IDEAL CONVERGENT SEQUENCE SPACES DEFINED BY MUSIELAK-ORLICZ FUNCTION OVER N-NORMED SPACES

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ABSTRACT. In the present paper we introduce sequence spaces using ideal convergence and Musielak-Orlicz function \( M = (M_k) \) over \( n \)-normed spaces and examine some properties of the resulting sequence spaces.

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1. Introduction and Preliminaries

The notion of ideal convergence was first introduced by P. Kostyrko [8] as a generalization of statistical convergence which was further studied in topological spaces by Das, Kostyrko, Wilczynski and Malik [2]. More applications of ideals can be seen in ([2], [3]). The concept of 2-normed spaces was initially developed by Gähler [4] in the mid of 1960’s, while that of \( n \)-normed spaces one can see in Misiak [11]. As an interesting non linear generalization of a normed linear space which was subsequently studied by many others ([5],[17]) and references therein. Recently a lot of activities have been started to study summability, sequence spaces and related topics in these non linear spaces (see [6],[18]). In particular Sahiner [18] combined these two concepts and investigated ideal summability in these spaces and introduced certain sequence spaces using 2-norm.

We continue in this direction, by using Musielak-Orlicz function, generalized sequences and also ideals we introduce \( I \)-convergence of generalized sequences with respect to Musielak-Orlicz function in \( n \)-normed spaces.

Let \( n \in \mathbb{N} \) and \( X \) be a real linear space of dimension \( d \), where \( d \geq n \geq 2 \). A real valued function \( ||\cdot,\cdots,\cdot|| \) on \( X^n \) satisfying the following four conditions:

1. \( ||x_1, x_2, \cdots, x_n|| = 0 \) if and only if \( x_1, x_2, \cdots, x_n \) are linearly dependent in \( X \);
2. \( ||x_1, x_2, \cdots, x_n|| \) is invariant under permutation;
3. $||\alpha x_1, x_2, \ldots, x_n|| = |\alpha| \ ||x_1, x_2, \ldots, x_n||$ for any $\alpha \in \mathbb{R}$, and

4. $||x + x', x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||x', x_2, \ldots, x_n||$

is called an $n$-norm on $X$, and the pair $(X, ||\cdot, \ldots, \cdot||)$ is called an $n$-normed space.

For example, we may take $X = \mathbb{R}^n$ being equipped with the $n$-norm $||x_1, x_2, \ldots, x_n||_E$ = the volume of the $n$-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \ldots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{i_n}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, ||\cdot, \ldots, \cdot||)$ be an $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in $X$. Then the following function $||\cdot, \ldots, \cdot||_\infty$ on $X^{n-1}$ defined by

$$||x_1, x_2, \ldots, x_{n-1}||_\infty = \max\{||x_1, x_2, \ldots, x_{n-1}, a_i|| : i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on $X$ with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence $(x_k)$ in a $n$-normed space $(X, ||\cdot, \ldots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \ldots, z_{n-1}|| = 0$$

for every $z_1, \ldots, z_{n-1} \in X$.

A sequence $(x_k)$ in a $n$-normed space $(X, ||\cdot, \ldots, \cdot||)$ is said to be Cauchy if

$$\lim_{k, p \to \infty} ||x_k - x_p, z_1, \ldots, z_{n-1}|| = 0$$

for every $z_1, \ldots, z_{n-1} \in X$.

If every cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be $n$-Banach space.

Let $(X, ||\cdot, \ldots, \cdot||)$ be a $n$-normed space. Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $X$ is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \left\{ n \in \mathbb{N} : ||x_n - x|| \geq \epsilon \right\}$ has natural density zero for each $\epsilon > 0$.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set $Y$ is said to be an ideal in $Y$ if

1. $\phi \in \mathcal{I}$
2. $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$
3. $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$,
while an admissible ideal $I$ of $Y$ further satisfies $\{x\} \in I$ for each $x \in Y$ see [5].
Given $I \subset 2^\mathbb{N}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ is said to be $I$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $I$ see [8].

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19], Theorem 10.4.2, P-183).

An orlicz function $M$ is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let $w$ be the space of all real or complex sequences $x = (x_k)$, then

$$
\ell_M = \{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}
$$

which is called as an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$
||x|| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.
$$

It is shown in [9] that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p(p \geq 1)$. The $\Delta_2$-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$.

A sequence $M = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([10],[14]). A sequence $N = (N_k)$ defined by

$$
N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \text{ for } k = 1, 2, \ldots
$$

is called the complementary function of a Musielak-Orlicz function $M$. For a given Musielak-Orlicz function $M$, the Musielak-Orlicz sequence space $t_M$ and its subspace
$h_M$ are defined as follows
\[
t_M = \{ x \in w : I_M(cx) < \infty \text{ for some } c > 0 \},
\]
\[
h_M = \{ x \in w : I_M(cx) < \infty \text{ for all } c > 0 \},
\]
where $I_M$ is a convex modular defined by
\[
I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_M.
\]

We consider $t_M$ equipped with the Luxemburg norm
\[
\|x\| = \inf \{ k > 0 : I_M\left(\frac{x}{k}\right) \leq 1 \}
\]
or equipped with the Orlicz norm
\[
\|x\|^0 = \inf \{ \frac{1}{k} \left(1 + I_M(kx)\right) : k > 0 \}.
\]

The notion of difference sequence spaces was introduced by Kizmaz [7], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [1] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let $m, n$ be non-negative integers, then for $Z = c, c_0$ and $l_\infty$, we have sequence spaces
\[
Z(\Delta^n_m) = \{ x = (x_k) \in w : (\Delta^n_m x_k) \in Z \}
\]
for $Z = c, c_0$ and $l_\infty$ where $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+m})$ and $\Delta^0_m x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation
\[
\Delta^n_m x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+m}.n.
\]

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [1]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [7]. For more details about sequence spaces(see [12],[13],[15],[16])and references therein.

Let $\Lambda = (\lambda_n)$ be non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \geq \lambda_n + 1$, $\lambda_1 = 0$. Let $I$ be an admissible ideal of $\mathbb{N}$, $M = (M_k)$ be a Musielak-Orlicz function and $(X, \| \cdot \|, \cdots, \| \cdot \|)$ is a $n$-normed space. Further, suppose $p = (p_k)$ is a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. By $S(n - X)$ we denote the space of all sequences
defined over \((X, ||·||, ·, ·, ·)||\). Now we define the following sequence spaces in this paper:

\[ W^I_\lambda \left( \lambda, M, \Delta, u, p, ||·||, \cdots, ||\right) = \]

\[ \{ x \in S(n - X) : \forall \epsilon > 0, \{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \frac{\Delta_n x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p \geq \epsilon \} \in I \text{ for some } \rho > 0 \text{ and each } z_1, \cdots, z_{n-1} \in X \}, \]

\[ W^I_0 \left( \lambda, M, \Delta, u, p, ||·||, \cdots, ||\right) = \]

\[ \{ x \in S(n - X) : \forall \epsilon > 0, \{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \frac{\Delta_n x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p \geq \epsilon \} \in I \text{ for some } \rho > 0 \text{ and each } z_1, \cdots, z_{n-1} \in X \}, \]

\[ W_\infty \left( \lambda, M, \Delta, u, p, ||·||, \cdots, ||\right) = \]

\[ \{ x \in S(n - X) : \exists K > 0 \text{ such that } \sup_{n \in \mathbb{N}} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \frac{\Delta_n x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p \leq K \text{ for some } \rho > 0 \text{ and each } z_1, \cdots, z_{n-1} \in X \} \]

and

\[ W_\infty^I \left( \lambda, M, \Delta, u, p, ||·||, \cdots, ||\right) = \]

\[ \{ x \in S(n - X) : \exists K > 0 \text{ such that } \{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \frac{\Delta_n x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p \geq K \} \in I \text{ for some } \rho > 0 \text{ and each } z_1, \cdots, z_{n-1} \in X \}. \]

The following inequality will be used throughout the paper. If \(0 \leq p_k \leq \sup p_k = H\), \(D = \max(1, 2^{H-1})\) then

\[ |a_k + b_k|^p \leq D^p \{ |a_k|^p + |b_k|^p \} \]

(1)
for all $k$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^p \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduce some sequence spaces using ideal convergence for Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces. We study some relevant algebraic and topological properties. Further some inclusion relations among these spaces are also examined.

2. Main Results

**Theorem 1.** Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of strictly positive real numbers and $I$ be an admissible ideal of $\mathbb{N}$. Then $W^I(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot||, \ldots, ||\cdot||)$, $W^I_\Delta(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot||, \ldots, ||\cdot||)$, $W^I_\infty(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot||, \ldots, ||\cdot||)$ and $W^I_\infty(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot||, \ldots, ||\cdot||)$ are linear spaces over the real field $\mathbb{R}$.

**Proof.** Let $x = (x_k), y = (y_k) \in W^I(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot||, \ldots, ||\cdot||)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive integers $\rho_1$ and $\rho_2$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m x_k - L}{\rho_1}, z_1, \ldots, z_{n-1} \right| \right) \right]^{p_k} \geq \epsilon \right\} \in I$$

and

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m y_k - L}{\rho_2}, z_1, \ldots, z_{n-1} \right| \right) \right]^{p_k} \geq \epsilon \right\} \in I.$$ 

Since $||\cdot||, \ldots, ||\cdot||$ is a $n$-norm and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function.

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m (\alpha x_k + \beta y_k - L)}{\alpha |\rho_1 + |\beta| \rho_2}, z_1, \ldots, z_{n-1} \right| \right) \right]^{p_k} \leq D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ \frac{\rho_1 |\alpha|}{(\alpha |\rho_1 + |\beta| \rho_2)} M_k \left( \left| \frac{\Delta^m x_k - L}{\rho_1}, z_1, \ldots, z_{n-1} \right| \right) \right]^{p_k}$$

$$+ D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ \frac{\rho_2 |\beta|}{(\alpha |\rho_1 + |\beta| \rho_2)} M_k \left( \left| \frac{\Delta^m y_k - L}{\rho_2}, z_1, \ldots, z_{n-1} \right| \right) \right]^{p_k}$$

$$\leq D F \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m x_k - L}{\rho_1}, z_1, \ldots, z_{n-1} \right| \right) \right]^{p_k}$$

$$+ D F \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m y_k - L}{\rho_2}, z_1, \ldots, z_{n-1} \right| \right) \right]^{p_k},$$

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where $F = \max \left[ 1, \left( \frac{\rho_1 |\alpha|}{|\alpha| + |\beta|\rho_2} \right)^H, \left( \frac{\rho_2 |\beta|}{|\alpha| + |\beta|\rho_2} \right)^H \right]$. From the above inequality, we get
\[
\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m (\alpha x_k + \beta y_k) - L}{\rho_1}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \geq \epsilon \right\}
\subseteq \left\{ n \in \mathbb{N} : DF \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m x_k - L}{\rho_1}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\}
\cup \left\{ n \in \mathbb{N} : DF \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m y_k - L}{\rho_2}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \geq \frac{\epsilon}{2} \right\}.
\]

Two sets on the right hand side belong to $I$ and this completes the proof.

Similarly, we can prove that $W_{0}^{f} \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||., .|| \right)$, $W_{\infty} \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||., .|| \right)$ and $W_{\infty}^{f} \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||., .|| \right)$ are linear spaces.

**Theorem 2.** Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. For any fixed $n \in \mathbb{N}$, $W_{\infty} \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||., .|| \right)$ is a paranormed space with
\[
g_{n}(x) \equiv \inf \left\{ \rho^{\frac{mp}{n}} : \rho > 0 : \sup_{k} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \leq 1, \right. \\
\left. \forall z_1, \cdots, z_{n-1} \in X \right\}.
\]

**Proof.** It is clear that $g_{n}(x) = g_{n}(-x)$. Since $M_k(0) = 0$, we get $\inf \left\{ \rho^{\frac{mp}{n}} \right\} = 0$ for $x = 0$ therefore, $g_{n}(0) = 0$. For $x = (x_k), y = (y_k) \in W_{\infty} \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||., .|| \right)$. Let
\[
B(x) = \left\{ \rho > 0 : \sup_{k} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \leq 1, \forall z_1, \cdots, z_{n-1} \in X \right\},
\]
\[
B(y) = \left\{ \rho > 0 : \sup_{k} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m y_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \leq 1, \forall z_1, \cdots, z_{n-1} \in X \right\}.
\]

Suppose $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$. If $\rho = \rho_1 + \rho_2$, then we have
\[
\sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k \left( \frac{\Delta^m(x_k + y_k)}{\rho}, z_1, \ldots, z_{n-1} \right) \\
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k \left( \frac{\Delta^m x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right) \\
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k \left( \frac{\Delta^m y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right).
\]

Thus, \( \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k \left( \frac{\Delta^m(x_k + y_k)}{\rho_1 + \rho_2}, z_1, \ldots, z_{n-1} \right) \leq 1 \) and

\[
g_n(x + y) \leq \inf \left\{ (\rho_1 + \rho_2)^{\frac{p_k}{n}} : \rho_1 \in B(x), \rho_2 \in B(y) \right\} \\
\leq \inf \left\{ \rho_1^{\frac{p_k}{n}} : \rho_1 \in B(x) \right\} + \inf \left\{ \rho_2^{\frac{p_k}{n}} : \rho_2 \in B(y) \right\} \\
= g_n(x) + g_n(y).
\]

Let \( \sigma^s \to \sigma \) where \( \sigma, \sigma^s \in \mathbb{C} \) and \( g_n(x^s - x) \to 0 \) as \( s \to \infty \). We show that \( g_n(\sigma^s x^s - \sigma x) \to 0 \) as \( s \to \infty \). For

\[
B(x^s) = \left\{ \rho_s > 0 : \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \frac{\Delta^m(x^s_k)}{\rho_s}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \leq 1, \forall z_1, \ldots, z_{n-1} \in X \right\},
\]

\[
B(x^s - x) = \left\{ \rho'_s > 0 : \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \frac{\Delta^m(x^s_k - x_k)}{\rho'_s}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \leq 1, \forall z_1, \ldots, z_{n-1} \in X \right\}.
\]

If \( \rho_s \in B(x^s) \) and \( \rho'_s \in B(x^s - x) \) then we observe that

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Theorem 3. Let $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$, $\mathcal{M}'' = (M''_k)$ are Musielak-Orlicz functions. Then we have

(i) $W_0^I\left(\lambda, \mathcal{M}', \Delta^m, u, p, ||\cdot, \cdots, ||\right) \subseteq W_0^I\left(\lambda, \mathcal{M} \circ \mathcal{M}', \Delta^m, u, p, ||\cdot, \cdots, ||\right)$ provided that $H_0 = \inf p_k > 0$.

(ii) $W_0^I\left(\lambda, \mathcal{M} + \mathcal{M}'', \Delta^m, u, p, ||\cdot, \cdots, ||\right) \cap W_0^I\left(\lambda, \mathcal{M}', \Delta^m, u, p, ||\cdot, \cdots, ||\right)$

$$\subseteq W_0^I\left(\lambda, \mathcal{M}' + \mathcal{M}'' + \Delta^m, u, p, ||\cdot, \cdots, ||\right).$$
Proof. (i) For given $\epsilon > 0$, first choose $\epsilon_0 > 0$ such that $\max\{\epsilon_0^H, \epsilon_0^H_0\} < \epsilon$. Now using the continuity of $(M_k)$. Choose $0 < \delta < 1$ such that $0 < t < \delta$, this implies that $M_k(t) < \epsilon_0$. Let $x = (x_k) \in W_0 \left( \lambda, \mathcal{M}', \Delta^m, u, p, ||\cdot||, \cdot, ||\cdot|| \right)$. Now from the definition

$$B(\delta) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p \geq \delta^H \right\} \in I.$$

Thus, if $n \notin B(\delta)$ then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p < \delta^H.$$ 

$$\Rightarrow \sum_{k \in I_n} u_k \left[ M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p < \lambda_n \delta^H.$$ 

$$\Rightarrow u_k \left[ M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p < \delta^H \text{ for all } k \in I_n.$$ 

Thus, $u_k \left[ M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right] < \delta$ for all $k \in I_n$. Hence,

$$u_k M_k \left( M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right) < \epsilon_0 \forall k \in I_n$$

which consequently implies that

$$\sum_{k \in I_n} u_k \left[ M_k \left( M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \right]^p < \lambda_n \max\{\epsilon_0^H, \epsilon_0^H_0\}$$

$$< \lambda_n \epsilon.$$ 

Thus,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \right]^p < \epsilon.$$ 

This shows that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \right]^p \geq \epsilon \right\} \subset B(\delta)$$

and thus belongs to $I$. This proves the result.

(ii) Let $x = (x_k) \in W_0^I \left( \lambda, \mathcal{M}', \Delta^m, u, p, ||\cdot||, \cdot, ||\cdot|| \right) \cap W_0^I \left( \lambda, \mathcal{M}^n, \Delta^m, u, p, ||\cdot||, \cdot, ||\cdot|| \right).$ Then the fact,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ (M_k' + M_k'') \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p$$

$$\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k'' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p + D \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k' \left( \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^p$$

completes the proof of the theorem.
Theorem 4. The sequence spaces \( W^f_0 \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \) and \( W^f_\infty \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \) are solid.

Proof. Let \( x = (x_k) \in W^f_0 \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \), let \( (\alpha_k) \) be a sequence of scalars such that \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \). Then we have

\[
\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m(\alpha_k x_k)}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \right\}^n \subset \left\{ n \in \mathbb{N} : \frac{C}{\lambda_n} \sum_{k \in I_n} u_k \left[ M_k \left( \left| \frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right]^{p_k} \geq \epsilon \right\} \in I,
\]

where \( C = \max \{1, |\alpha_k|^H\} \). Hence \( (\alpha_k x_k) \in W^f_0 \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \) for all sequences of scalars \( \alpha_k \) with \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \) whenever \( (x_k) \in W^f_0 \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \).

Similarly, we can prove that \( W^f_\infty \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \) is a solid space.

Theorem 5. The sequence spaces \( W^f_0 \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \) and \( W^f_\infty \left( \lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, || \right) \) are monotone.

Proof. It is obvious.

References


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