ON UNIVALENCE CRITERIA FOR ANALYTIC FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR AND RUSCHEWEYH DERIVATIVE

A. ALB LUPAŞ

Abstract. In this paper we obtain sufficient conditions for univalence of analytic functions defined by the linear operator $L_\alpha^n : \mathcal{A} \to \mathcal{A}$, $L_\alpha^n f(z) = (1 - \alpha) R^n f(z) + \alpha S^n f(z)$, $z \in U$, where $R^n f(z)$ is the Ruscheweyh derivative, $S^n f(z)$ the Sălăgean operator and $\mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots , z \in U \}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$.

2000 Mathematics Subject Classification: 30C45, 30A20, 34A40.

Keywords: differential operator, analytic functions, univalent functions.

1. Introduction

Denote by $U$ the unit disc of the complex plane, $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and $\mathcal{H}(U)$ the space of holomorphic functions in $U$.

Let $\mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots , z \in U \}$ with $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{S}$ the subclass of functions that are univalent in $U$.

Definition 1. Sălăgean [10]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator $S^n$ is defined by $S^n : \mathcal{A} \to \mathcal{A}$,

$$
S^0 f(z) = f(z) \\
S^1 f(z) = zf'(z) \\
\vdots \\
S^{n+1} f(z) = z (S^n f(z))', \quad z \in U.
$$

Remark 1. If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$. 

25
Definition 2. (Ruscheweyh [9]) For \( f \in A \), \( n \in \mathbb{N} \), the operator \( R^n \) is defined by \( R^n : A \to A \),

\[
R^0 f(z) = f(z) \\
R^1 f(z) = zf'(z) \\
\vdots \\
(n+1)R^{n+1} f(z) = z (R^n f(z))' + nR^n f(z), \quad z \in U.
\]

Remark 2. If \( f \in A \), \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j \), \( z \in U \).

Definition 3. [1], [2] Let \( \alpha \geq 0 \), \( n \in \mathbb{N} \). Denote by \( L^n_\alpha \) the operator given by \( L^n_\alpha : A \to A \),

\[
L^n_\alpha f(z) = (1 - \alpha)R^n f(z) + \alpha S^n f(z), \quad z \in U.
\]

Remark 3. If \( f \in A \), \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( L^n_\alpha f(z) = z + \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j \), \( z \in U \).

Our considerations are based on the following results.

Lemma 1. [4] Let \( f \in A \). If for all \( z \in U \)

\[
\left( 1 - |z|^2 \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,
\]

then the function \( f \) is univalent in \( U \).

Lemma 2. [7] Let \( f \in A \). If for all \( z \in U \)

\[
\left| \frac{zf'(z)}{f^2(z)} - 1 \right| \leq 1,
\]

then the function \( f \) is univalent in \( U \).

Lemma 3. [11] Let \( \mu \) be a real number, \( \mu > \frac{1}{2} \) and \( f \in A \). If for all \( z \in U \)

\[
\left| \left( 1 - |z|^{2\mu} \right) \frac{zf''(z)}{f'(z)} + 1 - \mu \right| \leq \mu,
\]

then the function \( f \) is univalent in \( U \).
Lemma 4. [6] If \( f(z) \in S \) and 
\[
\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,
\]
then 
\[
\sum_{n=1}^{\infty} (n-1) |b_n|^2 \leq 1.
\]

Lemma 5. [8] Let \( \nu \in \mathbb{C}, \Re(\nu) \geq 0 \) and \( f \in A \). If for all \( z \in U \) 
\[
\left(1 - |z|^{2\Re(\nu)}\right) \frac{|z f''(z)|}{|f'(z)|} \leq 1,
\]
then the function 
\[
F_\nu(z) = \left(\nu \int_0^z u^{\nu-1} f'(u) \, du\right)^{\frac{1}{\nu}}
\]
is univalent in \( U \).

2. The main result

Following the paper of M. Darus and R. Ibrahim [5], we establish the sufficient conditions to obtain a univalence for analytic function involving the differential operator \( RD_{\lambda,\alpha}^n \).

Theorem 6. Let \( f \in A \). If for all \( z \in U \),
\[
\sum_{j=2}^{\infty} \left\{ \alpha_j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} [j(2j-1)] |a_j| \leq 1. \tag{1}
\]
Then \( L_{\alpha}^n f(z) \) is univalent in \( U \).

Proof. Let \( f \in A \). Assume that (1) is hold. Then for all \( z \in U \) we have 
\[
\left(1 - |z|^2\right) \frac{z (L_{\alpha}^n f(z))''}{(L_{\alpha}^n f(z))'} \leq \left(1 + |z|^2\right) \left| \frac{z (L_{\alpha}^n f(z))''}{(L_{\alpha}^n f(z))'} \right| =
\]
\[
\left(1 + |z|^2\right) \left| \frac{z \sum_{j=2}^{\infty} \left\{ \alpha_j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} [j(j-1)] a_j |a_j|}{1 - \sum_{j=2}^{\infty} \left\{ \alpha_j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j |a_j|} \right| \leq 1.
\]

Thus, in view of Lemma 1, \( L_{\alpha}^n f(z) \) is univalent in \( U \).
Theorem 7. Let \( f \in A \). If for all \( z \in U \),
\[
\sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n + j - 1)!}{n!(j - 1)!} \right\} |a_j| \leq \frac{1}{\sqrt{7}}, \tag{2}
\]
then \( L_\alpha^n f(z) \) is univalent in \( U \).

Proof. Let \( f \in A \). Assume that (2) is hold. It is sufficient to show that
\[
\left| \frac{z^2 (L_\alpha^n f(z))'}{(L_\alpha^n f(z))^2} - 1 \right| \leq 1,
\]
which is equivalent to show that
\[
\left| \frac{z^2 (L_\alpha^n f(z))'}{2 (L_\alpha^n f(z))^2} \right| \leq 1.
\]
We have
\[
\left| \frac{z^2 (L_\alpha^n f(z))'}{2 (L_\alpha^n f(z))^2} \right| = \frac{z^2 \left( 1 + \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n + j - 1)!}{n!(j - 1)!} \right\} ja_j z^{j-1} \right)}{2 \left( z + \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n + j - 1)!}{n!(j - 1)!} \right\} a_j z^j \right)^2} =
\]
\[
\leq \frac{1 + \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n + j - 1)!}{n!(j - 1)!} \right\} j a_j z^{j-1}}{2 \left( 1 + 2 \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n + j - 1)!}{n!(j - 1)!} \right\} a_j z^j \right)^2}
\]
which is less than 1 if the assertion (2) is hold. Thus in view of Lemma 2, \( L_\alpha^n f(z) \) is univalent in \( U \).

Theorem 8. Let \( f \in A \). If for all \( z \in U \)
\[
\sum_{j=2}^{\infty} j \left[ (j - 1) + (2\mu - 1) \right] \left\{ \alpha j^n + (1 - \alpha) \frac{(n + j - 1)!}{n!(j - 1)!} \right\} |a_j| \leq 2\mu - 1, \quad \mu > \frac{1}{2}, \tag{3}
\]
then \( L_\alpha^n f(z) \) is univalent in \( U \).
Proof. Let \( f \in A \). Then for all \( z \in U \) we have

\[
\left| (1 - |z|^{2\mu}) \frac{z \left( L^n_\alpha f(z) \right)''}{(L^n_\alpha f(z))'} + 1 - \mu \right| \leq \left( 1 + |z|^{2\mu} \right) \frac{z \left( L^n_\alpha f(z) \right)''}{(L^n_\alpha f(z))'} \left| 1 - \mu \right|
\]

\[
2 \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j \left( j - 1 \right) \left| a_j \right| + \left| 1 - \mu \right|
\]

the last inequality is less than \( \mu \) if the assertion (3) is hold. thus, in view of Lemma 3, \( L^n_\alpha f(z) \) is univalent in \( U \).

As applications of Theorems 6, 7 and 8 we have the following result

**Theorem 9.** Let \( f \in A \). If for all \( z \in U \) one of the inequalities (1-3) holds, then

\[
\sum_{j=1}^{\infty} (j - 1) |b_j|^2 \leq 1,
\]

where

\[
\frac{z}{L^n_\alpha f(z)} = 1 + \sum_{j=1}^{\infty} b_j z^j.
\]

Proof. Let \( f \in A \). Then, in view of Theorems 6, 7 or 8, \( L^n_\alpha f(z) \) is univalent in \( U \).

Hence, by Lemma 4 we obtain the result.

**Theorem 10.** Let \( f \in A \). If for all \( z \in U \),

\[
\sum_{j=2}^{\infty} j \left[ 2 (j - 1) + \operatorname{Re}(\nu) \right] \left\{ \alpha j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} |a_j| \leq \operatorname{Re}(\nu), \quad \operatorname{Re}(\nu) > 0.
\]

Then

\[
G_\nu(z) = \left( \nu \int_0^z u^{\nu-1} \left( L^n_\alpha f(u) \right)' du \right)^{\frac{1}{\nu}}
\]

is univalent in \( U \).

Proof. Let \( f \in A \). Then for all \( z \in U \), we have

\[
\frac{\left( 1 - |z|^{2\operatorname{Re}(\nu)} \right)}{\operatorname{Re}(\nu)} \left| \frac{z \left( L^n_\alpha f(z) \right)''}{(L^n_\alpha f(z))'} \right| \leq \left( 1 + |z|^{2\operatorname{Re}(\nu)} \right) \frac{z \left( L^n_\alpha f(z) \right)''}{(L^n_\alpha f(z))'} \left| 1 - \mu \right|
\]

\[
2 \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j \left( j - 1 \right) \left| a_j \right| \frac{1}{\operatorname{Re}(\nu)} \frac{1}{1 - \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j \left( j - 1 \right) \left| a_j \right|}.
\]

29
The last inequality is less than 1 if the assertion (4) is hold. Thus, in view of Lemma 5, $G_{\nu}(z)$ is univalent.

References


Alina Alb Lupaş
Department of Mathematics and Computer Science,
University of Oradea,
Oradea, Romania
email: dalb@uoradea.ro