INCLUSION AND NEIGHBORHOOD PROPERTIES OF CERTAIN CLASSES OF MULTIVALENTLY ANALYTIC FUNCTIONS DEFINED BY USING A DIFFERENTIAL OPERATOR

M.K. AOUF AND S. BULUT

ABSTRACT. Making use of a differential operator, we introduce and investigate two classes of multivalently analytic functions of complex order. In this paper, we obtain coefficient estimates and inclusion relationships involving the \((j, \delta)\)-neighborhood of various subclasses of multivalently analytic functions of complex order.

2000 Mathematics Subject Classification: 30C45.

Keywords: Multivalent analytic functions, differential operator, complex order, inclusion properties, neighborhood.

1. Introduction

Let \(T(j, p)\) denote the class of functions of the form:

\[
f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \quad (a_k \geq 0; \ p, j \in N = \{1, 2, \ldots}\),
\]

which are analytic in the open unit disc \(U = \{z : |z| < 1\}\).

A function \(f(z) \in T(j, p)\) is said to be \(p\)-valently starlike of order \(\alpha\) if it satisfies the inequality:

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < p; p \in N). \tag{1.2}
\]

We denote by \(T^*_j(p, \alpha)\) the class of all \(p\)-valently starlike functions of order \(\alpha\).

Also a function \(f(z) \in T(j, p)\) is said to be \(p\)-valently convex of order \(\alpha\) if it satisfies the inequality:

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < p; p \in N). \tag{1.3}
\]
We denote by \( C_j(p, \alpha) \) the class of all \( p \)-valently convex functions of order \( \alpha \).

We note that (see for example Duren [10] and Goodman [12])

\[
f(z) \in C_j(p, \alpha) \iff z f'(z) / p \in T_j^*(p, \alpha) \quad (0 \leq \alpha < p; p \in \mathbb{N}).
\]

(1.4)

For each \( f(z) \in T(j, p) \), we have (see [9])

\[
f(q)(z) = \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q} \quad (q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p > q).
\]

(1.5)

For a function \( f(z) \) in \( T(j, p) \), we define

\[
D_0^q f(z) = f(q)(z),
\]

\[
D_1^p f(q)(z) = f(q)(z) = \frac{z}{(p-q)!} (f(q)(z))' = \frac{z}{(p-q)} f^{(1+q)}(z)
\]

\[
= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left( \frac{k-q}{p-q} \right) a_k z^{k-q},
\]

(1.6)

\[
D_2^p f(q)(z) = D(D_1^p f(q)(z))
\]

\[
= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left( \frac{k-q}{p-q} \right)^2 a_k z^{k-q},
\]

(1.7)

and

\[
D_n^p f(q)(z) = D(D_{n-1}^p f(q)(z)) \quad (n \in \mathbb{N})
\]

\[
= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left( \frac{k-q}{p-q} \right)^n a_k z^{k-q}
\]

(1.8)

The differential operator \( D_n^p f(q)(z) \) was introduced by Aouf [6, 7]. We note that, when \( q = 0 \) and \( p = 1 \), the differential operator \( D_1^q = D^q \) was introduced by Salagean [19]. Also when \( q = 0 \), the operator \( D_n^p \) was introduced by Kamali and Orhan [13], Aouf [5] and Aouf and Mostafa [8].

Now, making use of the differential operator \( D_n^p f(q)(z) \) given by (1.8), we introduce a new subclass \( R_j(p, q, b, \beta) \) of the \( p \)-valently analytic function class \( T(j, p) \) satisfying the following inequality:
\begin{align*}
\left| \frac{1}{b} \left( \frac{z(D_{p}^{n}f(q)(z))'}{D_{p}^{n}f(q)(z)} - (p-q) \right) \right| < \beta \tag{1.9}
\end{align*}

(z \in U; p, j \in N; q, n \in N_{0}; b \in C \setminus \{0\}; 0 < \beta \leq 1; p > q).

Now, following the earlier investigations by Goodman [11], Ruscheweyh [18], and others including Altintas and Owa [1], Altintas et al. ([2] and [3]), Murugusundaramoorthy and Srivastava [14], Raina and Srivastava [17], Aouf [4] and Srivastava and Orhan [20] (see also [15], [16] and [21]), we define the $(j, \delta)$-neighborhood of a function $f(z) \in T(j, p)$ by (see, for example, [3, p. 1668])

\begin{align*}
N_{j, \delta}(f) = \left\{ g : g \in T(j, p), g(z) = z^{p} - \sum_{k=j+p}^{\infty} b_{k}z^{k} \text{ and } \sum_{k=j+p}^{\infty} k |a_{k} - b_{k}| \leq \delta \right\}. \tag{1.10}
\end{align*}

In particular, if $h(z) = z^{p} \ (p \in N)$,

\begin{align*}
N_{j, \delta}(h) = \left\{ g : g \in T(j, p), g(z) = z^{p} - \sum_{k=j+p}^{\infty} b_{k}z^{k} \text{ and } \sum_{k=j+p}^{\infty} k |b_{k}| \leq \delta \right\}. \tag{1.12}
\end{align*}

Also, let $L_{j}(n, p, q, b, \beta)$ denote the subclass of $T(j, p)$ consisting of functions $f(z)$ which satisfy the inequality:

\begin{align*}
\left| \frac{1}{b} \left( \frac{(D_{p}^{n}f(q)(z))'}{(p-q)z^{p-q-1}} - \theta(p, q) \right) \right| < \beta
\end{align*}

(z \in U; p, j \in N; q, n \in N_{0}; b \in C \setminus \{0\}; 0 < \beta \leq 1; p > q), \tag{1.13}

where

\begin{align*}
\theta(p, q) = \frac{p!}{(p-q)!} = \begin{cases} 
1 & (q = 0), \\
p(p-1)\ldots(p-q+1) & (q \neq 0). 
\end{cases} \tag{1.14}
\end{align*}

**Remark 1.** Throughout our present paper, we assume that $\theta(p, q)$ is defined by (1.14).
2. Neighborhoods for the Classes \( R_j(n, p, q, b, \beta) \) and \( L_j(n, p, q, b, \beta) \)

In our investigation of the inclusion relations involving \( N_{j, \beta}(h) \), we shall require Lemmas 1 and 2 below.

**Lemma 1.** Let the function \( f(z) \in T(j, p) \) be defined by (1.1). Then \( f(z) \) is in the class \( R_j(n, p, q, b, \beta) \) if and only if

\[
\sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n (k + \beta |b| - p) \theta(k, q) a_k \leq \beta |b| \theta(p, q). \tag{2.1}
\]

**Proof.** Let a function \( f(z) \) of the form (1.1) belong to the class \( R_j(n, p, q, b, \beta) \). Then, in view of (1.8) and (1.9), we obtain the following inequality:

\[
\text{Re} \left\{ \frac{z(D^n_p f^{(q)}(z))'}{D^n_p f^{(q)}(z)} - (p-q) \right\} > -\beta |b| \quad (z \in U), \tag{2.2}
\]

or, equivalently,

\[
\text{Re} \left\{ \frac{- \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n (k - p) \theta(k, q) a_k z^{k-p}}{\theta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n \theta(k, q) a_k z^{k-p}} \right\} > -\beta |b| \quad (z \in U). \tag{2.3}
\]

Setting \( z = r \) (0 \( r < 1 \)) in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for \( r = 0 \) and also for all \( r \) (0 \( r < 1 \)). Thus, by letting \( r \to 1^- \) through real values, (2.3) leads us to the desired assertion (2.1) of Lemma 1.

Conversely, by applying the hypothesis (2.1) and letting \( |z| = 1 \), we find from (1.9) that

\[
\left| \frac{z(D^n_p f^{(q)}(z))'}{D^n_p f^{(q)}(z)} - (p-q) \right| = \frac{\sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n (k - p) \theta(k, q) a_k z^{k-p}}{\theta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n \theta(k, q) a_k z^{k-p}} \leq \frac{\beta |b| \left\{ \theta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n \theta(k, q) a_k \right\}}{\theta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n \theta(k, q) a_k} = \beta |b|.
\]

Hence, by the maximum modulus theorem, we have \( f(z) \in R_j(n, p, q, b, \beta) \), which evidently completes the proof of Lemma 1.
Similarly, we can prove the following lemma.

**Lemma 2.** Let the function \( f(z) \in T(j,p) \) be defined by \( (1.1) \). Then \( f(z) \in L_j(n,p,q,b,\beta) \) if and only if

\[
\sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^{n+1} \theta(k,q)a_k \leq \beta |b|.
\] (2.4)

Our first inclusion relation involving \( N_{j,\delta}(h) \) is given in the following theorem.

**Theorem 3.** Let

\[
\delta = \frac{(j+p)\beta |b| \theta(p,q)}{(j+\beta |b|) \theta(j+p,q)} \quad (p > |b|),
\] (2.5)

then

\[ R_j(n,p,q,b,\beta) \subset N_{j,\delta}(h). \] (2.6)

**Proof.** Let \( f(z) \in R_j(n,p,q,b,\beta) \). Then, in view of the assertion (2.1) of Lemma 1, we have

\[
\left( \frac{j+p-q}{p-q} \right)^n (j+\beta |b|) \theta(j+p,q) \sum_{k=j+p}^{\infty} a_k
\]
\[
\leq \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n (k+\beta |b|-p) \theta(k,q) a_k \leq \beta |b| \theta(p,q),
\] (2.7)

which readily yields

\[
\sum_{k=j+p}^{\infty} a_k \leq \frac{\beta |b| \theta(p,q)}{(j+\beta |b|) \theta(j+p,q)}.
\] (2.8)

Making use of (2.1) again, in conjunction with (2.8), we get

\[
\left( \frac{j+p-q}{p-q} \right)^n \theta(j+p,q) \sum_{k=j+p}^{\infty} k a_k
\]
\[
\leq \beta |b| \theta(p,q) + (p-\beta |b|) \left( \frac{j+p-q}{p-q} \right)^n \theta(j+p,q) \sum_{k=j+p}^{\infty} a_k
\]
\[
\leq \beta |b| \theta(p,q) + \frac{\beta |b| \theta(p,q) \theta(p,q)}{(j+\beta |b|)} = \frac{(j+p)\beta |b| \theta(p,q)}{(j+\beta |b|)}.
\]
Hence
\[ \sum_{k=j+p}^{\infty} ka_k \leq \frac{(j + p) \beta |b| \theta(p, q)}{(j + p - q \frac{p - q}{p - q})} = \delta \quad (p > |b|) \quad (2.9) \]
which, by means of the definition (1.12), establishes the inclusion relation (2.6) asserted by Theorem 3.

In a similar manner, by applying the assertion (2.4) of Lemma 2 instead of the assertion (2.1) of Lemma 1 to functions in the class \( L_j(n, p, q, b, \beta) \), we can prove the following inclusion relationship.

**Theorem 4.** If
\[ \delta = \frac{(j + p) \beta |b|}{\left( \frac{j + p - q}{p - q} \right)^{n+1} \theta(j + p, q)} , \quad (2.10) \]
then
\[ L_j(n, p, q, b, \beta) \subset N_{j, \delta}(h). \quad (2.11) \]

### 3. Neighborhoods for the classes \( R_j^{(\alpha)}(n, p, q, b, \beta) \) and \( L_j^{(\alpha)}(n, p, q, b, \beta) \)

In this section, we determine the neighborhood for each of the classes \( R_j^{(\alpha)}(n, p, q, b, \beta) \) and \( L_j^{(\alpha)}(n, p, q, b, \beta) \), which we define as follows.

A function \( f(z) \in T(j, p) \) is said to be in the class \( R_j^{(\alpha)}(n, p, q, b, \beta) \) if there exists a function \( g(z) \in R_j(n, p, q, b, \beta) \) such that
\[ \left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha \quad (z \in U; \ 0 \leq \alpha < p). \quad (3.1) \]

Analogously, a function \( f(z) \in T(j, p) \) is said to be in the class \( L_j^{(\alpha)}(n, p, q, b, \beta) \) if there exists a function \( g(z) \in L_j(n, p, q, b, \beta) \) such that the inequality (3.1) holds true.

**Theorem 5.** If \( g(z) \in R_j(n, p, q, b, \beta) \) and
\[ \alpha = p - \frac{\delta \left( \frac{j + p - q}{p - q} \right)^n (j + \beta |b|) \theta(j + p, q)}{(j + p) \left( \frac{j + p - q}{p - q} \right)^n (j + \beta |b|) \theta(j + p, q) - \beta |b| \theta(p, q)} , \quad (3.2) \]

40
then
\[ N_{j,\delta}(g) \subset R_j^{(\alpha)}(n, p, q, b, \beta), \quad (3.3) \]

where
\[ \delta \leq p(j + p) \left\{ 1 - \beta |b| \theta(p, q) \left[ \left( \frac{j + p - q}{p - q} \right)^n (j + \beta |b|)\theta(j + p, q) \right]^{-1} \right\}. \quad (3.4) \]

**Proof.** Suppose that \( f(z) \in N_{j,\delta}(g) \). We find from (1.10) that
\[ \sum_{k=j+p}^{\infty} k |a_k - b_k| \leq \delta, \quad (3.5) \]
which readily implies that
\[ \sum_{k=j+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{j + p} \quad (p, j \in \mathbb{N}). \quad (3.6) \]

Next, since \( g(z) \in R_j(n, p, q, b, \beta) \), we have [cf. equation (2.8)]
\[ \sum_{k=j+p}^{\infty} b_k \leq \frac{\beta |b| \theta(p, q)}{(j + p - q) (j + \beta |b|)\theta(j + p, q)}, \quad (3.7) \]
so that
\[ \frac{|f(z) - g(z)|}{g(z)} \leq \frac{\sum_{k=j+p}^{\infty} |a_k - b_k|}{1 - \sum_{k=j+p}^{\infty} b_k} \]
\[ \leq \frac{\delta}{j + p} \cdot \left\{ \left( \frac{j + p - q}{p - q} \right)^n (j + \beta |b|)\theta(j + p, q) - \beta |b| \theta(p, q) \right\} = p - \alpha, \quad (3.8) \]
provided that \( \alpha \) is given by (3.2). Thus, by the above definition, \( f(z) \in R_j^{(\alpha)}(n, p, q, b, \beta) \) for \( \alpha \) given by (3.2). This evidently proves Theorem 5.

The proof of Theorem 6 below is similar to that of Theorem 5.
**Theorem 6.** If \( g(z) \in L_j(n, p, q, b, \beta) \) and

\[
\alpha = p - \frac{\delta \left( \frac{j + p - q}{p - q} \right)^{n+1} \theta(j + p, q)}{(j + p) \left\{ \left( \frac{j + p - q}{p - q} \right)^{n+1} \theta(j + p, q) - \beta |b| \right\}},
\]

then

\[
N_{j, \delta}(g) \subset L_j^{(\alpha)}(n, p, q, b, \beta),
\]

where

\[
\delta \leq p(j + p) \left\{ 1 - \beta |b| \left[ \left( \frac{j + p - q}{p - q} \right)^{n+1} \theta(j + p, q) \right]^{-1} \right\}.
\]

**Acknowledgements.** The second author was supported by the Kocaeli University under Grant KOU-BAP-HD 2011/22.

**References**


M. K. Aouf
Department of Mathematics, Faculty of Science,
Mansoura University,
Mansoura 35516, Egypt
email: mkaouf127@yahoo.com

S. Bulut
Kocaeli University,
Civil Aviation College,
Arslanbey Campus,
41285 Izmit-Kocaeli, Turkey
email: serap.bulut@kocaeli.edu.tr