ON \((\in, \in \lor Q_k)\)-FUZZY KU-IDEALS OF KU-ALGEBRAS

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Abstract. We define \((\in, \in \lor q_k)\)-fuzzy KU-ideals of KU-algebras and then some related results have been provided.

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Keywords: KU-algebras, KU-ideals, \((\in, \in \lor q_k)\)-fuzzy KU-ideals.

1. Introduction

Fuzzy set theory was first introduced by Zadeh [9] in 1965. The concept of KU-algebras was given by Prabpayak and Leerawat [6, 7] in 2009. The study of fuzzy KU-algebras was first initiated by Mostafa et al. [4]. They also studied KU-algebras in terms of interval-valued fuzzy sets in [5]. Akram et al. [2] introduced the concept of interval valued \((\tilde{\theta}, \tilde{\delta})\)-fuzzy KU-ideals of KU-algebras and Yaqoob et al. [8] introduced the concept of cubic KU-ideals of KU-algebras.

In this article, we study the concept of \((\in, \in \lor q_k)\)-fuzzy KU-subalgebra and \((\in, \in \lor q_k)\)-fuzzy KU-ideal of KU-algebras.

2. Review of literature

Now we recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 1. [6] By a KU-algebra we mean an algebra with a binary operation ”∗”, satisfying the following conditions:

\[(i) : (l \ast m) \ast [(m \ast n) \ast (l \ast n)] = 0,\]
\[(ii) : l \ast 0 = 0, \forall l \in X,\]
\[(iii) : 0 \ast l = l, \forall l \in X,\]
\[(iv) : l \ast m = 0 = m \ast l \text{ implies } l = m, \forall l, m, n \in X.\]

We call it an algebra \((X, \ast, 0)\) of type \((2, 0)\). In further study of this article we denote a KU-algebra by \(X\). We define ”\(\leq\)” in \(X\) as if \(l \leq m\) if and only if \(m \ast l = 0\).
Definition 2. [7] A subset $S$ of KU-algebra $X$ is called KU-subalgebra of $X$ if $l \ast m \in S$, whenever $l, m \in S$.

Definition 3. [7] A non-empty subset $A$ of a KU-algebra $X$ is called a KU-ideal of $X$ if it satisfies the following conditions:

1. $0 \in A$,
2. $l \ast (m \ast n) \in A$, $m \in A$ implies $l \ast n \in A$, for all $l, m, n \in X$.

Definition 4. Fuzzy point in a KU-algebra $X$ is defined as

$$\psi(z) = \begin{cases} t & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$. The notation $x_t \alpha \psi$ means that $\psi(x) \geq t$ and $x_t q_k \psi$ means that $\psi(x) + t > 1$ and $x_t q_k \psi \Rightarrow \psi(x) + t + k > 1$, while the notation $x_t \alpha \psi \Rightarrow x_t q_k \psi$ does not hold.

3. $(\in, \in \lor q_k)$-fuzzy KU-ideals in KU-algebras

In this section we study the properties of $(\in, \in \lor q_k)$-fuzzy KU-ideals.

Definition 5. A fuzzy subset $\psi : X \to [0, 1]$ is said to be $(\in, \in \lor q_k)$-fuzzy KU-subalgebra of $X$ if it satisfies the following conditions:

(i) $[x, t] \in \psi \Rightarrow [0, t] \in \lor q_k \psi$,

(ii) $[x \ast (y \ast z), t_1] \in \psi, [y, t_2] \in \psi \Rightarrow [x, t_1 \land t_2] \in \lor q_k \psi$.

Example 1. Let us consider the KU-algebra $(X, \ast, 0)$ in which $\ast$ is defined as follows:

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Let us define $\psi(0) = 0.9$, $\psi(l) = 0.8$, $\psi(m) = 0.7$, $\psi(n) = 0.6$, $\psi(p) = 0.5$. Let $t = 0.49$ and $k = 0.48$ then by routine calculation it is clear that $\psi$ is an $(\in, \in \lor q_{0.48})$-fuzzy KU-subalgebra of $X$.

Definition 6. A fuzzy subset $\psi : X \to [0, 1]$ is said to be an $(\in, \in \lor q_k)$-fuzzy KU-ideal of $X$ if it satisfies the following conditions:

(i) $[x, t] \in \psi \Rightarrow [0, t] \in \lor q_k \psi$,

(ii) $[x \ast (y \ast z), t_1] \in \psi, [y, t_2] \in \psi \Rightarrow [x, t_1 \land t_2] \in \lor q_k \psi$. 

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Theorem 1. A fuzzy subset \( \psi \) of \( X \) is said to be an \((\varepsilon, \in \vee q_k)\)-fuzzy KU-ideal of \( X \) if and only if it satisfies:

(i) \( \psi(0) \geq \min \{ \psi(x), \frac{1-k}{2} \} \),

(ii) \( \psi(x \ast z) \geq \min \{ \psi(x \ast (y \ast z)), \psi(y), \frac{1-k}{2} \} \) \( \forall x, y, z \in X \).

Proof. Let \( \psi \) of \( X \) is an \((\varepsilon, \in \vee q_k)\)-fuzzy KU-ideal of \( X \). Let there exist some \( x, y, z \) in \( X \) such that

(i) \( \psi(0) < \min \{ \psi(x), \frac{1-k}{2} \} \),

(ii) \( \psi(x \ast z) < \min \{ \psi(x \ast (y \ast z)), \psi(y), \frac{1-k}{2} \} \).

Now consider (i) and if \( \psi(x) < \frac{1-k}{2} \Rightarrow \psi(0) < \psi(x) \) and \( \psi(0) < t \leq \psi(x) \) for some \( t \in (0, 1) \) \( \Rightarrow [x, t] \in \psi \) but \( [0, t] \notin \psi \). Moreover \( \psi(0) + t < 2t < 1 - k \) which implies that \([0, t] \notin q_k \psi \). Hence \([0, t] \notin \vee q_k \psi \), which contradicts the given hypothesis. Now if \( \psi(x) \geq \frac{1-k}{2} \) then it will imply that \( [x, \frac{1-k}{2}] \notin \psi \) and \( \psi(0) < \frac{1-k}{2} \Rightarrow [0, \frac{1-k}{2}] \notin \psi \). Moreover if \( \psi(0) + \frac{1-k}{2} < 1 - k \) \( \Rightarrow [0, \frac{1-k}{2}] \notin q_k \psi \) and consequently \([0, \frac{1-k}{2}] \notin q_k \psi \), which contradicts the given hypothesis and thus \( \psi(0) \geq \min \{ \psi(x), \frac{1-k}{2} \} \). Now consider (ii) and if

\[
\min \{ \psi(x \ast (y \ast z)), \psi(y) \} < \frac{1-k}{2} \Rightarrow \psi(x \ast z) < \min \{ \psi(x \ast (y \ast z)), \psi(y) \}
\]

and for some \( t \in (0, 1) \) we have

\[
\psi(x \ast z) < t \leq \min \{ \psi(x \ast (y \ast z)), \psi(y) \}.
\]

Which implies that \( [x \ast (y \ast z), t] \in \psi \) and \([y, t] \in \psi \) but \([x \ast z, t] \notin \psi \). And if \( \psi(x \ast z) + t < 2t < 1 - k \) and thus \([x \ast z, t] \notin q_k \psi \). Consequently \([x \ast z, t] \notin \vee q_k \psi \) which is contradiction and if \( \min \{ \psi(x \ast (y \ast z)), \psi(y) \} \geq \frac{1-k}{2} \) we get again \([x \ast z, t] \in \vee q_k \psi \), which again contradicts the given hypothesis and thus

\[
\psi(x \ast z) \geq \min \{ \psi(x \ast (y \ast z)), \psi(y), \frac{1-k}{2} \}.
\]

Conversely assume that (i) and (ii) are valid and we have to prove that \( \psi \) of \( X \) is \((\varepsilon, \in \vee q_k)\)-fuzzy KU-ideal of \( X \). For this let \([x, t] \in \psi \) for \( x \in X \) and \( t \in [0, 1] \). Which implies that \( \psi(x) \geq t \). But \( \psi(0) \geq \min \{ \psi(x), \frac{1-k}{2} \} \geq \min \{ t, \frac{1-k}{2} \} \). Now if \( t > \frac{1-k}{2} \) then \( \psi(0) \geq \frac{1-k}{2} \Rightarrow \psi(0) = t > 1 - k \nrightarrow [0, t] \notin q_k \psi \) and if \( t > \frac{1-k}{2} \) then it is obvious that \([0, t] \notin q_k \psi \), thus \([0, t] \notin \vee q_k \psi \). Hence \([x, t] \in \psi \Rightarrow [0, t] \in q_k \psi \). Similarly we can show that

\[
[x \ast (y \ast z), t_1] \in \psi \Rightarrow [x, t_2] \in \psi \Rightarrow [x \ast z, t_1 \wedge t_2] \in q_k \psi.
\]

This completes the proof.
Corollary 2. A fuzzy subset $\psi$ of $X$ is said to be an $(\in, \in \circ \text{q}_k)$-fuzzy KU-subalgebra of $X$ if and only if it satisfies:

(i) $\psi(0) \geq \min \left\{ \psi(x), \frac{1-k}{2} \right\}$,

(ii) $\psi(x) \geq \min \left\{ \psi(x \ast y), \psi(y), \frac{1-k}{2} \right\}$ for all $x, y \in X$.

Proof. By putting $z = 0$ in the proof of the above theorem we can easily prove it.

Next we characterize $(\in, \in \circ \text{q}_k)$-fuzzy KU-ideal of $X$ in terms of level sets.

Theorem 3. A fuzzy subset $\psi$ of $X$ is said to be $(\in, \in \circ \text{q}_k)$-fuzzy KU-ideal of $X$ if and only if the following set $U[\psi, t] = \{ x \in X | \psi(x) \geq t \}$ is a KU-ideal of $X$ where $t \in (0, \frac{1-k}{2}]$.

Proof. Assume that $\psi$ of $X$ is an $(\in, \in \circ \text{q}_k)$-fuzzy KU-ideal of $X$ and let $x \in U[\psi, t]$ which implies by definition that $\psi(x) \geq t$ for some $t \in (0, \frac{1-k}{2}]$. But $\psi(0) \geq \min \left\{ \psi(x), \frac{1-k}{2} \right\} \geq \min \left\{ t, \frac{1-k}{2} \right\} = t$, which implies that $0 \in U[\psi, t]$. Now again let $(x \ast (y \ast z)) \in U[\psi, t]$ and $y \in U[\psi, t]$ then by definition we get $\psi(x \ast (y \ast z)) \geq t$ and $\psi(y) \geq t$ but

$$\psi(x \ast z) \geq \min \left\{ \psi(x \ast (y \ast z)), \psi(y), \frac{1-k}{2} \right\} \geq \min \left\{ t, t, \frac{1-k}{2} \right\} = t,$$

which implies that $x \ast z \in U[\psi, t]$. Hence $U[\psi, t]$ is a KU-ideal of $X$ where $t \in (0, \frac{1-k}{2}]$.

Conversely let $U[\psi, t]$ is a KU-ideal of $X$ where $t \in (0, \frac{1-k}{2}]$ and we show that $\psi$ of $X$ is an $(\in, \in \circ \text{q}_k)$-fuzzy KU-ideal of $X$. For this let there exist some $t \in (0, \frac{1-k}{2}]$ such that $\psi(0) < t \leq \min \left\{ \psi(x), \frac{1-k}{2} \right\}$ which implies that $x \in U[\psi, t]$ but $0 \not\in U[\psi, t]$ which is contradiction and hence $\psi(0) \geq \min \left\{ \psi(x), \frac{1-k}{2} \right\}$. Similarly we can prove that $\psi(x \ast z) \geq \min \left\{ \psi(x \ast (y \ast z)), \psi(y), \frac{1-k}{2} \right\}$. This completes the proof.

Example 2. Let us consider the KU-algebra $(X, \ast, 0)$ in which $\ast$ is defined as follows

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Define a fuzzy subset $\psi$ of $X$ as $\psi(0) = 0.9, \psi(l) = 0.8, \psi(m) = 0.75, \psi(n) = 0.7, \psi(p) = 0.65, \psi(q) = 0.3$. Then

$$U[\psi, t] = \begin{cases} X & \text{if } t \in (0, 0.3] \text{ for } k = 0.4 \\ \{0, l, m, n, p\} & \text{if } t \in (0.3, 0.4] \text{ for } k = 0.2 \end{cases}$$
As $X$ and $\{0, l, m, n, p\}$ are $\text{KU}$-ideals of $X$, so by Theorem 3, $\psi$ of $X$ is $\langle \in, \in \vee q_k \rangle$-fuzzy $\text{KU}$-ideal of $X$.

**Corollary 4.** A fuzzy subset $\psi$ of $X$ is said to be $\langle \in, \in \vee q_k \rangle$-fuzzy $\text{KU}$-subalgebra of $X$ if and only if $U[\psi, t] = \{x \in X \mid \psi(x) \geq t\}$ is a $\text{KU}$-subalgebra of $X$ where $t \in (0, \frac{1-k^2}{2})$.

**Proof.** By putting $z = 0$ in the proof of the above theorem we can easily prove it.

**Theorem 5.** Every $\langle \in, \in \rangle$-fuzzy $\text{KU}$-subalgebra (resp., $\text{KU}$-ideal) implies $\langle \in, \in \vee q_k \rangle$-fuzzy $\text{KU}$-subalgebra (resp., $\text{KU}$-ideal) of $X$.

**Proof.** The proof is straightforward.

**Definition 7.** A fuzzy subset $\psi : X \to [0, 1]$ is said to be $\langle \in, q_k \rangle$-fuzzy $\text{KU}$-algebra of $X$ if it satisfy the following conditions:

(i) $[x, t] \in \psi \Rightarrow [0, t] \in q_k \psi$,

(ii) $[x \ast y, t_1] \in \psi, [y, t_2] \in \psi \Rightarrow [x, t_1 \wedge t_2] \in q_k \psi$.

**Definition 8.** A fuzzy subset $\psi : X \to [0, 1]$ is said to be $\langle \in, q_k \rangle$-fuzzy $\text{KU}$-ideal of $X$ if it satisfy the following conditions:

(i) $[x, t] \in \psi \Rightarrow [0, t] \in q_k \psi$,

(ii) $[x \ast (y \ast z), t_1] \in \psi, [y, t_2] \in \psi \Rightarrow [x \ast z, t_1 \wedge t_2] \in q_k \psi$.

**Theorem 6.** Every $\langle \in, q_k \rangle$-fuzzy $\text{KU}$-subalgebra (resp., $\text{KU}$-ideal) implies $\langle \in, \in \vee q_k \rangle$-fuzzy $\text{KU}$-subalgebra (resp., $\text{KU}$-ideal) of $X$.

**Proof.** The proof is straightforward.

**Example 3.** Let us consider the $\text{KU}$-algebra $(X, \ast, 0)$ in which $\ast$ is defined as follows

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Define a fuzzy subset $\psi$ as

$$
\psi(x) = \begin{cases} 
0.65 & \text{if } x = 0 \\
0.74 & \text{if } x = l \\
0.55 & \text{if } x \in \{m, n\} \\
0.35 & \text{if } x = p 
\end{cases}
$$

then $\psi$ is an $\langle \in, \in \vee q_k \rangle$-fuzzy $\text{KU}$-ideal of $X$ for $k = 0.2$ but $\psi$ is not an $\langle \in, q_k \rangle$-fuzzy $\text{KU}$-ideal of $X$ because $l_{0.7} \in \psi$ but $0_{0.7} \notin \psi$. 79
Theorem 7. Let \( \emptyset \neq A \subset X \), the characteristic function \( \psi_A \) of \( A \) is an \((\epsilon, \epsilon \cap q_k)\)-fuzzy KU-subalgebra (resp., KU-ideal) of \( X \) if and only if \( A \) is a KU-subalgebra (resp., KU-ideal) of \( X \).

Proof. Let \( A \) be a KU-ideal of \( X \), then it is obviously an \((\epsilon, \epsilon)\)-fuzzy KU-ideal of \( X \) which then implies that \( A \) is an \((\epsilon, \epsilon \cap q_k)\)-fuzzy KU-ideal of \( X \).

Conversely assume that \( A \) is an \((\epsilon, \epsilon \cap q_k)\)-fuzzy KU-ideal of \( X \) and we show that \( A \) is a KU-ideal of \( X \). For this let \( x \ast (y \ast z) \in A \), \( y \in A \) then by definition \( \psi_A(x \ast (y \ast z)) = 1 \) and \( \psi_A(y) = 1 \Rightarrow [(x \ast (y \ast z)), 1] \in \psi_A \) and \( [y, 1] \in \psi_A \). But by hypothesis

\[
\psi_A(x \ast z) \geq \min \left\{ \psi_A(x \ast (y \ast z)), \psi_A(y), \frac{1-k}{2} \right\} = \min \left\{ 1, 1, \frac{1-k}{2} \right\} = \frac{1-k}{2}
\]

and as \( k \in [0, 1) \) so \( \frac{1-k}{2} \neq 0 \) and hence \( \psi_A(x \ast z) \geq 1 \Rightarrow x \ast z \in A \). Moreover in the same way \( \psi_A(0) \geq \min \{ \psi_A(x), \frac{1-k}{2} \} = 1 \Rightarrow 0 \in A \). Hence \( A \) is a KU-ideal of \( X \).

The other case can be seen in a similar way.

Theorem 8. If \( \{ \psi_i : i \in \Lambda \} \) be a family of \((\epsilon, \epsilon \cap q_k)\)-fuzzy KU-subalgebra (resp., KU-ideal) of \( X \) then so is their intersection \( \psi = \bigcap_{i \in \Lambda} \psi_i \).

Proof. Let \( \{ \psi_i : i \in \Lambda \} \) be a family of \((\epsilon, \epsilon \cap q_k)\)-fuzzy KU-ideal of \( X \) and we have to show that \( \psi = \bigcap_{i \in \Lambda} \psi_i \) is an \((\epsilon, \epsilon \cap q_k)\)-fuzzy KU-ideal of \( X \). For this let \( [x, t] \in \psi \) and we have to show that \( [0, t] \in \bigcup_{i \in \Lambda} q_k \psi_i \). Assume that \( [0, t] \in \bigcup_{i \in \Lambda} q_k \psi_i \Rightarrow \psi(0) < t \) and \( \psi(0) + t < 1 - k \). Which implies that \( \psi(0) < \frac{1-k}{2} \).

Now let

\[
\Delta_1 = \{ i \in \Lambda \mid [0, t] \in q_k \psi_i \}
\]

and

\[
\Delta_2 = \{ i \in \Lambda \mid [0, t] \in q_k \psi_i \} \cap \{ i \in \Lambda \mid [0, t] \in \psi_i \}
\]

then we have \( \Lambda = \Delta_1 \cup \Delta_2 \) and \( \Delta_1 \cap \Delta_2 = \emptyset \). Let us suppose that if \( \Delta_2 = \emptyset \), then

\[
[0, t] \in \bigvee_{i \in \Lambda} \psi_i \Rightarrow \psi_i(0) \geq t, \forall i \in \Lambda \Rightarrow \psi(0) = \bigcap_{i \in \Lambda} \psi_i(0) \geq t,
\]

which contradicts the assumption and so \( \Delta_2 \neq \emptyset \). Thus for each \( i \in \Delta_2 \) we have \( [0, t] + t \geq 1 - k \) and \( [0, t] < t \), it implies that \( t > \frac{1-k}{2} \). Now since \( [x, t] \in \psi \Rightarrow \psi(x) \geq t \) and we can write it as \( \psi(x) \geq t > \frac{1-k}{2} \) for all \( i \in \Lambda \). Next assume that \( \psi_i(0) < \frac{1-k}{2} \) for all \( i \in \Lambda \) and hence \( \psi(0) \geq \frac{1-k}{2} \Rightarrow [0, t] \in q_k \psi \). Similarly we can show that if \( [x \ast (y \ast z), t_1] \in \psi \), \( [y, t_2] \in \psi \), then it implies that \( [x \ast z, t_1 \wedge t_2] \in q_k \psi \). Which shows that \( \psi = \bigcap_{i \in \Lambda} \psi_i \) is \((\epsilon, \epsilon \cup q_k)\)-fuzzy KU-ideal of \( X \). The other case can be seen in a similar way.
For any fuzzy subset \( \psi \) in \( X \) and \( t \in (0, 1] \), we denote \( \psi_t = \{ x \in X \mid [x, t]q_k \psi \} \) and \([\psi]_t = \{ x \in X \mid [x, t] \in Vq_k \psi \} \) then it is clear that \([\psi]_t = U [x, t] \cup \psi_t \).

**Theorem 9.** Let \( \psi : X \to [0, 1] \) be a fuzzy subset of \( X \) then \( \psi \) is an \((\in, \in \lor q)\)-fuzzy KU-subalgebra (resp., KU-ideal) of \( X \) if and only if \([\psi]_t \) is a KU-subalgebra (resp., KU-ideal) of \( X \) for all \( t \in (0, 1] \).

**Proof.** Let us assume that \( \psi \) is an \((\in, \in \lor q)\)-fuzzy KU-ideal of \( X \) and we aim to prove that \([\psi]_t \) is a KU-ideal of \( X \) for all \( t \in (0, 1] \). For this let \( x \in [\psi]_t = U [x, t] \cup \psi_t \), which then implies that \([x, t] \in Vq_k \psi \Rightarrow \psi (x) \geq t \) or \( \psi (x) + t > 1 - k \). As \( \psi (0) \geq \min \{ \psi (x), \frac{1-k}{2} \} \), so we have the following cases.

(i) If \( \psi (x) \geq t \) and \( t > \frac{1-k}{2} \) then \( \psi (0) \geq \frac{1-k}{2} \Rightarrow \psi (0) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k \),
which implies that \([0, t]q_k \psi \) and if \( t \leq \frac{1-k}{2} \) then \( \psi (0) \geq t \Rightarrow [0, t] \in \psi \). Hence
\([0, t] \in Vq_k \psi \).

(ii) If \( \psi (x) + t > 1 - k \) and \( t > \frac{1-k}{2} \) then \( \psi (0) \geq (1-k-t) \wedge \frac{1-k}{2} \Rightarrow \psi (0) \geq 1 - k - t \),
which implies that \([0, t]q_k \psi \) and if \( t \leq \frac{1-k}{2} \) then \( \psi (0) \geq (1-k-t) \wedge \frac{1-k}{2} = \frac{1-k}{2} \Rightarrow t \Rightarrow [0, t] \in \psi \). Hence \([0, t] \in Vq_k \psi \). Thus form both cases we get \( 0 \in [\psi]_t \).

Again let \((x \ast (y \ast z)) \in [\psi]_t \) and \( y \in [\psi]_t \Rightarrow [x \ast (y \ast z), t] \in Vq_k \psi \) and \([y, t] \in Vq_k \psi \Rightarrow [x \ast (y \ast z), t] \in \psi \) or \([x \ast (y \ast z), t]q_k \psi \) and \([y, t] \in \psi \) or \([y, t]q_k \psi \Rightarrow \psi (x \ast (y \ast z)) \geq t \) or \( \psi (x \ast (y \ast z)) + t + k > 1 \) and \( \psi (y) \geq t \) or \( \psi (y) + t + k > 1 \). So we discuss the following cases.

(i) If \( \psi (x \ast (y \ast z)) \geq t \) and \( \psi (y) \geq t \). So \( \psi (x \ast z) \geq \min \{ t, t, \frac{1-k}{2} \} \) and if \( t > \frac{1-k}{2} \Rightarrow \psi (x) \geq \frac{1-k}{2} \) and hence \( \psi (x \ast z) + t > 1 - k \Rightarrow [x \ast z, t]q_k \psi \) and if \( t \leq \frac{1-k}{2} \) then \( \psi (x \ast z) \geq t \Rightarrow [x \ast z, t] \in \psi \). Hence \([x \ast z, t] \in Vq_k \psi \).

Similarly from all other cases we get \([x \ast z, t] \in Vq_k \psi \). Which shows that \([\psi]_t \) is a KU-ideal of \( X \) for all \( t \in (0, 1] \).

Conversely assume that \([\psi]_t \) is a KU-ideal of \( X \) for all \( t \in (0, 1] \) and we have to show that \( \psi \) is an \((\in, \in \lor q)\)-fuzzy KU-ideal of \( X \). Suppose there exist some \( t \in (0, 1] \) such that
\[ \psi (0) < t \leq \min \left\{ \psi (x), \frac{1-k}{2} \right\}, \psi (x \ast z) < t \]
\[ \leq \min \left\{ \psi (x \ast (y \ast z)), \psi (y), \frac{1-k}{2} \right\} \Rightarrow x \in U [\psi, t] \subseteq [\psi]_t \Rightarrow 0 \in [\psi]_t \]
by hypothesis. Which then implies that \( \psi (0) \geq t \) or \( \psi (0) + t + k > 1 \), this is a contradiction. Similarly
\[ \psi (x \ast z) < t \leq \min \left\{ \psi (x \ast (y \ast z)), \psi (y), \frac{1-k}{2} \right\} \]
leads to a contradiction. Thus $\forall x, y, z \in X$ we have

$$\psi(0) \geq \min \left\{ \psi(x), \frac{1-k}{2} \right\} \text{ and } \psi(x \ast z) \geq \min \left\{ \psi(x \ast y) \ast z, \psi(y), \frac{1-k}{2} \right\},$$

which shows that $\psi$ is an $(\in, \in \lor \lambda)$-fuzzy KU-ideal of $X$. The other case can be seen in a similar way.

**Theorem 10.** Let there is an $(\in, \in \lor \lambda)$-fuzzy KU-subalgebra (resp., KU-ideal) of $X$ such that $\{\psi(x) \mid \psi(x) < \frac{1-k}{2}\} \geq 2$ then $\psi$ can be expressed as the union of two proper non-equivalent $(\in, \in \lor \lambda)$-fuzzy KU-subalgebra (resp., KU-ideal) of $X$.

**Proof.** Let us define the fuzzy sets as

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in [\psi]_{t_1}, \\ t_2 & \text{if } x \in [\psi]_{t_2} \setminus [\psi]_{t_1}, \\ \vdots & \vdots \\ t_r & \text{if } x \in [\psi]_{t_r} \setminus [\psi]_{t_{r-1}}, \end{cases}$$

and

$$\psi(x) = \begin{cases} \psi(x) & \text{if } x \in [\psi]_{\frac{1-k}{2}}, \\ t_2 & \text{if } x \in [\psi]_{t_2} \setminus [\psi]_{\frac{1-k}{2}}, \\ \vdots & \vdots \\ t_r & \text{if } x \in [\psi]_{t_r} \setminus [\psi]_{t_{r-1}}. \end{cases}$$

for $[\psi]_{\frac{1-k}{2}} \subseteq [\psi]_{t_1} \subseteq \ldots \subseteq [\psi]_{t_r} = X$ and $\{\psi(x) \mid \psi(x) < \frac{1-k}{2}\} = \{t_1, t_2, \ldots, t_r\}$ for $t_1 > t_2 > \ldots > t_r$ with $r \geq 2$. Then by level cut theorem $\mu$ and $\lambda$ are $(\in, \in \lor \lambda)$-fuzzy KU-ideal of $X$ and the chain of $(\in, \in \lor \lambda)$-level KU-ideals $\mu$ and $\lambda$ are given by respectively as $[\psi]_{t_1} \subseteq [\psi]_{t_2} \subseteq \ldots \subseteq [\psi]_{t_r}$ and $[\psi]_{\frac{1-k}{2}} \subseteq [\psi]_{t_2} \subseteq \ldots \subseteq [\psi]_{t_r}$. They are non-equivalent and $\psi = \mu \cup \lambda$. This completes the proof. The other case can be seen in a similar way.

**References**


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