A SUBCLASS OF ANALYTIC FUNCTIONS AND A GENERALIZED LINEAR DIFFERENTIAL OPERATOR

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Abstract. In this article a sub class of analytic function is introduced, which is defined using a generalized differential operator and various properties of this sub class are discussed.

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1. Introduction

Let $A$ denote the class of analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

defined on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Silverman introduced and studied about a sub class $T$ of $A$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

(1.2)

Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be a function in $A$ then the convolution or Hadamard product of $f$ given by (1.1) and $g(z)$ is defined as

$$(f \ast g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Modified Hadamard product for functions with negative coefficients $f(z)$ given by (1.2) and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0)$ is defined as

$$(f \ast g)(z) := z - \sum_{n=2}^{\infty} a_n b_n z^n.$$
For complex numbers \( \alpha_1, \alpha_2, \ldots, \alpha_q \) and \( \beta_1, \beta_2, \ldots, \beta_s; (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_{0}^{-} \setminus \mathbb{Z}_{0}^{-} = \{0, -1, -2, \ldots\}; \) for \( j = 1, 2, \ldots, s \), the generalized hypergeometric function, denoted as \( _qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) \), defined as

\[
_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_q)_n z^n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_s)_n n!}
\]

where \( q \leq s + 1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U \) and \( \mathbb{N} \) denotes the set of all positive integers and \( (x)_n \) is the Pochhammer symbol defined in terms of gamma function, as

\[
(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{if } n = 0 \\ x(x+1)\ldots(x+n-1) & \text{if } n \in \mathbb{N}. \end{cases}
\]

Corresponding to the function \( g_{q,s}(\alpha_1, \beta_1; z) \), defined by

\[
g_{q,s}(\alpha_1, \beta_1; z) := z_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z),
\]

let us introduce a generalized differential operator \( D_{\lambda,\mu}^m(\alpha_1, \beta_1) f(z) : \mathcal{A} \to \mathcal{A} \) as follows

\[
D_{\lambda,\mu}^0(\alpha_1, \beta_1) f(z) := f(z) \ast g_{q,s}(\alpha_1, \beta_1; z)
\]

\[
D_{\lambda,\mu}^1(\alpha_1, \beta_1) f(z) := (1 - \lambda + \mu)(f(z) \ast g_{q,s}(\alpha_1, \beta_1; z)) + (\lambda - \mu)z(f(z) \ast g_{q,s}(\alpha_1, \beta_1; z))' + \lambda \mu z^2(f(z) \ast g_{q,s}(\alpha_1, \beta_1; z))^''
\]

\[
D_{\lambda,\mu}^m(\alpha_1, \beta_1) f(z) := D_{\lambda,\mu}^{m-1}(\alpha_1, \beta_1) f(z)
\]

where \( 0 \leq \mu \leq \lambda \leq 1 \) and \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). It is easy to observe that

\[
D_{\lambda,\mu}^m(\alpha_1, \beta_1) f(z) = z + \sum_{n=2}^{\infty} \frac{[1+(n-1)(\lambda - \mu + n\mu\lambda)]^m (\alpha_1)_n (\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_s)_n (n-1)!} \alpha_n z^n.
\]

For brevity let us take

\[
B_n = \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_s)_n (n-1)!}.
\]

Hence

\[
D_{\lambda,\mu}^m(\alpha_1, \beta_1) f(z) = z + \sum_{n=2}^{\infty} [1+(n-1)(\lambda - \mu + n\mu\lambda)]^m B_n \alpha_n z^n.
\]

This operator \( D_{\lambda,\mu}^m(\alpha_1, \beta_1) f(z) \) generalizes several earlier operators for proper choices of the parameters. For \( \mu = 0 \), we find \( D_{\lambda,0}^m(\alpha_1, \beta_1) f(z) \) reduces to the operator introduced and studied by Selvaraj et al., [27]. For \( q = 2, s = 1, \alpha_1 = \beta_1 \)
we see that this operator reduces to the operator introduced and studied by Dorina Răducanu et al., [4]. For \( \lambda = 1, \mu = 0, q = 2, s = 1, \alpha_1 = \beta_1 \) we obtain the differential operator defined by Al Oboudi [16]. For \( \lambda = \mu = 0 \) we obtain Dziok-Srivatsava operator [5].

Further by specializing the parameters we can find Ruscheweyh derivative operator [25], Carlson-Shaffer operator [3], fractional calculus operators [18, 19], Hohlov linear operator [8] and the generalized Bernardi-Libera-Livingston linear integral operator [2, 11, 13] and Sălăgean derivative operator [26].

**Definition 1.1.** Let \( 0 \leq \gamma \leq 1, \alpha \geq 1, k \geq 0 \) and \( 0 \leq \beta < 1 \). A function \( f \in \mathcal{A} \) is said to be in the class \( S(\lambda, \mu, m, \gamma, \alpha, k, \beta) \), if it satisfies

\[
\Re \left\{ \frac{\alpha z G'(z)}{G(z)} - (\alpha - 1) \right\} > k \left| \frac{z G'(z)}{G(z)} - \alpha \right| + \beta \tag{1.4}
\]

where

\[
G(z) = (1 - \gamma) D_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z) + \gamma z [D_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)]' \tag{1.5}
\]

Also we define \( TS(\lambda, \mu, m, \gamma, \alpha, k, \beta) = T \cap S(\lambda, \mu, m, \gamma, \alpha, k, \beta) \).

By specializing the parameters involved in \( S(\lambda, \mu, m, \gamma, \alpha, k, \beta) \) and \( TS(\lambda, \mu, m, \gamma, \alpha, k, \beta) \) one could result in known classes of analytic functions which were studied earlier such as Starlike functions, parabolic starlike functions and \( k \)-starlike functions. Further several new subclasses of analytic functions could be defined by specializing the parameters involved.

In this investigation various properties of the functions belonging to the classes \( S(\lambda, \mu, m, \gamma, \alpha, k, \beta) \) and \( TS(\lambda, \mu, m, \gamma, \alpha, k, \beta) \).

2. COEFFICIENT ESTIMATES

**Lemma 2.1.** [4] Let \( \beta \) be a real number and let \( w \) be a complex number. Then \( \Re w \geq \beta \) if and only if

\[
|w + (1 - \beta)| - |w - (1 + \beta)| \geq 0.
\]

**Theorem 2.2.** Let \( f(z) \in \mathcal{A} \) as given by (1.1). If

\[
\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(1+k)] B_n |a_n| \leq 1 - \beta \tag{2.1}
\]

then \( f \in S(\lambda, \mu, m, \gamma, \alpha, k, \beta) \).
Proof. It is sufficient to show that

\[
\left| \frac{\alpha z G'(z)}{G(z)} - (\alpha - 1) - k \right| |\frac{\alpha z G'(z)}{G(z)} - \alpha| - (1 + \beta) \leq \left| \frac{\alpha z G'(z)}{G(z)} - (\alpha - 1) - k \right| \\left| \frac{z G'(z)}{G(z)} - \alpha \right| + (1 - \beta). \tag{2.2}
\]

Consider

\[
\left| \frac{\alpha z G'(z)}{G(z)} - (\alpha - 1) - k \right| \\left| \frac{z G'(z)}{G(z)} - \alpha \right| + (1 - \beta) = \frac{1}{G(z)} \left| \alpha z G'(z) - (\alpha - 1) G(z) - ke^{i\theta} |\alpha z G'(z) - \alpha G(z)| + (1 - \beta) G(z) \right|
\]

\[
> \frac{|z|}{|G(z)|} [2 - \beta - \sum_{n=2}^{\infty} [2 - \beta + \alpha(n-1)(1+k)] B_n |a_n|].
\]

In similar manner

\[
\left| \frac{\alpha z G'(z)}{G(z)} - (\alpha - 1) - k \right| \\left| \frac{\alpha z G'(z)}{G(z)} - \alpha \right| - (1 + \beta) < \frac{|z|}{|G(z)|} [\beta + \sum_{n=2}^{\infty} [\alpha(n-1)(1+k) - \beta] B_n |a_n|].
\]

Therefore

\[
\left| \frac{\alpha z G'(z)}{G(z)} - (\alpha - 1) - k \right| \\left| \frac{\alpha z G'(z)}{G(z)} - \alpha \right| + (1 - \beta)
\]

\[
- \left| \frac{\alpha z G'(z)}{G(z)} - (\alpha - 1) - k \right| \\left| \frac{\alpha z G'(z)}{G(z)} - \alpha \right| - (1 + \beta)
\]

\[
> \frac{|z|}{|G(z)|} [2(1 - \beta) - 2 \sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(1+k)] B_n |a_n|] \geq 0.
\]

Hence the proof.

**Theorem 2.3.** If \( f \in T \) as given in (1.2), then \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \) if and only if

\[
\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(k+1)] B_n |a_n| \leq 1 - \beta. \tag{2.3}
\]

The result is sharp.
Proof. Assume that the condition (2.3) holds. In view of Theorem 2.2 and by the definition of \( TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \), we see that \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \).

Conversely suppose that \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \), then (1.4) reduces to

\[
1 - \sum_{n=2}^{\infty} \left[ 1 + \alpha(n-1) \right] B_n a_n z^{n-1} - \beta > k \left| \frac{\sum_{n=2}^{\infty} \alpha(n-1) B_n a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} B_n a_n z^{n-1}} \right|.
\]

By letting \( z \to 1^- \) through real axis we get the desired result.

Also the result is sharp for the functions given by

\[
f(z) = z - \frac{1 - \beta}{[1 - \beta + \alpha(n-1)(1+k)]B_n} z^n \quad \text{for } n \geq 2.
\]

Corollary 2.4. If \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \), then

\[
a_n \leq \frac{1 - \beta}{[1 - \beta + \alpha(n-1)(1+k)]B_n} \quad \text{for } n \geq 2.
\]

3. Growth and Distortion Theorem

Theorem 3.1. Let \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \). Then for \( |z| < 1 \),

\[
r - \frac{1 - \beta}{[1 - \beta + \alpha(1+k)]B_2} r^2 \leq |f(z)| \leq r + \frac{1 - \beta}{[1 - \beta + \alpha(1+k)]B_2} r^2
\]

and

\[
1 - \frac{2(1 - \beta)}{[1 - \beta + \alpha(1+k)]B_2} r \leq |f'(z)| \leq 1 + \frac{1 - \beta}{[1 - \beta + \alpha(1+k)]B_2} r.
\]

The result is sharp for the function \( f(z) \) given by

\[
f(z) = z - \frac{1 - \beta}{[1 - \beta + \alpha(1+k)]B_2} z^2.
\]

Proof. By Theorem(2.3) we have for \( f(z) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \),

\[
\sum_{n=2}^{\infty} \left[ 1 - \beta + \alpha(n-1)(k+1) \right] B_n a_n \leq 1 - \beta.
\]

Note that

\[
[1 - \beta + \alpha(k+1)]B_2 \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} [1 - \beta + \alpha(k+1)]B_2 a_n
\]
\[ \leq \sum_{n=2}^{\infty} [1 - \beta + \alpha(n - 1)(k + 1)]B_na_n \leq 1 - \beta. \]

Hence
\[ \sum_{n=2}^{\infty} a_n \leq \frac{1 - \beta}{[1 - \beta + \alpha(1 + k)]B_2}. \quad (3.1) \]

Therefore
\[
|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \\
\leq r + r^2 \sum_{n=2}^{\infty} a_n \\
\leq r + \frac{1 - \beta}{[1 - \beta + \alpha(1 + k)]B_2 r^2}
\]

and
\[
|f(z)| \geq r - \sum_{n=2}^{\infty} a_n r^n \\
\geq r - r^2 \sum_{n=2}^{\infty} a_n \\
\geq \frac{1 - \beta}{[1 - \beta + \alpha(1 + k)]B_2 r^2}.
\]

In view of Theorem 2.3 we have,
\[
\frac{1 - \beta + \alpha(1 + k)}{2} \sum_{n=2}^{\infty} n a_n = \sum_{n=2}^{\infty} \frac{[1 - \beta + \alpha(1 + k)]B_2 n a_n}{2} \\
\leq \sum_{n=2}^{\infty} \frac{[1 - \beta + \alpha(n - 1)(1 + k)]B_n n a_n}{n} \\
\leq 1 - \beta.
\]

Therefore
\[
|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n a_n r^{n-1} \\
\leq 1 + r \sum_{n=2}^{\infty} n a_n \\
\leq 1 + \frac{2(1 - \beta)}{[1 - \beta + \alpha(1 + k)]B_2 r}.
\]
Similarly we can prove
\[ |f'(z)| \geq 1 - \frac{2(1 - \beta)}{[1 - \beta + \alpha(1 + k)]B_2} r. \]

4. Extreme points

Theorem 4.1. Let

\[ f_1(z) := z \quad \text{and} \quad f_n(z) := z - \frac{1 - \beta}{[1 - \beta + \alpha(n - 1)(1 + k)]B_n} z^n. \] (4.1)

Then \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \) if and only if
\[ f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \quad (z \in U). \] (4.2)

where \( \lambda_n \geq 0, \ (n \geq 1) \) and \( \sum_{n=1}^{\infty} \lambda_n = 1. \) Also the extreme points of \( TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \) are given by (4.1).

Proof. Suppose (4.2) holds for \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \), then
\[ f(z) = z + \sum_{n=2}^{\infty} \lambda_n \frac{1 - \beta}{[1 - \beta + \alpha(n - 1)(1 + k)]B_n} z^n. \]

Since
\[
\sum_{n=2}^{\infty} \frac{[1 - \beta + \alpha(n - 1)(1 + k)]B_n \lambda_n (1 - \beta)}{[1 - \beta + \alpha(n - 1)(1 + k)]B_n} = (1 - \beta) \sum_{n=2}^{\infty} \lambda_n \\
= (1 - \beta)(1 - \lambda_1) \\
\leq 1 - \beta,
\]
we have \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \).

Conversely, suppose that \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \) and take

\[ \lambda_n = \frac{[1 - \beta + \alpha(n - 1)(1 + k)]B_n}{1 - \beta} a_n \quad \text{for} \quad n \geq 2 \]

and \( \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n. \)

Then \( f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z). \)

Hence the proof.
5. Closure Theorem

**Theorem 5.1.** Let the functions $f_j \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (z \in U) \quad (5.1)$$

and let $c_j \geq 0 \quad (j = 1, 2, \ldots, p)$ such that $\sum_{j=1}^{p} c_j = 1$. Then $h(z) = \sum_{j=1}^{p} c_j f_j(z) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$.

**Proof.** Now $h(z)$ can be written as

$$h(z) = \sum_{j=1}^{p} c_j \left[ z - \sum_{n=2}^{\infty} a_{n,j} z^n \right]$$

$$= z - \sum_{n=2}^{\infty} \left[ \sum_{j=1}^{p} c_j a_{n,j} \right] z^n.$$

Since $f_j \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ for every $j = 1, 2, \ldots, p$ we have

$$\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(k+1)] B_n a_{n,j} \leq 1 - \beta.$$

Therefore

$$\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(k+1)] B_n \left[ \sum_{j=1}^{p} c_j a_{n,j} \right] \leq \sum_{j=1}^{p} c_j (1 - \beta) = 1 - \beta.$$

Hence $h \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$.

**Corollary 5.2.** The class $TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ is closed under convex linear combination.

6. Convolution and Integral Properties

**Theorem 6.1.** Let $g(z) \in T$ of the form

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (0 \leq b_n \leq 1 \quad \text{for} \quad n \geq 2)$$

be analytic in $U$. If the function $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, then $(f \ast g)(z) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, where $\ast$ denotes the modified Hadamard product.
Proof. Consider
\[
\sum_{n=2}^{\infty} \left[1 - \beta + \alpha(n-1)(k+1)\right] B_n a_n b_n
\]
\[
\leq \sum_{n=2}^{\infty} \left[1 - \beta + \alpha(n-1)(k+1)\right] B_n a_n \leq 1 - \beta
\]
and hence \( f * g \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \).

Definition 6.1. Let \( I_c : T \to T \) be an integral operator defined as
\[
I_c(f(z)) = c + 1 \int_0^z t^{c-1} f(t) \, dt \quad (c > -1, z \in U). \tag{6.1}
\]

Note that for \( f(z) \) given by (1.2),
\[
I_c(f(z)) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n.
\]

By taking \( g(z) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} z^n \) where \( 0 \leq \frac{c+1}{c+n} \leq 1 \) in Theorem 6.1, we have the following result.

Corollary 6.2. If the function \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \) then \( I_c(f(z)) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \).

7. Radii of Starlikeness, Convexity and Close to convexity

Theorem 7.1. Let the function \( f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1) \). Then \( f \) is starlike of order \( \rho \) \((0 \leq \rho < 1)\) in \(|z| < r_1\) where
\[
r_1 = \inf_{n \geq 2} \left[ \frac{(1-\rho)[1 - \beta + \alpha(n-1)(k+1)] B_n}{(n-\rho)(1-\beta)} \right]^{\frac{1}{n-1}}.
\]

Proof. To prove the result we have to show that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad (0 \leq \rho \leq 1)
\]
for \( z \in U \) with \(|z| < r_1\). We have \( \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \) if \( \sum_{n=2}^{\infty} \frac{n-\rho}{1-\rho} a_n z^{n-1} \leq 1 \). By (2.3) we have
\[
\sum_{n=2}^{\infty} \frac{[1 - \beta + \alpha(n-1)(k+1)] B_n |a_n|}{1 - \beta} \leq 1.
\]
Hence the result will follow if
\[
\frac{n-\rho}{1-\rho}|z|^{n-1} \leq \frac{[1-\beta + \alpha(n-1)(k+1)]B_n|a_n|}{1-\beta}
\]
or if \(|z| \leq \left[ \frac{(1-\rho)[1-\beta + \alpha(n-1)(k+1)]B_n}{(n-\rho)(1-\beta)} \right]^\frac{1}{n-1} \).

**Corollary 7.2.** Let the function \(f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)\). Then \(f\) is convex of order \(\rho (0 \leq \rho < 1)\) in \(|z| < r_2\) where
\[
r_2 = \inf_{n \geq 2} \left[ \frac{(1-\rho)[1-\beta + \alpha(n-1)(k+1)]B_n}{n(n-\rho)(1-\beta)} \right]^\frac{1}{n-1}.
\]

**Corollary 7.3.** Let the function \(f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)\). Then \(f\) is close to convex of order \(\rho (0 \leq \rho < 1)\) in \(|z| < r_3\) where
\[
r_3 = \inf_{n \geq 2} \left[ \frac{(1-\rho)[1-\beta + \alpha(n-1)(k+1)]B_n}{n(1-\beta)} \right]^\frac{1}{n-1}.
\]

**References**


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