \beta \) we obtain

\[
|w(z)| = \left| \sum_{n=2}^{\infty} \frac{2(4 - \cos \alpha - \beta)}{\cos \alpha - \beta} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{n(2n - \cos \alpha - \beta)}{\cos \alpha - \beta} a_n \leq |z|.
\]

This completes the proof by Lemma 1.
For completeness, we now give the integral means inequality for the class \( \text{SP}_pT(\alpha, \beta) \).

**Theorem 7.** Let \( f \in \text{SP}_pT(\alpha, \beta) \), \( |\alpha| < \pi/2 \), \( 0 \leq \beta < 1 \) and \( f_2(z) \) is defined by
\[
f(z) = z - \frac{\cos \alpha - \beta}{4 - \cos \alpha - \beta} z^2.
\]
Then for \( z = re^{i\theta}, \) \( 0 < r < 1 \) we have
\[
\int_0^{2\pi} |f(z)|^n d\theta \leq \int_0^{2\pi} |f_2(z)|^n d\theta.
\]

4. Radii of close-to-convexity, starlikeness and convexity

**Theorem 8.** If \( f(z) \in \text{UCSPT}(\alpha, \beta) \) then \( f \) is close-to-convex of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in \( |z| < r_1(\alpha, \beta, \gamma) \) where
\[
r_1(\alpha, \beta, \gamma) = \inf_n \left\{ \left( \frac{1}{n-\gamma} \left( \frac{n-1}{2n-\cos \alpha - \beta} \right) \right) \right\}^{\frac{1}{n-1}}, \ n \geq 2.
\]

*Proof.* By a computation we have
\[
|f'(z) - 1| = \left| - \sum_{n=2}^{\infty} na_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.
\]

Now, \( f \) is close-to-convex of order \( \gamma \) if we have the condition
\[
\sum_{n=2}^{\infty} \left( \frac{n}{1-\gamma} \right) a_n |z|^{n-1} \leq 1.
\]

Considering the coefficient conditions required by Lemma 1 the above inequality (3) is true if
\[
\left( \frac{n}{1-\gamma} \right) |z|^{n-1} \leq \frac{n(2n-\cos \alpha - \beta)}{\cos \alpha - \beta} \quad \text{or if}
\]
\[
|z| \leq \left\{ \frac{(1-\gamma)(2n-\cos \alpha - \beta)}{\cos \alpha - \beta} \right\}^{\frac{1}{n-1}}, \ n \geq 2.
\]

This expression yields the bounds required by the above theorem.

**Theorem 9.** If \( f(z) \in \text{UCSPT}(\alpha, \beta) \) then \( f \) is starlike of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in \( |z| < r_2(\alpha, \beta, \gamma) \) where
\[
r_2(\alpha, \beta, \gamma) = \inf_n \left\{ \left( \frac{1-\gamma}{n} \frac{(2n-\cos \alpha - \beta)}{(n-\gamma)(\cos \alpha - \beta)} \right) \right\}^{\frac{1}{n-1}}, \ n \geq 2.
\]
Proof. By a computation we have

\[ \left| \frac{zf''(z)}{f(z)} - 1 \right| = \frac{\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}. \]

Now \( f \) is starlike of order \( \gamma \) if we have the condition

\[ \sum_{n=2}^{\infty} \left( 1 - \gamma \right) a_n |z|^{n-1} \leq 1. \]  

(4)

Considering the coefficient conditions required by Lemma 1, the above inequality is true if \( \left( \frac{n - \gamma}{1 - \gamma} \right) |z|^{n-1} \leq \frac{n(2n - \cos \alpha - \beta)}{\cos \alpha - \beta} \) or if

\[ |z| \leq \left\{ \frac{(1 - \gamma) n(2n - \cos \alpha - \beta)}{(n - \gamma)(\cos \alpha - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2. \]

This last expression yields the bound required.

Theorem 10. If \( f(z) \in UCSPT(\alpha, \beta) \) then \( f \) is convex of order \( \gamma \) \((0 \leq \gamma < 1)\) in \(|z| < r_{3}(\alpha, \beta, \gamma)\) where

\[ r_{3}(\alpha, \beta, \gamma) = \inf_{n} \left\{ \left( 1 - \gamma \right) \frac{(2n - \cos \alpha - \beta)}{(n - \gamma)(\cos \alpha - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2. \]

Proof. By a computation we have

\[ \left| \frac{zf''(z)}{f''(z)} - 1 \right| = \frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} na_n z^{n-1}} \leq \frac{\sum_{n=2}^{\infty} n(n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n |z|^{n-1}}. \]

Now \( f \) is convex of order \( \gamma \) if we have the condition

\[ \sum_{n=2}^{\infty} \frac{n(n - \gamma)}{1 - \gamma} a_n |z|^{n-1} \leq 1. \]  

(5)
Considering the coefficient conditions required by Lemma 1, the above inequality (5) is true if
\[ \left( \frac{n(n - \gamma)}{1 - \gamma} \right) |z|^{n-1} \leq \frac{n(2n - \cos \alpha - \beta)}{\cos \alpha - \beta} \]
or if
\[ |z| \leq \left\{ \frac{(1 - \gamma)(2n - \cos \alpha - \beta)}{(n - \gamma)(\cos \alpha - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2. \]

This gives the bound required by the above theorem.

For completeness, we give without proof, theorems concerning the radii of close-to-convexity, starlikeness and convexity for the class $SP_p T(\alpha, \beta)$.

**Theorem 11.** If $f(z) \in SP_p T(\alpha, \beta)$ then $f$ is close-to-convex of order $\gamma$ ($0 \leq \gamma < 1$) in $|z| < r_4(\alpha, \beta, \gamma)$ where
\[ r_4(\alpha, \beta, \gamma) = \inf_n \left\{ \frac{(1 - \gamma)(2n - \cos \alpha - \beta)}{n(\cos \alpha - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2. \]

**Theorem 12.** If $f(z) \in SP_p T(\alpha, \beta)$ then $f$ is starlike of order $\gamma$ ($0 \leq \gamma < 1$) in $|z| < r_5(\alpha, \beta, \gamma)$ where
\[ r_5(\alpha, \beta, \gamma) = \inf_n \left\{ \frac{(1 - \gamma)(2n - \cos \alpha - \beta)}{(n - \gamma)(\cos \alpha - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2. \]

**Theorem 13.** If $f(z) \in SP_p T(\alpha, \beta)$ then $f$ is convex of order $\gamma$ ($0 \leq \gamma < 1$) in $|z| < r_6(\alpha, \beta, \gamma)$ where
\[ r_6(\alpha, \beta, \gamma) = \inf_n \left\{ \frac{(1 - \gamma)(2n - \cos \alpha - \beta)}{n(n - \gamma)(\cos \alpha - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2. \]

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