SOME SUBORDINATION THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING A LINEAR OPERATOR

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Abstract. By using the subordination theorem for analytic functions we derive interesting subordination results for certain class of analytic functions defined by new linear operator.

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1. Introduction

Let \( A \) denote the class of functions \( f(z) \) of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic and univalent in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). If \( f(z) \) and \( g(z) \) are analytic in \( U \), we say that \( f(z) \) is subordinate to \( g(z) \), written \( f \prec g \) or \( f(z) \prec g(z) \) \( (z \in U) \), if there exists a Schwarz function \( w(z) \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in U) \), such that \( f(z) = g(w(z)) \), \( (z \in U) \). In particular, if \( g(z) \) is univalent in \( U \), then \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \) and \( f(U) \subset g(U) \) (see [16] and [17]).

For the functions \( f \in A \) given by (1) and \( g \in A \) given by

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k,
\]

the Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).
\]
Let $CV$ and $ST$ be the subclasses of $A$ which are starlike and convex functions, respectively. A function $f(z) \in A$ is said to be in the class of uniformly starlike functions of order $\gamma$ and type $\beta$, denoted by $SP(\beta, \gamma)$ if

$$\Re \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|,$$

where $\beta \geq 0, -1 \leq \gamma < 1, \beta + \gamma \geq 0$. Similarly, if $f(z) \in A$ satisfies

$$\Re \left\{ \frac{zf''(z)}{f'(z)} - \gamma \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right|,$$

where $\beta \geq 0, -1 \leq \gamma < 1, \beta + \gamma \geq 0$, then $f(z)$ is said to be in the class of uniformly convex functions of order $\gamma$ and type $\beta$, and is denoted by $UCV(\beta, \gamma)$. The classes $SP(\beta, \gamma)$ and $UCV(\beta, \gamma)$ were studied by Bharti et al. [8].

For functions $f, g \in A$, we define the linear operator $D_n^\lambda : A \to A$ ($\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}$) by:

$$D_0^\lambda(f * g)(z) = (f * g)(z),$$

$$D_1^\lambda(f * g)(z) = D_\lambda(f * g)(z) = (1 - \lambda)(f * g)(z) + \lambda z ((f * g)(z))',$$

and (in general)

$$D_n^\lambda(f * g)(z) = D_\lambda(D_{n-1}^\lambda(f * g)(z)) \quad (\lambda \geq 0; n \in \mathbb{N}).$$

If $f$ and $g$ are given by (1) and (2), respectively, then from (6), we see that

$$D_n^\lambda(f * g)(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k b_k z^k \quad (\lambda \geq 0; n \in \mathbb{N}_0).$$

From (7), we can easily deduce that

$$\lambda z (D_n^\lambda(f * g)(z))' = D_{n+1}^\lambda(f * g)(z) - (1 - \lambda)D_n^\lambda(f * g)(z) \quad (\lambda > 0).$$

The operator $D_n^\lambda(f * g)(z)$ was introduced by Aouf and Seoudy [5]. We observe that the linear operator $D_n^\lambda(f * g)(z)$ reduces to several interesting many other linear operators considered earlier for different choices of $n$, $\lambda$ and the function $g(z)$:

(i) For $b_k = 1$ (or $g(z) = \frac{z}{1-z}$), we have $D_n^\lambda(f * g)(z) = D_n^\lambda f(z)$, where $D_n^\lambda$ is the generalized Sălăgean operator (or Al-Oboudi operator [1]) which yield Sălăgean operator $D^n$ for $\lambda = 1$ introduced and studied by Sălăgean [22];
For $n = 0$ and

$$b_k = \Gamma_k = \frac{(a_1)_{k-1} \cdots (a_l)_{k-1}}{(b_1)_{k-1} \cdots (b_m)_{k-1}(1)_{k-1}}$$

where $a_i \in \mathbb{C}; i = 1, \ldots, l; b_j \in \mathbb{C}\backslash\mathbb{Z}^+_0 = \{0, -1, \ldots\}; j = 1, \ldots, m; l \leq m + 1; l, m \in \mathbb{N}_0$,

we have $D^{\lambda}_a(f * g)(z) = (f * g)(z) = H_{a,m} (a_1; b_1) f(z)$, where the operator $H_{a,m} (a_1; b_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [10] (see also [11] and [12]). The operator $H_{a,m} (a_1; b_1)$ contains in turn many interesting operators such as, Hohlov linear operator (see [13]), the Carlson-Shaffer linear operator (see [9] and [21]), the Ruscheweyh derivative operator (see [20]), the Bernardi-Libera-Livingston operator (see [7], [14] and [15]) and Owa-Srivastava fractional derivative operator (see [18]);

(iii) For $g(z)$ of the form (9), the operator $D^{\lambda}_a(f * g)(z) = D^{n_1}_{\alpha}(a_1, b_1) f(z)$, introduced and studied by Selvaraj and Karthikeyan [23];

(iv) For

$$b_k = \left[ \frac{\Gamma (k + 1) \Gamma (2 - \alpha)}{\Gamma (k + 1 - \alpha)} \right]^{n_1}$$

we have $D^{n_1}_{\alpha}(f * g)(z) = D^{n_1}_{\alpha} f(z)$, where $D^{n_1}_{\alpha} f(z)$ is a linear operator which was introduced and studied by Al-Oboudi and Al-Amoudi ([2] and [3], see also [4]);

(v) For

$$b_k = \left[ \frac{(a)_{k-1}}{(c)_{k-1}} \right]^{n_1}$$

we note that $D^{n_1}_{\alpha}(f * g)(z) = F_{n_1,c}^{a,c}(z)$, where $F_{n_1,c}^{a,c}(z)$ is a linear multiplier operator which introduced by Prajapat and Riana [19];

(vi) For $b_k = [\Gamma_k]^{n_1}$, where $\Gamma_k$ is given by (1.9), we obtain the linear operator $D^{n_1}_{\alpha}(f * g)(z) = L_{a,m}^{n_1} (a_1; b_1) f(z)$, where $L_{a,m}^{n_1} (a_1; b_1)$ is defined by Srivastava et al. [24]. The operator $L_{a,m}^{n_1} (a_1; b_1)$ contains Al-Oboudi and Al-Amoudi operator [2, 3] and Prajapat and Riana operator [19].

Let $SP^{n_1}_{\alpha}(f, g; \gamma, \beta)$ be the class of functions $f, g \in A$ satisfying the following condition:

$$\Re \left\{ \frac{z(D^{n_1}_{\alpha}(f * g)(z))^{'}}{D^{n_1}_{\alpha}(f * g)(z)} - \gamma \right\} > \beta \left| \frac{z(D^{n_1}_{\alpha}(f * g)(z))'}{D^{n_1}_{\alpha}(f * g)(z)} - 1 \right| (z \in U), \quad (10)$$
where $-1 \leq \gamma < 1$, $\beta \geq 0$, $\beta + \gamma \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.

Let $UCV^n(f, g; \gamma, \beta)$ be the class of function $f, g \in A$ satisfying the following condition:

\[
\Re \left\{ 1 + \frac{z(D^n_\lambda(f \ast g)(z))''}{(D^n_\lambda(f \ast g)(z))'} - \gamma \right\} > \beta \left| \frac{z(D^n_\lambda(f \ast g)(z))''}{(D^n_\lambda(f \ast g)(z))'} \right| \quad (z \in U),
\]

(11)

where $-1 \leq \gamma < 1$, $\beta \geq 0$, $\beta + \gamma \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.

From (10) and (11), we have

\[
f(z) \in UCV^n(f, g; \gamma, \beta) \iff z f'(z) \in SP^n_\lambda(f, g; \gamma, \beta).
\]

(12)

Taking $b_k = [\Gamma_k]^n$, where $\Gamma_k$ is given by (9), we note that $SP^n_\lambda(f, g; \gamma, \beta) = SP^n_{\lambda, l, m}(a_1; b_1; \gamma, \beta)$ and $UCV^n_\lambda(f, g; \gamma, \beta) = UCV^n_{\lambda, l, m}(a_1; b_1; \gamma, \beta)$.

**Definition 1.** [25] A sequence $\{c_k\}^\infty_{k=1}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1) is analytic, univalent and convex in $U$, we have

\[
\sum^\infty_{k=1} a_k c_k z^k \prec f(z) \quad (z \in U; a_1 = 1).
\]

(13)

2. Main Results

To state and prove our main results, we need the following lemma.

**Lemma 1.** [25] The sequence $\{c_k\}^\infty_{k=1}$ is a subordinating factor sequence if and only if

\[
\Re \left( 1 + 2 \sum^\infty_{k=1} c_k z^k \right) > 0 \quad (z \in U).
\]

(14)

**Theorem 2.** A function $f(z) \in A$ of the form (1) is in the class $SP^n_\lambda(f, g; \gamma, \beta)$ if

\[
\sum^\infty_{k=2} [k(1 + \beta) - (\alpha + \beta)] [1 + \lambda(k - 1)]^n |b_k| |a_k| \leq 1 - \gamma,
\]

(15)

where $g(z)$ is given by (2), $-1 \leq \gamma < 1$, $\beta \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.  

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Proof. It suffices to show that

$$\beta \left| \frac{z(D_{\lambda}^{n}(f * g)(z))'}{D_{\lambda}^{n}(f * g)(z)} - 1 \right| - \Re \left\{ \frac{z(D_{\lambda}^{n}(f * g)(z))'}{D_{\lambda}^{n}(f * g)(z)} - 1 \right\} < 1 - \gamma \quad (z \in U),$$

we have

$$\frac{\beta \left| \frac{z(D_{\lambda}^{n}(f * g)(z))'}{D_{\lambda}^{n}(f * g)(z)} - 1 \right| - \Re \left\{ \frac{z(D_{\lambda}^{n}(f * g)(z))'}{D_{\lambda}^{n}(f * g)(z)} - 1 \right\}}{(1 + \beta) \sum_{k=2}^{\infty} (k-1) [1 + \lambda(k-1)]^n |b_k| |a_k| |z|^{k-1}} \leq \frac{(1 + \beta) \sum_{k=2}^{\infty} (k-1) [1 + \lambda(k-1)]^n |b_k| |a_k| |z|^{k-1}}{1 - \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n |b_k| |a_k|}. $$

This last expression is bounded above by $(1 - \gamma)$ if (14) is satisfied.

By virtue of (12) and Theorem 2, we have

**Corollary 3.** A function $f(z) \in \mathcal{A}$ of the form (1) is in the class $UCV_{\alpha}^{n}(f; g; \gamma, \beta)$ if

$$\sum_{k=2}^{\infty} k [(1 + \beta) - (\alpha + \beta)] [1 + \lambda(k-1)]^n |b_k| |a_k| \leq 1 - \gamma,$$

where $g(z)$ is given by (2), $-1 \leq \gamma < 1$, $\beta \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.

Let $SP_{\alpha}^{n*}(f; g; \gamma, \beta)$ and $UCV_{\alpha}^{n*}(f; g; \gamma, \beta)$ denote the classes of functions $f(z) \in \mathcal{A}$ of the form (1) whose coefficients satisfy the conditions (15) and (16), respectively. We note that $SP_{\alpha}^{n*}(f; g; \gamma, \beta) \subseteq SP_{\alpha}^{n}(f; g; \gamma, \beta)$ and $UCV_{\alpha}^{n*}(f; g; \gamma, \beta) \subseteq UCV_{\alpha}^{n}(f; g; \gamma, \beta)$.

**Theorem 4.** Let the function $f(z)$ defined by (1) be in the class $SP_{\alpha}^{n*}(f; g; \gamma, \beta)$, where $g(z)$ is given by (2), $\beta \geq 0$, $-1 \leq \gamma < 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$. Then

$$\frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} (f * h)(z) < h(z) \quad (z \in U; h \in CV) \quad (16)$$

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and
\[
\Re(f(z)) > -\frac{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|} (z \in \mathbb{U}).
\] (17)

The constant \(\frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]}\) is the best estimate.

**Proof.** Let \(f(z) \in SP_{\lambda}^{\alpha}(f, g; \gamma, \beta)\) and suppose that \(h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in CV\). Then we readily have
\[
\Re(f(z)) > \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} (f * h)(z)
\]
\[
= \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} \left( z + \sum_{k=2}^{\infty} a_k c_k z^k \right).\] (18)

Thus, by Definition 1, the assertion of our theorem will hold if the sequence
\[
\left\{ \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} a_k \right\}_{k=1}^{\infty}
\]
is a subordinating factor sequence, with \(a_1 = 1\). In view of Lemma 1, this is equivalent to the following inequality
\[
\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|} a_k z^k \right\} > 0 \quad (z \in \mathbb{U}).\] (20)

Now since
\[
[k(1 + \beta) - (\gamma + \beta)][1 + \lambda(k - 1)]^n \quad (\beta \geq 0; -1 \leq \gamma < 1; \lambda > 0; n \in \mathbb{N}_0)
\]
is an increasing function of \(k\), we have
\[
\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|} a_k z^k \right\}
\]
\[
= \Re \left\{ 1 + \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|} z + \sum_{k=2}^{\infty} \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2| a_k z^k}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|} \right\}
\]
\[
\geq 1 - \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|} r^n
\]
\[
= \sum_{k=2}^{\infty} \frac{[k(1 + \beta) - (\alpha + \beta)][1 + \lambda(k - 1)]^n |b_k| a_k r^k}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|}
\]
\[
= \sum_{k=2}^{\infty} \frac{k \lambda |b_k| a_k r^k}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|}
\]
\[
= 1.
\]
\[1 - \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|^r} - \frac{1 - \gamma}{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|^r} = 1 - r > 0 \quad (|z| = r < 1), \quad (21)\]

where we have used the assertion (15) of Theorem 2. Thus (20) holds true in \( U \). This proves the first assertion. The inequality (17) follows from (16) by taking

\[ h(z) = \frac{z}{1 - z} = z + \sum_{k=2}^{\infty} z^k \in CV. \quad (22) \]

To prove the sharpness of the constant \( \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} \), we consider the function \( f_0(z) \) defined by

\[ f_0(z) = z - \frac{1 - \gamma}{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|^2} z^2 \quad (\beta \geq 0; -1 \leq \gamma < 1; \lambda > 0; n \in \mathbb{N}_0), \quad (23) \]

which is a member of the class \( SP_{\lambda^*}^n(f, g; \gamma, \beta) \). Then from the relation (16), we obtain

\[ \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} f_0(z) \prec \frac{z}{1 - z}. \quad (24) \]

It can be easily verified that

\[ \min_{|z| \leq 1} \Re \left( \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} \right) = -\frac{1}{2}, \quad (25) \]

this shows that the constant \( \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n |b_2|]} \) is best possible, and the proof of Theorem 4 is completed.

Similarly from (12) and Theorem 4, we can prove the following theorem.

**Theorem 5.** Let the function \( f(z) \) defined by (1) be in the class \( UCV_{\lambda^*}^n(f, g; \gamma, \beta) \), where \( g(z) \) is given by (2), \( \beta \geq 0, -1 \leq \gamma < 1, \lambda \geq 0 \) and \( n \in \mathbb{N}_0 \). Then

\[ \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{1 - \gamma + 2(2 + \beta - \gamma)(1 + \lambda)^n |b_2|} (f * h)(z) \prec h(z) \quad (z \in U; h \in CV) \quad (26) \]

and

\[ \Re(f(z)) > \frac{1 - \gamma + 2(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{2(2 + \beta - \gamma)(1 + \lambda)^n |b_2|} \quad (z \in U). \quad (27) \]

The constant \( \frac{(2 + \beta - \gamma)(1 + \lambda)^n |b_2|}{1 - \gamma + 2(2 + \beta - \gamma)(1 + \lambda)^n |b_2|} \) is the best estimate.
Remark 1. (i) Taking $b_k = 1$ in Theorem 4, we obtain the result of Aouf et al. [6, Theorem 1];
(ii) Taking

$$b_k = \left[ \frac{\Gamma(k+1) \Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \right]^n$$

($\alpha \neq 2, 3, 4, ...$),

in Theorems 4 and 4, respectively, we obtain the results of Aouf and Mostafa [4, Theorems 2.4 and 2.8, respectively];
(iii) Taking

$$b_k = \left[ \frac{(a)_{k-1}}{(a+c)_{k-1}} \right]^n$$

($a, c \in \mathbb{R}^+$),

in Theorem 4, we obtain the result of Prajapat and Riana [19, Theorem 1].

Taking $b_k = [\Gamma_k]^n$, where $\Gamma_k$ is given by (9), in Theorems 4 and 5, we obtain the following results for the classes $SP_{\lambda,m}^n(a_1; b_1; \gamma, \beta)$ and $UCV_{\lambda,m}^{n*}(a_1; b_1; \gamma, \beta)$, respectively.

Corollary 6. Let the function $f(z)$ defined by (1) be in the class $SP_{\lambda,m}^n(a_1; b_1; \gamma, \beta)$, where $g(z)$ is given by (2), $\beta \geq 0$, $-1 \leq \gamma < 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$. Then

$$\frac{(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|]} (f \ast h)(z) \prec h(z) \quad (z \in U; h \in CV)$$

and

$$\Re(f(z)) > -\frac{1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|}{(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|} \quad (z \in U).$$

The constant $\frac{(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|}{2[1 - \gamma + (2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|]}$ is the best estimate.

Corollary 7. Let the function $f(z)$ defined by (1) be in the class $UCV_{\lambda,m}^{n*}(a_1; b_1; \gamma, \beta)$, where $g(z)$ is given by (2), $\beta \geq 0$, $-1 \leq \gamma < 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$. Then

$$\frac{(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|}{1 - \gamma + 2(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|} (f \ast h)(z) \prec h(z) \quad (z \in U; h \in CV)$$

and

$$\Re(f(z)) > -\frac{1 - \gamma + 2(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|}{2(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|} \quad (z \in U).$$

The constant $\frac{(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|}{1 - \gamma + 2(2 + \beta - \gamma)(1 + \lambda)^n \|\Gamma_2^n\|}$ is the best estimate.
References


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