Recurrence Generalized $(\kappa, \mu)$ Space Forms

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Abstract. In this paper we study generalized $(k, \mu)$ space forms by considering certain curvature conditions like generalized recurrent, ricci recurrent, generalized $\phi$ recurrent conditions. We found relations among associated functions $f_1, f_2, f_3, f_4, f_5, f_6$ in $\phi$-concorcular recurrent, quasi-conformally $\phi$-flat and quasi-conformally flat generalized $(k, \mu)$ space forms.

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1. Introduction

A generalized Sasakian space form was defined by Carriazo et al. in [1], as an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ whose curvature tensor $R$ is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3,$$

where $f_1, f_2, f_3$ are some differentiable functions on $M$ and

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$
$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$
$$R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,$$

for any vector fields $X, Y, Z$ on $M$.

In [7], the authors defined a generalized $(k, \mu)$ space form as an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ whose curvature tensor can be written as

$$R = f_4R_4 + f_5R_5 + f_6R_6,$$
where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on $M$ and $R_1, R_2, R_3$ are tensors defined above and

\[
R_4(X, Y)Z = g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y,
\]
\[
R_5(X, Y)Z = g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX,
\]
\[
R_6(X, Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi,
\]
for any vector fields $X, Y, Z$, where $2h = L_\xi \phi$ and $L$ is the usual Lie derivative. This manifold was denoted by $M(f_1, f_2, f_3, f_4, f_5, f_6)$.

Natural examples of generalized $(k, \mu)$ space forms are $(k, \mu)$ space forms and generalized Sasakian space forms. The authors in [1] proved that contact metric generalized $(k, \mu)$ space forms are generalized $(k, \mu)$ spaces and if dimension is greater than or equal to 5, then they are $(k, \mu)$ spaces with constant $\phi$-sectional curvature $2f_6 - 1$. They gave a method of constructing examples of generalized $(k, \mu)$ space forms and proved that generalized $(k, \mu)$ space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. Further in [3], it is proved that under $D_a$-homothetic deformation generalized $(k, \mu)$ space form structure is preserved for dimension 3, but not in general. In this paper, we study generalized $(k, \mu)$ space forms under the curvature conditions like generalized recurrent, ricci recurrent, generalized $\phi$-recurrent, flat and $\phi$-flat conditions. The paper is organised as follows.

After preliminaries in section 2, we study generalized recurrent generalized $(k, \mu)$ space forms in section 3 and found the condition for $M(f_1, f_2, f_3, f_4, f_5, f_6)$ to be co-symplectic. In section 4 we study generalized $\phi$-recurrent generalized $(k, \mu)$-space forms and found relations among associated functions. In sections 5 and 6 we study concircular curvature tensor and quasi-conformal curvature of generalized $(k, \mu)$ space forms.

### 2. Preliminaries

In this section, some general definitions and basic formulas are presented which will be used later. A $(2n+1)$-dimensional Riemannian manifold $(M, g)$ is said to be an almost contact metric manifold if it admits a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$, and a 1-form $\eta$ satisfying

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad (3)
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (4)
\]
\[
g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \phi X) = 0, \quad g(X, \xi) = \eta(X). \quad (5)
\]

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is the fundamental 2-form of $M$. 

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It is well known that on a contact metric manifold \((M, \phi, \xi, \eta, g)\), the tensor \(h\) is defined by \(2h = L_{\xi} \phi\) which is symmetric and satisfies the following relations.

\[
h\xi = 0, \quad h\phi = -\phi h, \quad trh = 0, \quad \eta \circ h = 0, \quad (6)
\]

\[
\nabla_X \xi = -\phi X - \phi hX, \quad (\nabla_X \eta)Y = g(X + hX, \phi Y).
\]

In a \((2n + 1)\)-dimensional \((k, \mu)\)-contact metric manifold, we have [6]

\[
h^2 = (k - 1)\phi^2, \quad k \leq 1, \quad (8)
\]

\[
(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (9)
\]

\[
(\nabla_X h)(Y) = [(1 - k)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi hX) - \mu \eta(X)\phi hY.
\]

**Definition 1.** A contact metric manifold \(M\) is said to be

(i) Einstein if \(S(X,Y) = \lambda g(X,Y)\), where \(\lambda\) is a constant and \(S\) is the Ricci tensor,

(ii) \(\eta\)-Einstein if \(S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y)\), where \(\alpha\) and \(\beta\) are smooth functions on \(M\).

In a \((2n + 1)\)-dimensional generalized \((k, \mu)\) space form, the following relations hold.

\[
R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \quad (11)
\]

\[
QX = [2nf_1 + 3f_2 - f_3]X + [(2n - 1)f_4 - f_6]hX - [3f_2 + (2n - 1)f_3]\eta(X)\xi, \quad (12)
\]

\[
S(X,Y) = [2nf_1 + 3f_2 - f_3]g(X,Y) + [(2n - 1)f_4 - f_6]g(hX,Y) - [3f_2 + (2n - 1)f_3]\eta(X)\eta(Y), \quad (13)
\]

\[
S(X,\xi) = 2n(f_1 - f_3)\eta(X), \quad (14)
\]

\[
r = 2n[(2n + 1)f_1 + 3f_2 - 2f_3], \quad (15)
\]

where \(Q\) is the Ricci operator, \(S\) is the Ricci tensor and \(r\) is the scalar curvature of \(M(f_1, ..., f_6)\).

The relation between the associated functions \(f_i, i = 1, ..., 6\) of \(M(f_1, ..., f_6)\) was recently discussed by Carriazo et al. [7].
3. Generalized recurrent generalized \((k, \mu)\) space forms

A generalized \((k, \mu)\) space form \(M(f_1, ..., f_6)\) is called generalized recurrent \([8]\) if its curvature tensor \(R\) satisfies the condition

\[
(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)[g(Z, W)Y - g(Y, W)Z],
\]

where \(A\) and \(B\) are two \(-1\)–forms and \(B\) is non-zero.

**Theorem 1.** A generalized recurrent \(M(f_1, ..., f_6)\) is co-symplectic provided \(f_1 \neq f_3\).

**Proof.** Taking \(Y = W = \xi\) in (1), we obtain

\[
(\nabla_X R)(\xi, Z)\xi = A(X)R(\xi, Z)\xi + B(X)[\eta(Z)\xi - Z].
\]

By the definition of covariant derivative, we have

\[
(\nabla_X R)(\xi, Z)\xi = \nabla_X R(\xi, Z)\xi - R(\nabla_X \xi, Z)\xi - R(\xi, \nabla_X Z)\xi - R(\xi, Z)\nabla_X \xi.
\]

Using (2), (7) and (10), we get

\[
(\nabla_X R)(\xi, Z)\xi = X(f_1 - f_3)[\eta(Z)\xi - Z] - X(f_4 - f_6)hZ - (f_4 - f_6)[(1 - k)g(X, \phi Z)\xi - g(X, h\phi Z)\xi]

+ g(X, h\phi Z)\xi - \mu\eta(X)\phi hZ] + (f_1 - f_3)\nabla_X Z - (f_4 - f_6)g(-\phi X - \phi h X, hZ)\xi.
\]

Now comparing (17) and (19), we obtain

\[
[(X - A(X))(f_1 - f_3) - B(X)][\eta(Z)\xi - Z]

+ [(A(X) - X)(f_4 - f_6)]hZ - (f_4 - f_6)[(1 - k)g(X, \phi Z)\xi - g(X, h\phi Z)\xi]

+ g(X, h\phi Z)\xi - \mu\eta(X)\phi hZ] + (f_1 - f_3)\nabla_X Z + (f_4 - f_6)g(\phi X + \phi h X, hZ)\xi = 0.
\]

Taking \(Z = \xi\) in (20), we obtain

\[
(f_1 - f_3)(\nabla_X \xi) = 0.
\]

Thus \(M\) is co-symplectic if \(f_1 \neq f_3\). Hence the proof.

**Ricci-recurrent generalized \((k, \mu)\) space forms**

A generalized \((k, \mu)\)-space form \(M(f_1, ..., f_6)\) is generalized Ricci-recurrent \([9]\), if its Ricci tensor \(S\) satisfies the condition

\[
(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + 2nB(X)g(Y, Z),
\]

where \(A\) and \(B\) are two non-zero \(-1\)–forms.
Theorem 2. In a generalized Ricci-recurrent \( M(f_1, \ldots, f_6) \), \( f_1 \neq f_3 \) holds. Further the \( 1 \)-\( \text{forms} \ A(X) \) and \( B(X) \) are related by (28).

Proof. By the definition of covariant derivative, we have
\[
(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi). \tag{23}
\]
Using (7) and (14) in (23), we get
\[
(\nabla_X S)(Y, \xi) = 2n(d(f_1 - f_3)(X)\eta(Y) + 2n(f_1 - f_3)g(X + hX, \phi Y)
+ (2nf_1 + 3f_2 - f_3)g(Y, \phi X + \phi h X) + [(2n - 1)f_4 - f_6]g(h Y, \phi X + \phi h X).
\tag{24}
\]
Taking \( Z = \xi \) in (22) and using (5) and (14), we obtain
\[
2n(d(f_1 - f_3)(X)\eta(Y) = 2n(f_1 - f_3)A(X)\eta(Y) + 2nB(X)\eta(Y).
\tag{25}
\]
From (24) and (25), we obtain
\[
2n(d(f_1 - f_3)(X)\eta(Y) = 2n(f_1 - f_3)A(X) - 2nB(X).
\tag{26}
\]
Taking \( Y = \xi \) in (26), we obtain
\[
2n(d(f_1 - f_3)(X) = 2n(f_1 - f_3)A(X) - 2nB(X).
\tag{27}
\]
If \( f_1 - f_3 = c \), a constant, then (27) reduces to
\[
B(X) = cA(X).
\tag{28}
\]
Since \( B(X) \) is not zero, we have \( f_1 \neq f_3 \).

It is easy to see that a generalized recurrent \( M(f_1, \ldots, f_6) \) is always generalized Ricci-recurrent. It follows from theorem 1 and theorem 2 that

Corollary 3. A generalized recurrent \( M(f_1, \ldots, f_6) \) is always co-symplectic.
Definition 2. A generalized $\phi$–Ricci recurrent \cite{4,9}, if
\[ \phi^2((\nabla_X Q)(Y)) = A(X)QY + 2nB(X)Y \] (29)

Definition 3. $\phi$–Ricci symmetric, if
\[ \phi^2((\nabla_X Q)(Y)) = 0, \] (30)
where $Q$ is the Ricci operator, $A(X)$ and $B(X)$ are non-zero 1–forms.

Theorem 4. In a generalized $(k,\mu)$ space form which is $\phi$–Ricci recurrent the relation $3f_2 + (2n - 1)f_3 = 0$ holds.

Proof. Using (4) and (3), we have
\[ -\nabla_X QY + Q(\nabla_X Y) + \eta((\nabla_X Q)Y) = A(X)QY + 2nB(X)Y. \] (31)
Taking $Y = \xi$ in (31) and contracting with respect to $Z$, we obtain
\[ -g(\nabla_X Q\xi, Z) + g(Q(\nabla_X \xi), Z) + \eta((\nabla_X Q)\xi)\eta(Z) = A(X)g(Q\xi, Z) + 2nB(X)\eta(Z) \] (32)
Using (7) and (12) in (32), we obtain
\[ 2n(f_1 - f_3)[g(\phi X, Z) + g(\phi h X, Z)] - S(\phi X, Z) - S(\phi h X, Z) = 2n[(f_1 - f_3)A(X) + B(X)]\eta(Z). \] (33)
Replacing $X$ by $\phi X$ in (33), we get
\[ 2n(f_1 - f_3)[g(\phi^2 X, Z) + g(\phi h X, Z)] - S(\phi^2 X, Z) - S(\phi h X, Z) = 2n[(f_1 - f_3)A(\phi X) + B(\phi X)]\eta(Z). \] (34)
Using (3), (13) and (14) in (34), we get
\[ S(X, Z) = 2n(f_1 - f_3) - ((2n - 1)f_4 - f_6)(k - 1)g(X, Z) + [3f_2 + (2n - 1)f_3]g(h X, Z) + (k - 1)(2n - 1)f_4 - f_6] \eta(X)\eta(Z) + 2n[(f_1 - f_3)A(\phi X) + B(\phi X)]\eta(Z). \] (35)
Replacing $Z$ by $\phi Z$ in (35), we get
\[ S(X, \phi Z) = [2n(f_1 - f_3) - ((2n - 1)f_4 - f_6)(k - 1)]g(X, \phi Z) + [3f_2 + (2n - 1)f_3]g(h X, \phi Z). \] (36)
Again from (13), we have
\[ S(X, \phi Z) = [2nf_1 + 3f_2 - f_3]g(X, \phi Z) + [(2n - 1)f_4 - f_6]g(hX, \phi Z). \] (37)

From (37) and (36), we get
\[ 3f_2 + (2n - 1)f_3 = 0. \] (38)

If \( A(X) \) and \( B(X) \) are zero in (35), then \( M(f_1, ..., f_6) \) is called \( \phi \)-Ricci symmetric \[9\].

It is easy to see that relation (38) holds in \( \phi \)-Ricci symmetric generalized \((k, \mu)\) space form.

Conversely suppose \( 3f_2 + (2n - 1)f_3 = 0 \) holds in \( \phi \)-Ricci symmetric generalized \((k, \mu)\) space form, then from (12)
\[ QY = (2nf_1 + 3f_2 - f_3)Y + [(2n - 1)f_4 - f_6]hY. \]

Differentiating covariantly with respect to \( X \), we obtain
\[ (\nabla_X Q)Y = \nabla_X((2nf_1 + 3f_2 - f_3)Y) + \nabla_X([(2n - 1)f_4 - f_6]hY). \]

Applying \( \phi^2 \) on both sides, we obtain
\[ \phi^2((\nabla_X Q)Y) = d(2nf_1 + 3f_2 - f_3)(X)\phi^2Y + d((2n - 1)f_4 - f_6)(X)\phi h^2Y. \]

Therefore \( M(f_1, ..., f_6) \) is \( \phi \)-Ricci symmetric if and only if \( 2nf_1 + 3f_2 - f_3 \) and \( (2n - 1)f_4 - f_6 \) are constants.

5. CONCIRCULAR CURVATURE TENSOR OF GENERALIZED \((k, \mu)\) SPACE FORMS

The Concircular curvature tensor of \( M(f_1, ..., f_6) \) is given by \[11\]
\[ \tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y]. \] (39)

\( M(f_1, ..., f_6) \) is said to be

**Definition 4.** \( \phi \)-concircular recurrent\[12\], if
\[ \phi^2((\nabla_W \tilde{C})(X, Y)Z) = A(W)\tilde{C}(X, Y)Z, \] (40)
where \( A(W) \) is a non-zero \( 1 \)-form.
Definition 5. :φ−concircular symmetric, if

\[ \phi^2((\nabla W \tilde{C})(X, Y)Z) = 0. \]  

(41)

Theorem 5. In a φ−concircular recurrent generalized \((k, \mu)\) space form, the relation \((2n - 1)f_3 + 3f_2 = 0\) holds.

Proof. Taking the covariant differentiation of (5), we get

\[ (\nabla W \tilde{C})(X, Y)Z = (\nabla W R)(X, Y)Z - \frac{dr(W)}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y]. \]  

(42)

Applying \(\phi^2\) on both sides, we get

\[ \phi^2((\nabla W \tilde{C})(X, Y)Z) = \phi^2((\nabla W R)(X, Y)Z) - \frac{dr(W)}{2n(2n + 1)}[g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y]. \]  

(43)

Suppose \(M(f_1, ..., f_6)\) is \(\phi−\)concircular recurrent. Then from (3) and (40) in (43) and taking \(X = \xi\), we obtain

\[ A(W)\tilde{C}(\xi, Y)Z = (\nabla W R)(\xi, Y)Z + \eta((\nabla W R)(\xi, Y)Z)\xi + \frac{dr(W)}{2n(2n + 1)}\eta(Z)\phi^2 Y. \]  

(44)

Suppose the vector fields \(X, Y\) and \(Z\) are orthogonal to \(\xi\). Then taking \(X = \xi\) in (5) and using (2) and (3), we get

\[ \tilde{C}(\xi, Y)Z = [(f_1 - f_3) - \frac{r}{2n(2n + 1)}]g(Y, Z)\xi + (f_4 - f_6)g(hZ, Y)\xi. \]  

(45)

Now using (2), (3) and (45) in (44) and contracting with respect to \(\xi\), we obtain

\[ A(W)\left[ \left( (f_1 - f_3) - \frac{r}{2n(2n + 1)} \right) g(Y, Z) + (f_4 - f_6)g(hZ, Y) \right] = 0. \]  

(46)

Taking \(Z = \xi\) in (46) and using (15), we obtain

\[ (2n - 1)f_3 + 3f_2 = 0. \]  

(47)
5.1. Concircular curvature tensor of \((k, \mu)\) space forms

In a \((k, \mu)\) space form \(M\), curvature tensor \(R\) is given by

\[
R(X,Y)Z = (c + 3/4)[g(Y,Z)X - g(X,Z)Y]
+ (c - 1/4)[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z]
+ (c + 3/4 - k)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]
+ \frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX]
+ (1 - \mu)[\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi],
\]

where \(c\) is the constant \(\phi\)–sectional curvature of \(M\).

From (48), we have

\[
R(\xi,Y)Z = k[g(Y,Z)\xi - \eta(Z)Y] + \mu[g(Y,hZ)\xi - \eta(Z)hY],
\]

\[
r = n[c(n + 1) + 3(n - 1) + 4k].
\]

**Theorem 6.** In a \(\phi\)–concircular recurrent \((k, \mu)\) space form, the constant \(\phi\)–sectional curvature of \(M\) is given by \(c = \frac{k(4n - 2) - 3(n - 1)}{n + 1}\).

**Proof.** Suppose \(M\) is \(\phi\)–concircular recurrent. Then from (3) and (40) in (43) and taking \(X = \xi\), we obtain

\[
A(W)\tilde{C}(\xi,Y)Z = -(\nabla_W R)(\xi,Y)Z + \eta((\nabla_W R)(\xi,Y)Z)\xi + \frac{dr(W)}{2n(2n + 1)}\eta(Z)\phi^2 Y.
\]

Suppose the vector fields \(X\), \(Y\) and \(Z\) are orthogonal to \(\xi\). Then taking \(X = \xi\) in (5) and using (49), (50) and (3), we get

\[
\tilde{C}(\xi,Y)Z = \left( k - \frac{c(n + 1) + 3(n - 1) + 4k}{2(2n + 1)} \right) g(Y,Z)\xi + \mu g(hZ,Y)\xi.
\]

Using (52), (49) and (3) in (51) and contracting with respect to \(\xi\), we obtain

\[
A(W) \left[ \left( k - \frac{c(n + 1) + 3(n - 1) + 4k}{2(2n + 1)} \right) g(Y,Z) + \mu g(hZ,Y) \right] = 0.
\]

Taking \(Z = \xi\) in (53), we get

\[
c = \frac{k(4n - 2) - 3(n - 1)}{n + 1}.
\]
6. Quasi-conformal curvature tensor on generalized $(k,\mu)$ space forms

In a $(2n+1)$-dimensional generalized $(k,\mu)$ space form $M(f_1,\ldots,f_6)$, the quasi-conformal curvature tensor \[11\] is given by

$$W(X,Y)Z = aR(X,Y)Z + b[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX]$$

$$- \frac{a + 2b(2n)}{2n(2n+1)}r[g(Y,Z)X - g(X,Z)Y],$$

(55)

where $a$ and $b$ are arbitrary constants such that $ab \neq 0$.

**Definition 6.** A generalized $(k,\mu)$ space form $M(f_1,\ldots,f_6)$ is said to be quasi-conformally $\phi$-flat if

$$g(W(X,Y)Z,\phi W) = 0.$$  

(56)

**Definition 7.** A generalized $(k,\mu)$ space form $M(f_1,\ldots,f_6)$ is said to be quasi-conformally flat if

$$W(X,Y)Z = 0.$$  

(57)

6.1. Quasi-conformally $\phi$-flat generalized $(k,\mu)$ space forms

In a $(2n+1)$-dimensional almost contact metric manifold $M$, [10], if $\{e_1,\ldots,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in $M$, then $\{\phi e_1,\ldots,\phi e_{2n},\xi\}$ is also a local orthonormal basis and

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n,$$

(58)

$$\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) - S(Y, \xi)\eta(Z),$$

(59)

$$\sum_{i=1}^{2n} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z),$$

(60)

for all $Y, Z \in TM$. In a generalized $(k,\mu)$ space form, we have

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r - 2n(f_1 - f_3),$$

(61)
\[
\sum_{i=1}^{2n} R(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} R(\phi e_i, Y, Z, \phi e_i) \\
= S(Y, Z) - ((f_1 - f_3)[g(Y, Z) - \eta(Z)\eta(Y)] + (f_4 - f_6)g(hZ, Y)).
\] (62)

**Theorem 7.** A quasi-conformally \(\phi\)-flat generalized \((k, \mu)\) space form is an \(\eta\)-Einstein manifold.

**Proof.** From (55), we have
\[
g(W(X, Y)Z, \phi W) = \\
ag(R(X, Y)Z, \phi W) + b[S(X, Z)g(Y, \phi W) - S(Y, Z)g(X, \phi W) + g(X, Z)S(Y, \phi W) \\
- g(Y, Z)S(X, \phi W) - a + 2b(2n) \frac{n(2n+1)}{2n(2n+1)} r[g(Y, Z)g(X, \phi W) - g(X, Z)g(Y, \phi W)],
\] (63)
for \(X, Y, Z, W \in TM\).

For a local orthonormal basis \(\{e_1, \ldots, e_{2n}, \xi\}\) of vector fields in \(M(f_1, \ldots, f_6)\), putting \(X = \phi e_i\) and \(W = e_i\) in (63) and using (58), (59), (60), (61) and (62), we obtain
\[
\sum_{i=1}^{2n} g(W(\phi e_i, Y)Z, \phi e_i) = a[S(Y, Z) - (f_1 - f_3)(g(Y, Z) - \eta(Z)\eta(Y)) \\
- (f_4 - f_6)g(hZ, Y)] + b[(2 - 2n)S(Y, Z) - S(\xi, Z)\eta(Y) - S(\xi, Z)\eta(Z) \\
- g(Y, Z)(r - 2n(f_1 - f_3))] - a + 2b(2n) \frac{n(2n+1)}{2n(2n+1)} r[g(Y, Z)2n - g(\phi Y, \phi Z)].
\] (64)

If \(M(f_1, \ldots, f_6)\) is quasiconformally \(\phi\)-flat, then (64) reduces to
\[
[b(2n-2) - a]S(Y, Z) = pg(Y, Z) + q\eta(Y)\eta(Z) - a(f_4 - f_6)g(hZ, Y),
\] (65)
where
\[
p = -a(f_1 - f_3) - b(r - 2n(f_1 - f_3)) - m(2n - 1),
q = a(f_1 - f_3) - 4nb(f_1 - f_3) - m,
m = \frac{a + 2b(2n)}{2n(2n+1)} r.
\]

Replacing \(Z\) by \(hZ\) in (65) and using (13) and (8), we obtain
\[
g(hZ, Y) = \frac{-a(f_4 - f_6) - (b(2n-2) - a)e}{[b(2n-2) - a][k - p]}(k - 1)[\eta(Z)\eta(Y) - g(Y, Z)],
\] (66)
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with
\[
t = 2nf_1 + 3f_2 - f_3, \\
e = (2n - 1)f_4 - f_6.
\]

Now substituting for \(g(hZ,Y)\) in (65), we obtain
\[
S(Y,Z) = \alpha g(Y,Z) + \beta \eta(Y)\eta(Z),
\]
where
\[
\alpha = \frac{p + l}{b(2n - 2) - a}, \\
\beta = \frac{q - l}{b(2n - 2) - a}, \\
l = \frac{a(f_4 - f_6)|-a(f_4 - f_6) - (b(2n - 2) - a)e|}{(b(2n - 2) - a)t - p}(k - 1).
\]

Therefore \(M(f_1, ..., f_6)\) is an \(\eta\)–Einstein.

Putting \(Z = \xi\) in (65) and using (14), we obtain
\[
2nb(f_1 - f_3)(2n - 1) - 2na(f_1 - f_3) = -br - \left(\frac{a + 4nb}{2n + 1}r\right).
\]

If \(f_1 = f_3\), then from (68), we have
\[
r = 0 \text{ or } a + b + 6nb = 0.
\]

Thus we have

**Proposition 8.** In a quasi-conformally \(\phi\)-flat \(M(f_1, ..., f_6)\), either \(r = 0\) or \(a + b + 6nb = 0\) provided \(f_1 = f_3\).

### 6.2. Quasi-conformally flat generalized \((k,\mu)\) space forms

**Theorem 9.** In a quasi-conformally flat generalized \((k,\mu)\) space form which is \(\phi\)–ricci recurrent the scalar curvature is given by \(-\frac{(2n + m)}{a}\).

**Proof.** Suppose \(M(f_1, ..., f_6)\) is Quasi-conformally flat, then from (55) and (57), we obtain
\[
aR(X,Y)Z = -b[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] \\
+ \frac{a + 2b(2n)}{2n(2n + 1)}r[g(Y,Z)X - g(X,Z)Y].
\]
Using (12) and (13) in the above equation, it reduces to

\[
R(X, Y)Z = \frac{1}{a} \left\{ -(2bt + m)[g(X, Z)Y - g(Y, Z)X] - be\{g(hX, Z)Y - g(hY, Z)X] \\
+ bs\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi \\
- be\{g(X, Z)hY - g(Y, Z)hX] + bs[\eta(X)Y - \eta(Y)\eta(X)]\xi(Z),
\right. 
\]  

(70)

where

\[
t = 2nf_1 + 3f_2 - f_3, \\
s = 3f_2 + (2n - 1)f_3, \\
e = (2n - 1)f_4 - f_6, \\
m = a + 2b(2n). \\
\]

If \(M(f_1, ..., f_6)\) is \(\phi\)-Ricci recurrent, then \(s = 0\) and \(e = 0\). Then from (70), we obtain

\[
R(X, Y)Z = \left(\frac{-(2bt + m)}{a}\right)[g(X, Z)Y - g(Y, Z)X]. 
\]  

(71)

References


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