UNIFICATION OF $\pi$-GENERALIZED CLOSED SETS BY HEREDITARY CLASSES IN GENERALIZED TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we introduce the notions of $\pi g^*$-closed and $\pi g^*$-open sets by using the notion of $\pi$-open sets. We generalize many concepts which is defined in ideal topological spaces and topological spaces by using these new notions. Also we study quasi $\mu_H$- normality and characterizations of quasi $\mu_H$- normal spaces are obtained. Several preservation theorems for quasi $\mu_H$- normal spaces are given.

2000 Mathematics Subject Classification: 54A05, 54C05

Keywords: generalized topology, hereditary class, $\pi g^*$-closed, quasi $\mu g^*$-$\mathcal{H}$-normal space.

1. Introduction and Preliminaries

The idea of generalized topology and hereditary classes was introduced and studied by Császár [5, 6]. He generalized ideal topology on a set by using these structures. In this paper, we introduce the notions of $\pi g^*$-closed and $\pi g^*$-open sets by using the notion of $\pi$-open sets. We generalize many concepts which is defined in ideal topological spaces and topological spaces by using these new notions.

Let $A$ be a subset of a topological space $X$. The closure of $A$ and the interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A subset $A$ of a topological space $(X, \tau)$ is said to be regular open [24](resp. regular closed) if $A = \text{Int}(\text{Cl}(A))$ (resp. $A = \text{Cl}(\text{Int}(A))$). The finite union of regular open sets is said to be $\pi$-open [28] in $(X, \tau)$. The complement of a $\pi$-open set is $\pi$-closed. A subset $A$ of a topological space $(X, \tau)$ said to be semi-open [12] (resp., $\alpha$-open [16], pre-open [14], $b$-open [2], $\beta$-open [1]) if $A \subset \text{Cl}(\text{Int}(A))$ (resp. $A \subset \text{Int}(\text{Cl}(A))$, $A \subset \text{Int}(\text{Cl}(A))$, $A \subset \text{Cl}(\text{Int}(A))$). The family of all semi-open (resp. $\alpha$-open, pre-open, $b$-open, $\beta$-open) sets in $(X, \tau)$ is denoted by $\text{SO}(X)$ (resp. $\alpha\text{O}(X)$, $\text{PO}(X)$, $\text{BO}(X)$, $\beta\text{O}(X)$). A function $f:(X, \tau) \to (Y, \sigma)$ is said to be $m-\pi$-closed [9]
if \( f(V) \) is \( \pi \)-closed in \((Y, \sigma)\) for every \( \pi \)-closed in \((X, \tau)\). A function \( f: (X, \tau) \to (Y, \sigma) \) is said \( \pi \)-continuous \([8]\) if \( f^{-1}(V) \) is \( \pi \)-closed in \((X, \tau)\) for every closed set \( V \) in \((Y, \sigma)\).

An ideal topological space is a topological space \((X, \tau, I)\) with an ideal \( I \) on \( X \), and is denoted by \((X, \tau, I)\). \( A^*(I) = \{ x \in X \mid U \cap A \notin I \text{ for each open neighborhood } U \text{ of } x \} \) is called the local function of \( A \) with respect to \( I \) and \( \tau \) \([11]\). When there is no chance for confusion \( A^*(I) \) is denoted by \( A^* \). For every ideal topological space \((X, \tau, I)\), there exists a topology \( \tau^*(I) \), finer than \( \tau \). Observe additionally that \( Cl^*(A) = A^* \cup A \) defines a Kuratowski closure operator for \( \tau^*(I) \) \([27]\). A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to be \( \text{semi}^*\text{-I-open} \) \([10]\) if \( A \subseteq Cl(I\text{nt}^*(A)) \). The family of all \( \text{semi}^*\text{-I-open} \) sets in \((X, \tau, I)\) is denoted by \( S^*\text{IO}(X) \).

Let \( X \) be a non-empty set and \( \text{exp} X \) denote the power set of \( X \). We call a class \( \mu \subset \text{exp} X \) a generalized topology \([5]\) (briefly, GT) if \( \emptyset \notin \mu \) and the union of elements of \( \mu \) belongs to \( \mu \). And let us say that a hereditary class \( H \subset X \) satisfying \( A \subset \mathcal{B} \subset X \) implies \( A \in H \). If \( \mu \) is a GT on \( X \) and \( A \subset X \), \( x \in X \) then \( x \in A^*_\mu \) \([6]\) iff \( x \in M \in \mu \Rightarrow M \cap A \notin H \). There is a GT \( \mu^* \) \([6]\) such that \( c_{\mu^*}(A) = A \cup A^*_\mu \) is intersection of all \( \mu^*\text{-closed} \) supersets of \( A \); \( M \in \mu^* \) iff \( X - M = c_{\mu^*}(X - M) \).

If one takes \( H = \emptyset \), then \( c_{\mu^*} \) becomes \( c_\mu \). If one takes \( \tau \) as GT and \( H = \emptyset \), then \( c_{\mu^*} \) becomes the usual closure operator. Similarly \( c_{\mu^*} \) becomes \( sc \) (resp. pcl, bcl, \( \beta \text{cl} \)) if \( \mu^* \) stands for \( SO(X) \) (resp. \( PO(X) \), \( BO(X) \), \( \beta O(X) \)). Likewise, if one takes \( \tau \) as GT and \( H = I \), then \( c_{\mu^*} \) becomes closure operator for \( \tau^*(I) \). Likewise, \( c_{\mu^*} \) becomes \( \text{scl} \) if \( \mu \) stands for \( S^*\text{IO}(X) \).

Given a topological space \((X, \tau)\) and a GT \( \mu \) on \( X \), \((X, \tau)\) is said to be \( \mu\text{-normal} \) \([18]\) if for any two disjoint closed sets \( A \) and \( B \) there exist two disjoint \( \mu\text{-open} \) sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

### 2. \( \pi\mu^*\text{-closed sets} \)

**Definition 1.** Let \( \mu \) be a GT and \( H \neq \emptyset \) be a hereditary class on a topological space \((X, \tau)\). A subset \( A \) of \( X \) is called a \( \pi \) generalized \( \mu^*\text{-closed} \) set (or simply \( \pi\mu^*\text{-closed} \)) if \( c_{\mu^*}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( \pi\text{-open} \).

The complement of a \( \pi\mu^*\text{-closed} \) set is said to be \( \pi\mu^*\text{-open} \).

**Remark 1.** (a) Let \( \mu \) be a GT and \( H \neq \emptyset \) be a hereditary class on topological space \((X, \tau)\). Then every \( \pi\mu^*\text{-closed} \) set reduces to \( \pi\gamma\text{-closed} \) \([8]\) (resp., \( \pi\gamma\text{s}\text{-closed} \) \([3]\), \( \pi\gamma\text{p}\text{-closed} \) \([19]\), \( \pi\gamma\text{b}\text{-closed} \) \([22]\), \( \pi\gamma\text{g}\text{-closed} \) \([25]\) ) if \( \mu \) is taken to be \( \tau \) (resp., \( SO(X) \), \( PO(X) \), \( BO(X) \), \( \beta O(X) \)) and \( H = \emptyset \).

(b) Let \( \mu \) be a GT and \( H \neq \emptyset \) be a hereditary class on \( X \). Then every \( \pi\mu^*\text{-closed} \)
Proof. It is obvious that every \( \pi \)-open set is open.

**Remark 2.** The following example shows that the reverse of Theorem 2.1 is not true.

**Example 1.** Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}\} \), \( \mathcal{H} = \{\emptyset, \{c\}\} \) and \( \mu = \{X, \emptyset, \{a, c\}, \{b, d\}\} \). Then the set \( \{a, d\} \) is \( \pi \mu \)-closed but not \( \mu \)-closed.

**Remark 3.** Finite intersection (union) of \( \pi \mu \)-closed sets need not be \( \pi \mu \)-closed by the following examples.

**Example 2.** Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \emptyset, \{a, c\}, \{b, d\}\} \), \( \mathcal{H} = \{\emptyset, \{a\}\} \) and \( \mu = \{X, \emptyset, \{a, b\}, \{a, c\}\} \). \( A = \{b, c\} \) and \( B = \{c, d\} \). Clearly \( A \) and \( B \) are \( \pi \mu \)-closed sets but \( A \cap B \) is not \( \pi \mu \)-closed.

**Example 3.** Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \), \( \mathcal{H} = \{\emptyset, \{a\}\} \) and \( \mu = \{X, \emptyset, \{a, d\}\} \). \( A = \{a, c\} \) and \( B = \{b\} \). Clearly \( A \) and \( B \) are \( \pi \mu \)-closed sets but \( A \cup B \) is not \( \pi \mu \)-closed.

**Theorem 1.** Let \( \mu \) be a GT and \( \mathcal{H} \neq \emptyset \) be a hereditary class on a topological space \((X, \tau)\). If \( A \) is \( \pi \mu \)-closed, \( B \) is \( \pi \)-closed and \( \mu \)-closed then \( A \cap B \) is \( \pi \mu \)-closed.

**Proof.** Let \( U \) be \( \pi \)-open such that \( A \cap B \subseteq U \). Then \( A \subseteq U \cup (X \setminus B) \). Since \( A \) is \( \pi \mu \)-closed and \( B \) is \( \pi \)-closed then \( c_{\mu \cdot}^{-1}(A) \subseteq (U \cup (X \setminus B)) \). Hence \( c_{\mu \cdot}(A \cap B) \subseteq (U \cup (X \setminus B)) \). Since \( B \) is \( \mu \)-closed, \( c_{\mu \cdot}(A \cap B) \subseteq U \). Hence \( A \cap B \) is \( \pi \mu \)-closed.

**Theorem 2.** Let \( \mu \) be a GT and \( \mathcal{H} \neq \emptyset \) be a hereditary class on a topological space \((X, \tau)\). For every \( A \in \mathcal{H} \), \( A \) is \( \pi \mu \)-closed.

**Proof.** Let \( A \subseteq U \) where \( U \) is \( \pi \)-open. Since \( A_{\mu}^* = \emptyset \) for every \( A \in \mathcal{H} \), \( c_{\mu \cdot}(A) = A \cup A_{\mu}^* = A \subseteq U \). Therefore \( A \) is \( \pi \mu \)-closed.

**Theorem 3.** Let \( \mu \) be a GT and \( \mathcal{H} \neq \emptyset \) be a hereditary class on a topological space \((X, \tau)\). For every subset \( A \) of \( X \), \( A_{\mu}^* \) is \( \pi \mu \)-closed.

**Proof.** Let \( A_{\mu}^* \subseteq U \) where \( U \) is \( \pi \)-open. Since \( (A_{\mu}^*)^* \subseteq A_{\mu}^* \), we have \( c_{\mu \cdot}(A_{\mu}^*) \subseteq U \). Hence \( A_{\mu}^* \) is \( \pi \mu \)-closed.

**Theorem 4.** Let \( \mu \) be a GT and \( \mathcal{H} \neq \emptyset \) be a hereditary class on a topological space \((X, \tau)\). If \( A \) is \( \pi \mu \)-closed, then \( c_{\mu \cdot}(A) \setminus A \) does not contain any nonempty \( \pi \)-closed set.
Proof. Let $F$ be $\pi$-closed subset of $X$, such that $F \subset c_{\mu^*}(A) \setminus A$ where $A$ is $\pi g\mu^*$-closed. Then $c_{\mu^*}(A) \subset (X \setminus F)$. Thus $F \subset (X \setminus c_{\mu^*}(A)) \cap c_{\mu^*}(A)$ and hence $F = \emptyset$.

**Theorem 5.** Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$ and $A \subset B \subset c_{\mu^*}(A)$, where $A$ is $\pi g\mu^*$-closed. Then $B$ is $\pi g\mu^*$-closed.

*Proof.* Let $B \subset U$ and $U$ is $\pi$-open. Since $A$ is $\pi g\mu^*$-closed and $B \subset U$, then $c_{\mu^*}(A) \subset U$. Now, $A \subset B \subset c_{\mu^*}(A)$, $c_{\mu^*}(A) = c_{\mu^*}(B)$ and hence $c_{\mu^*}(B) \subset U$. Thus $B$ is $\pi g\mu^*$-closed.

**Theorem 6.** Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$. Every $\pi$-open set is $\mu^*$-closed set if and only if every subset of $X$ is $\pi g\mu^*$-closed.

*Proof.* Suppose every $\pi$-open set is $\mu^*$-closed. Let $A$ be a subset of $X$. If $U$ is $\pi$-open such that $A \subset U$, then $A^* \subset U^* \subset U$ and $c_{\mu^*}(A) \subset U$. So $A$ is $\pi g\mu^*$-closed.

Conversely, suppose that every subset of $X$ is $\pi g\mu^*$-closed. If $U$ is $\pi$-open then by hypothesis, $U$ is $\pi g\mu^*$-closed and so $c_{\mu^*}(U) \subset U$. Thus, $U^* \subset U$ and so $U$ is $\mu^*$-closed.

**Theorem 7.** Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$. For each $x \in X$, either $\{x\}$ is $\pi$-closed or $\{x\}^c$ is $\pi g\mu^*$-closed.

*Proof.* Suppose that $\{x\}$ is not $\pi$-closed, then $\{x\}^c$ is not $\pi$-open and only $\pi$-open set containing $\{x\}$ is set $X$ itself. So $\{x\}^c$ is $\pi g\mu^*$-closed.

**Theorem 8.** Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$. If $A$ is $\pi g\mu^*$-closed in $X$, such that $A \subset Y \subset X$, then $A$ is $\pi g\mu^*$-closed in $Y$.

*Proof.* Let $U$ be a $\pi$-open set in $Y$ such that $A \subset U$, then $A \subset U = V \cap Y$ where $V$ is $\pi$-open in $X$. Since $A$ is $\pi g\mu^*$-closed in $X$, $c_{\mu^*}(A) \subset V$ . Therefore $c_{\mu^*}(A) \subset U$. Then $A$ is $\pi g\mu^*$-closed in $Y$.

**Theorem 9.** Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$. A subset $A$ of $X$ is $\pi g\mu^*$-open if and only if $F \subset int_{\mu^*}(A)$ whenever $F \subset A$ and $F$ is $\pi$-closed.

*Proof.* Let $F \subset A$ and $F$ be $\pi$-closed. Then $(X \setminus A) \subset (X \setminus F)$ and $X \setminus F$ $\pi$-open. Since $X \setminus A$ is $\pi g\mu^*$-closed $(c_{\mu^*}(A \setminus X)) \subset (X \setminus F)$. So $F \subset int_{\mu^*}(A)$. Conversely suppose that $F \subset int_{\mu^*}(A)$ whenever $F \subset A$ and $F$ is $\pi$-closed. Let $X \setminus A \subset U$ where $U$ is $\pi$-open. Then $X \setminus U \subset A$. Then by hypothesis $X \setminus U \subset int_{\mu^*}(A)$ and hence $c_{\mu^*}(X \setminus A) \subset U$. Therefore $A$ is $\pi g\mu^*$-open.
Theorem 10. Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$. If a subset of $X$ is $\pi\mu^*$-open then $U = X$ whenever $U$ is $\pi$-open and $\text{int}_{\mu^*}(A) \cup (X \setminus A) \subset U$.

Proof. Let $U$ be $\pi$-open and $\text{int}_{\mu^*}(A) \cup (X \setminus A) \subset U$ for $\pi\mu^*$-open $A$. Then $X \setminus U \subset (X \setminus \text{int}_{\mu^*}(A)) \cap A$. Since $X \setminus A$ is $\pi\mu^*$-closed and by the Theorem 2.5 $X \setminus U = \emptyset$, hence $X = U$.

3. Quasi $\mu^*$-$\mathcal{H}$-Normal Spaces

Definition 2. Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$. A topological space is called a quasi $\mu^*$-$\mathcal{H}$-normal space if for every pair of disjoint $\pi$-closed sets $A$ and $B$ of $X$, there exist disjoint $\mu^*$-open sets $U$ and $V$ such that $A \setminus U \in \mathcal{H}$ and $B \setminus V \in \mathcal{H}$.

Remark 4. Let $\mu$ be a GT and $\mathcal{H} = \{\emptyset\}$ on a topological space $(X, \tau)$. Then every quasi $\mu^*$-$\mathcal{H}$-normal space reduces to be quasi-normal [8] (resp., quasi-s-normal [4], quasi-p-normal [26]) space if $\mu$ is taken to be $\tau$ (resp., $SO(X)$, $PO(X)$).

Theorem 11. Every $\mu^*$-normal space is a quasi $\mu^*$-$\mathcal{H}$-normal space.

Proof. It is obvious by every $\mu$-open set is $\mu^*$-open.

The following example shows that the reverse of Theorem 3.1 is not true.

Example 4. Observe that the Countable Extension Topological space [Example 63, [23]] in which $X$ is real line, and if $\tau_1$ is the Euclidean topology on $X$ and $\tau_2$ is the topology of countable complements on $X$, we define $\tau$ to be the smallest topology generated by $\tau_1 \cup \tau_2$. Let $\mu = \{\emptyset\} \cup \{[n, \infty) | n \in N\} \cup \{(-\infty, n] | n \in N\}$ be generalized topology on $(X, \tau)$ and $\mathcal{H} = \{H | H \subset [a, b]\}$ be hereditary on $X$. Then $X$ is a quasi $\mu^*$-$\mathcal{H}$-normal space but not $\mu^*$-normal space.

Theorem 12. Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a topological space $(X, \tau)$. Then the followings are equivalent:

(a) $X$ is a quasi $\mu^*$-$\mathcal{H}$-normal space.
(b) For every $\pi$-closed set $F$ and $\pi$-open set $G$ containing $F$, there exists a $\mu$-open set $V$ such that $F \setminus V \in \mathcal{H}$ and $c_{\mu^*}(V) \setminus G \in \mathcal{H}$.
(c) For each pair of disjoint $\pi$-closed sets $A$ and $B$, there exists an $\mu$-open set $U$ such that $A \setminus U \in \mathcal{H}$ and $c_{\mu^*}(U) \setminus B \in \mathcal{H}$.

Proof. (a) $\Rightarrow$ (b) Let $F$ be a $\pi$-closed and $G$ be a $\pi$-open subset of $X$. Since $X \setminus G$ is $\pi$-closed and $F \subseteq G$, $F \cap (X \setminus G) = \emptyset$. $X$ is a quasi $\mu^*$-$\mathcal{H}$-normal space, so there
exist disjoint $\mu$-open sets $U$ and $V$ such that $F \setminus V \in \mathcal{H}$ and $(X \setminus G) \setminus U \in \mathcal{H}$. Then $c_{\mu^*}(V) \subset X \setminus U$ and $(X \setminus G) \cap c_{\mu^*}(V) \subset (X \setminus G) \cap (X \setminus U)$. Hence $c_{\mu^*}(V) \setminus G \in \mathcal{H}$.

(b) $\Rightarrow$ (c) Obvious by the hypothesis.

(c) $\Rightarrow$ (a) Let $A$ and $B$ be disjoint $\pi$-closed sets. By the hypothesis there exists a $\mu$-open set $U$ such that $A \setminus U \in \mathcal{H}$ and $c_{\mu^*}(U) \cap B \in \mathcal{H}$. Let $V = X \setminus c_{\mu^*}(U)$. Since $V$ is $\mu$-open and $U \cap V = \emptyset$, $X$ is a quasi $\mu_\mathcal{g} \mathcal{H}$-normal space.

**Theorem 13.** Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on $(X, \tau)$ topological space. If $X$ is a quasi $\mu_\mathcal{g} \mathcal{H}$-normal space then for every pair of disjoint $\pi$-closed sets $A$ and $B$ of $X$, there exist disjoint $\pi g_{\mu^*}$-open sets $U$ and $V$ such that $A \setminus U \in \mathcal{H}$ and $B \setminus V \in \mathcal{H}$.

**Proof.** It is obvious by every $\mu$-open set is $\pi g_{\mu^*}$-open.

**Theorem 14.** Let $\mu$ be a GT and $\mathcal{H} = \{\emptyset\}$ be a hereditary class on a $(X, \tau)$ topological space. Then the following are equivalent.

(a) $X$ is a quasi $\mu_\mathcal{g} \mathcal{H}$-normal space

(b) If for every pair of disjoint $\pi$-closed sets $A$ and $B$ of $X$, there exist disjoint $\pi g_{\mu^*}$-open sets $U$ and $V$ such that $A \setminus U \in \mathcal{H}$ and $B \setminus V \in \mathcal{H}$.

**Proof.** (a) $\Rightarrow$ (b) It is obvious by the previous theorem.

(b) $\Rightarrow$ (a) Let $A$ and $B$ be disjoint $\pi$-closed. By the hypothesis there exist $U$ and $V$ are disjoint $\pi g_{\mu^*}$-open subsets of $X$ such that $A \setminus U = \emptyset$ $B \setminus V = \emptyset$. Then $A \subset U$. Since $U$ is $\pi g_{\mu^*}$-open, $A \subset int_{\mu^*}(U)$ by Theorem 2.10. Similarly $B \subset int_{\mu^*}(V)$. Finally, since $int_{\mu^*}(U)$ and $int_{\mu^*}(V)$ are $\mu$-open sets and $\mathcal{H} = \{\emptyset\}$, $X$ is a quasi $\mu_\mathcal{g} \mathcal{H}$-normal space.

**Definition 3** ([13]). Let $(X, \mu)$ and $(Y, \lambda)$ be GTSs, then a function $f : X \rightarrow Y$ is called $(\mu, \lambda)$-open if $f(G) \in \lambda$ for each $G \in \mu$.

**Lemma 1.** If $\mathcal{H} \neq \emptyset$ is a hereditary class on $X$ and $f : X \rightarrow Y$ is a function, then $f(\mathcal{H}) = \{f(H) | H \in \mathcal{H}\}$ is a hereditary class on $Y$.

**Theorem 15.** Let $\mu$ be a GT and $\mathcal{H} \neq \emptyset$ be a hereditary class on a $(X, \tau)$, $\lambda$ be a GT on $(Y, \sigma)$ and $f : X \rightarrow Y$ is a bijection, $\pi$-continuous and $(\mu, \lambda)$-open. If $X$ is a quasi $\mu_\mathcal{g} \mathcal{H}$-normal space, then $Y$ is a quasi $\lambda_\mathcal{g} f(\mathcal{H})$-normal space.

**Proof.** Let $A$ and $B$ be disjoint $\pi$-closed subsets of $Y$, since $f$ is $\pi$-continuous function $f^{-1}(A)$, $f^{-1}(B)$ are $\pi$-closed subsets of $X$. Since $X$ is quasi $\mu_\mathcal{g} \mathcal{H}$-normal space, there exist disjoint $\mu$-open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \setminus U \in \mathcal{H}$ and $f^{-1}(B) \setminus V \in \mathcal{H}$. Then $f((f^{-1}(A) \setminus U) \in f(\mathcal{H})$ and $f((f^{-1}(A)) \setminus f(U) \in f(\mathcal{H})$. Because of the hereditary of $\mathcal{H}$, $A \setminus f(U) \in f(\mathcal{H})$. Similarly, $B \setminus f(U) \in f(\mathcal{H})$. Since $f(U)$ and $f(V)$ are disjoint $\lambda$-open subsets of $Y$, it follows that $Y$ is a quasi $\lambda_\mathcal{g} f(\mathcal{H})$-normal space.
Definition 4 ([5]). Let \((X, \mu)\) and \((Y, \lambda)\) be GTSs, then a function \(f : X \to Y\) is called \((\mu, \lambda)\)-continuous if \(f^{-1}(G) \in \mu\) for each \(G \in \lambda\).

Lemma 2. If \(\mathcal{H} \neq \emptyset\) is a hereditary class on \(X\) and \(f : X \to Y\) is a function, then \(f^{-1}(\mathcal{H}) = \{f^{-1}(H) | H \in \mathcal{H}\}\) is a hereditary class on \(X\).

Theorem 16. Let \(\mu\) be a GT and \(\mathcal{H} \neq \emptyset\) be a hereditary class on \(X, \tau, \lambda\) be a GT on \((Y, \sigma)\) and \(f : X \to Y\) is an injection, \(m\)-\(\pi\)-closed and \((\mu, \lambda)\)-continuous. If \(Y\) is a quasi \(\lambda_g\)-\(H\)-normal space, then \(X\) is a quasi \(\mu_g\)-\(f^{-1}(\mathcal{H})\)-normal space.

**Proof.** Let \(A\) and \(B\) be disjoint \(\pi\)-closed subsets of \(X\), since \(f\) is \(m\)-\(\pi\)-closed injection, \(f(A)\) and \(f(B)\) are disjoint \(\pi\)-closed subset of \(Y\). Since \(Y\) is quasi \(\lambda\)-normal space, there exist disjoint \(\lambda\)-open \(U\) and \(V\) such in \(Y\) that \(f(A) \setminus U \in \mathcal{H}\) and \(f(B) \setminus V \in \mathcal{H}\). Then \(f^{-1}(f(A)) \setminus f^{-1}(U) \in f^{-1}(\mathcal{H})\) and \(f^{-1}(f(B)) \setminus f^{-1}(V) \in f^{-1}(\mathcal{H})\). Similarly, \(B \setminus f^{-1}(V) \in f^{-1}(\mathcal{H})\). Since \(f\) is \((\mu, \lambda)\)-continuous, \(f^{-1}(U)\) and \(f^{-1}(V)\) are disjoint \(\mu\)-open subsets of \(X\). It follows that \(X\) is a quasi \(\mu_g\)-\(f^{-1}(\mathcal{H})\)-normal space.

Lemma 3. If \(\mathcal{H} \neq \emptyset\) is a hereditary class on \(X\) and \(Y\) is a subset of \(X\), then \(\mathcal{H}_Y = \{Y \cap H | H \in \mathcal{H}\}\) is a hereditary class on \(Y\).

Theorem 17. Let \(\mu\) be a GT and \(\mathcal{H} \neq \emptyset\) be a hereditary class on a \((X, \tau)\) topological space. If \(X\) is a quasi \(\mu_g\)-\(H\)-normal space and \(Y \subset X\) is \(\pi\)-closed, then \(Y\) is a quasi \(\mu_g\)-\(\mathcal{H}_Y\)-normal space.

**Proof.** Let \(A\) and \(B\) be disjoint \(\pi\)-closed subsets of \(Y\). Since \(Y\) is \(\pi\)-closed \(A\) and \(B\) are disjoint \(\pi\)-closed subsets of \(X\). By hypothesis, there exist disjoint open sets \(U\) and \(V\) such that \(A \setminus U \in \mathcal{H}\) and \(B \setminus V \in \mathcal{H}\). If \(A \setminus U = H \in \mathcal{H}\) and \(B \setminus V = G \in \mathcal{H}\), then \(A \setminus U \cap H\) and \(B \setminus V \cap G\). Since \(A \setminus Y\) and \(A \setminus (U \cup H)\) and so \(A \setminus (Y \cap H) \cup (Y \cap U)\). Therefore \(A \setminus (Y \setminus U) \cap (Y \setminus H) \in \mathcal{H}_Y\). Similarly, \(B \setminus (Y \setminus V) \cap (Y \setminus G) \in \mathcal{H}_Y\). If \(U_1 = Y \setminus U\) and \(V_1 = Y \setminus V\), then \(U_1\) and \(V_1\) are disjoint \(\mu\)-open sets such that \(A \setminus U_1 \in \mathcal{H}_Y\) and \(B \setminus V_1 \in \mathcal{H}_Y\). Hence \(Y\) is a quasi \(\mu_g\)-\(\mathcal{H}_Y\)-normal space.

**References**


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53