NEW RESULTS RELATED TO STARLIKENESS AND CONVEXITY OF THE BERNARDI INTEGRAL OPERATOR

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ABSTRACT. In this paper we extend some known results related to starlikeness and convexity of the Bernardi integral operator given by

\[ L_\beta[f](z) = \frac{\beta + 1}{z^\beta} \int_0^z f(t)t^{\beta - 1} dt \]  

2010 Mathematics Subject Classification: 30C45, 30A10, 30C80.

Keywords: univalent function, analytic function, integral operator, starlike functions, convex functions.

1. Introduction and preliminaries

Let \( \mathcal{H}(U) \) denote the set of holomorphic functions in the open disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and let

\[ \mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots \} \]

with \( \mathcal{A}_1 = \mathcal{A} \). Also, for a positive integer \( n \) and \( a \in \mathbb{C} \), let

\[ \mathcal{H}[a,n] = \{ f \in \mathcal{H}(U, f(z) = a + a_n z^n + \ldots, z \in U \} \]

and \( \mathcal{S} = \{ f \in \mathcal{A} : f \text{ is univalent in } U \} \).

Let

\[ \mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) + 1 > \alpha, z \in U \right\} \]

denote the class of normalized convex functions of order \( \alpha \), where \( \alpha \in \mathbb{R}, \alpha < 1 \). For \( \alpha = 0 \), \( \mathcal{K}(0) = \mathcal{K} \) denote the class of normalized convex functions in \( U \).
\[ S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\} \]

denote the class of starlike function of order \( \alpha \), with \( \alpha \in \mathbb{R}, \alpha < 1 \). For \( \alpha = 0 \), \( S^*(0) = S^* \) denote the class of starlike functions in \( \mathcal{U} \).

**Theorem 1.** [[2][10][7] Theorem 9.5.5., p. 218] If \( L_\gamma : \mathcal{A} \rightarrow \mathcal{A} \) is the integral operator defined by
\[
L_\gamma[f](z) = F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt,
\]
and \( \Re \gamma \geq 0, z \in \mathcal{U} \), then it is well known that:
(i) \( L_\gamma(S^*) \subset S^* \);
(ii) \( L_\gamma(K) \subset K \).

**Theorem 2.** ([8], Theorem 1) Let \( f \in \mathcal{A}, \beta \geq 1 \) and let
\[
F(z) = L_\beta[f](z) = \frac{\beta + 1}{z^\beta} \int_0^z f(t)t^{\beta-1}dt, \quad z \in \mathcal{U},
\]
If
\[
\Re \left[ \frac{zf''(z)}{f'(z)} + 1 \right] > -\frac{1}{2\beta}, \quad z \in \mathcal{U},
\]
then the function \( F \) is convex.

**Theorem 3.** ([9], Theorem 1) Let \( f \in \mathcal{A}, z \in \mathcal{U}, \beta \geq 1 \) and
\[
F(z) = L_\beta[f](z) = \frac{\beta + 1}{z^\beta} \int_0^z f(t)t^{\beta-1}dt, \quad z \in \mathcal{U},
\]
then the function \( F \) is starlike.

### 2. Main Results

In [9], Georgia Irina Oros was proved that if \( f \in S^* \left( -\frac{1}{n^2} \right), \beta \geq 1 \), then \( F \) given by (1) is starlike. We will extend this result from the next theorem:
Theorem 4. Let $\beta \geq 1$, $f \in A_n$, $F(z) = L_\beta[f](z)$ where $L_\beta$ is given by (1).

If

$$\Re \frac{zf'(z)}{f(z)} > -\frac{\beta}{2}, \; z \in \mathcal{U}$$

then

$$\Re \frac{zf'(z)}{F(z)} > -\beta, \; z \in \mathcal{U}.$$ 

Proof. Since $f \in A_n$, we have $F(z) = z + b_{n+1}z^{n+1} + ..., F(0) = 0, F'(0) = 1$.

From (1) we have

$$z^\beta F(z) = (\beta + 1) \int_0^z f(t)t^{\beta-1}dt, \; z \in \mathcal{U}. \quad (3)$$

By differentiating (3) and by a simple calculation we obtain

$$F(z) \left[ \beta + \frac{zF'(z)}{F(z)} \right] = (\beta + 1)f(z), \; z \in \mathcal{U}. \quad (4)$$

We let

$$p(z) = \frac{1}{\beta + 1} \left[ \frac{zF'(z)}{F(z)} + \beta \right] = 1 + c_nz^n + ..., \; p(0) = 1, \; p \in \mathcal{H}[1,n]. \quad (5)$$

Using (5), then (4) becomes

$$F(z) \cdot p(z) = f(z), \; z \in \mathcal{U}. \quad (6)$$

By differentiating (6) and using (5), we obtain

$$(1 + \beta)p(z) - \beta + \frac{zp'(z)}{p(z)} = z\frac{f'(z)}{f(z)}, \; z \in \mathcal{U}. \quad (7)$$

Using (2) and (7), we have

$$\Re \left[ (1 + \beta)p(z) - \beta + \frac{zp'(z)}{p(z)} \right] = \Re \frac{zf'(z)}{f(z)} > -\frac{\beta}{2}$$

which is equivalent to

$$\Re \left[ (1 + \beta)p(z) + \frac{zp'(z)}{p(z)} - \frac{\beta}{2} \right] > 0, \; z \in \mathcal{U}. \quad (8)$$

We let $\psi : \mathbb{C}^2 \times \mathcal{U} \rightarrow \mathbb{C}$,
\[ \psi(p(z), zp'(z), z) = (1 + \beta)p(z) + \frac{zp'(z)}{p(z)} - \frac{\beta}{2}, \quad z \in \mathcal{U}. \quad (9) \]

Then (8) is equivalent to
\[ \text{Re} \psi(p(z), zp'(z), z) > 0, \quad z \in \mathcal{U}. \quad (10) \]

In order to prove our theorem, we use a well known Lemma due to S.S. Miller and P.T. Mocanu (see [3]-[6]). For that we calculate
\[ \text{Re} \psi(i\rho, \sigma, z) = \text{Re} \left[ (1 + \beta)i\rho + \frac{\sigma}{i\rho} - \frac{\beta}{2} \right] = -\frac{\beta}{2} \leq 0. \]

Now, using the above mentioned Lemma, we get that \( \text{Re} p(z) > 0, \quad z \in \mathcal{U} \), i.e
\[ \text{Re} \frac{1}{1 + \beta} \left[ \frac{zf''(z)}{f(z)} + \beta \right] > 0, \]
which imply that
\[ \text{Re} \frac{zf'(z)}{f(z)} > -\beta, \quad z \in \mathcal{U} \]
hence \( F \in \mathcal{S}^*(-\beta), \quad \beta \geq 0. \)

**Remark 1.** This result improves the results in Theorem 1.

**Remark 2.** For \( \beta = 1 \), Theorem 4 extend the results obtained in [7], Theorem 9.5.2, p. 214, (R. J. Libera Theorem) for the Libera operator.

In [8], Georgia Irina Oros showed that if \( f \in \mathcal{K} \left( -\frac{1}{2\beta} \right) \), \( \beta \geq 1 \), then \( F \in \mathcal{K} \), where \( F \) is given by (1). We will extend this result by the following theorem:

**Theorem 5.** If \( \beta \geq 0, \ f \in \mathcal{A}_n \) and satisfies
\[ \text{Re} \frac{zf''(z)}{f(z)} + 1 > -\frac{\beta}{2} \quad (11) \]
then \( L_\beta[f](z) = F(z) \in \mathcal{K}(-\beta) \), where \( L_\beta \) given by (1).
Proof. By differentiating (3), and by a simple calculation we obtain that

$$F'(z)\left[\beta + 1 + \frac{zF''(z)}{F'(z)}\right] = (\beta + 1)f'(z), \ z \in \mathcal{U}. \quad (12)$$

We let

$$(1 + \beta)p(z) = \beta + 1 + \frac{zF''(z)}{F'(z)} = \beta + 1 + c_nz^n + ..., \ p(0) = 1, \ p \in \mathcal{H}[1, n]. \quad (13)$$

Using (13) in (12), we have

$$F'(z)p(z) = f'(z), \ z \in \mathcal{U}. \quad (14)$$

By differentiating (14) and by a simple calculation we obtain

$$\frac{zF''(z)}{F'(z)} + \beta + 1 + \frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} + 1 + \beta, \ z \in \mathcal{U}. \quad (15)$$

Using (13) in (15) we obtain

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} + 1 + \beta, \ z \in \mathcal{U}. \quad (16)$$

From (11), we have:

$$\text{Re}\left[p(z) + \frac{zp'(z)}{p(z)} - \frac{\beta^2}{2}\right] > 0. \quad (17)$$

We let $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C},$

$$\psi(p(z), zp'(z), z) = p(z) + \frac{zp'(z)}{p(z)} - \frac{\beta}{2}, \ z \in \mathcal{U}. \quad (18)$$

Then (17) becomes

$$\text{Re}\psi(p(z), zp'(z), z) > 0, \ z \in \mathcal{U}. \quad (19)$$

In order to prove our theorem, we use a well known Lemma due to S.S. Miller and P.T. Mocanu (see [3]-[6]). For that we calculate

$$\text{Re}\psi(i\rho, \sigma, z) = \text{Re}\left[i\rho + \frac{\sigma}{i\rho} - \frac{\beta}{2}\right] = -\frac{\beta}{2} < 0.$$

Now, using the above mentioned Lemma, we get that $\text{Re}\psi(z) > 0,$ $z \in \mathcal{U},$ i.e.
\[ \text{Re} \left[ \frac{z F''(z)}{F'(z)} + 1 \right] > -\beta, \quad z \in \mathcal{U} \]

hence \( F \in \mathcal{K}(-\beta) \).

**Remark 3.** The results of this theorem extend the results obtained in Theorem 1.

**Remark 4.** For \( \beta = 1 \), the results extend the results of Th. 9.5.2, Th. 9.5.3.[7], p. 214-215.

**REFERENCES**


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