COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS USING SALAGEAN OPERATOR

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ABSTRACT. In the present paper, we introduce new subclasses $ST_\Sigma(b, \phi)$ and $CV_\Sigma(b, \phi)$ of bi-univalent functions defined in the open disk. Furthermore, we find upper bounds for the second and third coefficients for functions in these new subclasses using Salagean operator.

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1. Introduction, Definitions and Preliminaries

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  (1)

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, by $S$ we shall denote the class of functions $f \in A$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. However, the famous Koebe one-quarter theorem ensures that the image of the unit disk $\mathbb{U}$ under every function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z$, ($z \in \mathbb{U}$) and $f(f^{-1}(w)) = w$, ($|w| < r_0(f), r_0(f) \geq \frac{1}{4}$) where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$  (2)

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We let $\Sigma$ to denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). If $f(z)$ is bi-univalent, it must be analytic in the boundary of the domain and
such that it can be continued across the boundary of the domain so that \( f^{-1}(z) \) is defined and analytic throughout \(|w| < 1\). Examples of functions in the class \( \Sigma \) are 
\[
\frac{z}{1-z}, -\log(1-z)
\]
and so on.

The coefficient estimate problem for the class \( S \), known as the Bieberbach conjecture, is settled by de-Branges [4], who proved that for a function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in the class \( S \), \(|a_n| \leq n, \) for \( n = 2, 3, \ldots \), with equality only for the rotations of the Koebe function
\[
K_0(z) = \frac{z}{(1-z)^2}.
\]

In 1967, Lewin [7] introduced the class \( \Sigma \) of bi-univalent functions and showed that \(|a_2| < 1.51\) for the functions belonging to \( \Sigma \). It was earlier believed that for \( f \in \Sigma \), the bound was \(|a_n| < 1\) for every \( n \) and the extremal function in the class was \( \frac{z}{1-z^2} \). E. Netanyahu [9] in 1969, ruined this conjecture by proving that in the set \( \Sigma \), \( \max_{f \in \Sigma} |a_2| \leq 4/3 \). In 1969, Suffridge [13] gave an example of \( f \in \Sigma \) for which \( a_2 = 4/3 \) and conjectured that \(|a_2| \leq 4/3\). In 1981, Styer and Wright [12] disproved the conjecture that \(|a_2| > 4/3\). Brannan and Clunie [2] conjectured that \(|a_2| \leq \sqrt{2}\). Kedzierawski [6] in 1985 proved this conjecture for a special case when the function \( f \) and \( f^{-1} \) are starlike functions. Brannan and Clunie [2] conjectured that \(|a_2| \leq \sqrt{2}\).

Tan [14] in proved that \(|a_2| \leq 1.485\) which is the best known estimate for functions in the class of bi-univalent functions.

Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses \( S^* (\alpha) \) and \( C (\alpha) \) of the univalent function class \( \Sigma \). Recently, Ali et al. [1] extended the results of Brannan and Taha [3] by generalising their classes using subordination.

An analytic function \( f \) is subordinate to an analytic function \( g \), written \( f(z) \prec g(z) \), provided there is a Schwarz function \( w \) defined on \( U \) with \( w(0) = 0 \) and \(|w(z)| < 1\) satisfying \( f(z) = g(w(z)) \). Ma and Minda [8], unified various subclasses of starlike and convex functions for which either of the quantity \( \frac{zf'(z)}{f(z)} \) or \( 1 + \frac{zf''(z)}{f'(z)} \) is subordinate to a more general superordinate function. For this purpose, they considered an analytic function \( \phi \) with positive real part in the unit disk \( U \), \( \phi(0) = 1, \phi'(0) > 0 \) and \( \phi \) maps \( U \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form
\[
\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, (B_1 > 0).
\]

216
Let a differential operator be defined [11] on a class of analytic functions of the form (1) as follows:

\[ D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z) \]

and in general

\[ D^n f(z) = D \left( D^{n-1} f(z) \right) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \]

We easily find that

\[ D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (n \in \mathbb{N}_0). \]  

(4)

**Definition 1.** Let \( b \) be a non-zero complex number. A function \( f(z) \) given by (1) is said to be in the class \( ST_\Sigma(b, \phi) \) if the following conditions are satisfied:

\[ f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{z (D^m f(z))'}{D^m f(z)} - 1 \right) \prec \phi(z), \quad z \in \mathbb{U} \]  

(5)

and

\[ 1 + \frac{1}{b} \left( \frac{w (D^m g(w))'}{D^m g(w)} - 1 \right) \prec \phi(w), \quad w \in \mathbb{U}, \]  

(6)

where the function \( g \) is given by (2).

**Definition 2.** Let \( b \) be a non-zero complex number. A function \( f(z) \) given by (1) is said to be in the class \( CV_\Sigma(b, \phi) \) if the following conditions are satisfied:

\[ f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{z (D^m f(z))'}{(D^m f(z))'} \right) \prec \phi(z), \quad z \in \mathbb{U} \]  

(7)

and

\[ 1 + \frac{1}{b} \left( \frac{w (D^m g(w))''}{(D^m g(w))''} \right) \prec \phi(w), \quad w \in \mathbb{U}, \]  

(8)

where the function \( g \) is given by (2).

2. Coefficient estimates

**Lemma 1.** [10] If \( p \in \varphi \), then \( |c_k| \leq 2 \) for each \( k \), where \( \varphi \) is the family of functions \( p \) analytic in \( \mathbb{U} \) for which \( \Re p(z) > 0 \), \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) for \( z \in \mathbb{U} \).

217
Theorem 2. Let the function \( f(z) \in A \) be given by (1). If \( f \in \text{ST}_\Sigma(b, \phi) \), then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1} |b|}{\sqrt{(2(3^m) - 2^{2m}) B_1^2 b + (B_1 - B_2) 2^{2m}}} \quad \text{and} \quad |a_3| \leq \frac{(B_1 + |B_2 - B_1|) |b|}{2(3^m) - 2^{2m}}.
\]

Proof. Since \( f \in \text{ST}_\Sigma(b, \phi) \), there exists two analytic functions \( r, s : U \to U \), with \( r(0) = 0 = s(0) \), such that

\[
1 + \frac{1}{b} \left( \frac{z (D^m f(z))'}{D^m f(z)} - 1 \right) = \phi(r(z)) \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{w (D^m g(w))'}{D^m g(w)} - 1 \right) = \phi(s(z)).
\]

Define the functions \( p \) and \( q \) by

\[
p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad \text{and} \quad q(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + \cdots.
\]

Or equivalently,

\[
r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1}{2} \left( p_2^2 - \frac{p_1^2}{2} \right) - \frac{p_1^2 p_2}{2} \right) z^3 + \cdots \right)
\]

and

\[
s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left( q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \left( q_3 + \frac{q_1}{2} \left( q_2^2 - \frac{q_1^2}{2} \right) - \frac{q_1 q_2}{2} \right) z^3 + \cdots \right).
\]

It is clear that \( p \) and \( q \) are analytic in \( U \) and \( p(0) = 1 = q(0) \). Also \( p \) and \( q \) have positive real part in \( U \) and hence \( |p_i| \leq 2 \) and \( |q_i| \leq 2 \). In the view of (11), (12) and (13), clearly,

\[
1 + \frac{1}{b} \left( \frac{z (D^m f(z))'}{D^m f(z)} - 1 \right) = \phi \left( \frac{p(z) - 1}{p(z) + 1} \right) \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{w (D^m g(w))'}{D^m g(w)} - 1 \right) = \phi \left( \frac{q(w) - 1}{q(w) + 1} \right).
\]

Using (13) and (14) together with (3), one can easily verify that

\[
\phi \left( \frac{p(z) - 1}{p(z) + 1} \right) = 1 + \frac{B_1 p_1}{2} z + \left( \frac{B_1}{2} p_2 - \frac{p_1^2}{2} \right) z^2 + \left( \frac{1}{4} B_2 p_1^2 \right) z^3 + \cdots
\]

(15)
and

\[
\phi \left( \frac{q(w) - 1}{q(w) + 1} \right) = 1 + B_1 q_1 w + \left( B_1 \frac{1}{2} \left( q_2 - \frac{q_1^2}{2} \right) + \frac{B_2 q_1^2}{4} \right) w^2 + \cdots. \tag{16}
\]

Since \( f \in \Sigma \) has the Maclaurin series given by (1), computation shows that its inverse \( g = f^{-1} \) has the expansion given by (2). It follows from (14), (15) and (16) that

\[
2^m a_2 = \frac{1}{2} B_1 p_1 b, \tag{17}
\]

\[
2 \left( 3^m \right) a_3 - \left( 2^{2m} \right) a_2^2 = \frac{1}{2} b B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} b B_2 p_1^2 \tag{18}
\]

and

\[
-2^m a_2 = \frac{1}{2} B_1 b q_1, \tag{19}
\]

\[
\left( 4 \left( 3^m \right) - \left( 2^{2m} \right) \right) a_2^2 - 2 \left( 3^m \right) a_3 = \frac{1}{2} b B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} b B_2 q_1^2. \tag{20}
\]

From (17) and (19), it follows that

\[
p_1 = -q_1. \tag{21}
\]

Now (18), (20) and (21) gives

\[
a_2^2 = \frac{B_1^3 (p_2 + q_2) b}{4 \left( (2.3^m - 2^{2m}) B_1^2 b + 2^{2m} (B_1 - B_2) \right)}. \tag{22}
\]

Using the fact that \( |p_2| \leq 2 \) and \( |q_2| \leq 2 \) gives the desired estimate on \( |a_2| \),

\[
|a_2| \leq \frac{B_1 \sqrt{B_1} |b|}{\sqrt{\left| (2.3^m - 2^{2m}) B_1^2 b + (B_1 - B_2) 2^{2m} \right|}}.
\]

From (18)-(20), gives

\[
a_3 = \frac{b B_1}{2} \left( \left( 4 \left( 3^m \right) - 2^{2m} \right) p_2 + 2^{2m} q_2 \right) + \frac{3^m p_1^2 (B_2 - B_1) b}{4 \left( 2 \left( 3^m \right) - 3^m 2^{2m} \right)}.
\]

Using the inequalities \( |p_1| \leq 2, |p_2| \leq 2 \) and \( |q_2| \leq 2 \) for functions with positive real part yields the desired estimation of \( |a_3| \).
For a choice of \( \phi(z) = \frac{1 + Az}{1 + Bz} \), \(-1 \leq B < A \leq 1\), we have the following corollary.

**Corollary 3.** Let \(-1 \leq B < A \leq 1\). If \( f \in ST_{\Sigma} \left( b, \frac{1+A}{1+B} \right) \), then

\[
|a_2| \leq \frac{|b| (A - B)}{\sqrt{|(2 (3^m) - 2^{2m}) (A - B) b + (1 + B) 2^{2m}|}}
\]

and

\[
|a_3| \leq \frac{|A - B| (1 + |1 + B|) |b|}{(2 (3^m) - 2^{2m})}.
\]

If we let \( \phi(z) = \left( \frac{1+z}{1-z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \), \( 0 < \alpha \leq 1 \), in the above theorem, we get the following:

**Corollary 4.** Let \( 0 < \alpha \leq 1 \). If \( f \in ST_{\Sigma} (b, \alpha) \), then

\[
|a_2| \leq \frac{|b| 2\alpha}{\sqrt{|2\alpha (2 (3^m) - 2^{2m}) b + (1 - \alpha) 2^{2m}|}}
\]

and

\[
|a_3| \leq \frac{(1 + |\alpha - 1|) 2\alpha |b|}{2 (3^m) - 2^{2m}}.
\]

**Theorem 5.** Let the function \( f(z) \in A \) be given by (1). If \( f \in CV_{\Sigma} (b, \phi) \), then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1} |b|}{\sqrt{2 (3^{m+1} - 2^{2m+1}) B_2^2 b + 2 (B_1 - B_2) 2^{2m}}}
\]

and

\[
|a_3| \leq \frac{(B_1 + |B_2 - B_1|) |b|}{2 (3^{m+1} - 2^{2m+1})}.
\]

**Proof.** Since \( f \in CV_{\Sigma} (b, \phi) \), there exists two analytic functions \( r, s : \mathbb{U} \rightarrow \mathbb{U} \), with \( r(0) = 0 = s(0) \), such that

\[
1 + \frac{1}{b} \left( \frac{z (D^m f(z))''}{(D^m f(z))'} \right) = \phi(r(z)) \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{w (D^m g(w))''}{(D^m g(w))'} \right) = \phi(s(z)).
\]

Using (11), (12), (15) and (16), one can easily verified that

\[
2^{m+1} a_2 = \frac{1}{2} B_1 p_1 b,
\]

\[
6 (3^m) a_3 - 4 (2^m) a_2^2 = \frac{1}{2} b B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} b B_2 p_1^2
\]
and

\[ -2^{m+1}a_2 = \frac{1}{2} B_1 b q_1, \quad (27) \]

\[ (12 \cdot 3^m - 4 \cdot 2^{2m}) a_2^2 - 6 \cdot 3^m a_3 = \frac{1}{2} b B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} b B_2 q_1^2. \quad (28) \]

From (25) and (27), it follows that

\[ p_1 = -q_1. \quad (29) \]

Now (26), (28) and (29) give

\[ a_2^2 = \frac{B_3^3 (p_2 + q_2) b}{8 \left( (3.3^m - 2.2^{2m}) B_2^2 b + 2 (B_1 - B_2) (2^{2m}) \right)}. \quad (30) \]

Using the fact that \(|p_2| \leq 2\) and \(|q_2| \leq 2\) gives the desired estimate on \(|a_2|\),

\[ |a_2| \leq \frac{B_1 \sqrt{B_2} |b|}{\sqrt{2 \left| (3^{m+1} - 2^{2m+1}) B_2^2 b + 2 (B_1 - B_2) 2^{2m} \right|}}. \]

From (26)-(28), gives

\[ a_3 = \frac{b B_1}{2} \left( \left( (12 \cdot 3^m - 4 \cdot 2^{2m}) p_2 + 4 \cdot 2^{2m} q_2 \right) + (B_2 - B_1) b p_1^2 \cdot 3^{m+1} \right) \]

\[ 24(3^m) (3^{m+1} - 2^{2m+1}) \]

Using the inequalities \(|p_1| \leq 2\), \(|p_2| \leq 2\) and \(|q_2| \leq 2\) for functions with positive real part yields

\[ |a_3| \leq \frac{(B_1 + |B_2 - B_1|) |b|}{2 \left( (3^{m+1} - 2^{2m+1}) \right)}. \]

For a choice of \(\phi(z) = \frac{1 + A z}{1 + B z}, -1 \leq B < A \leq 1\), we have the following corollary.

**Corollary 6.** Let \(-1 \leq B < A \leq 1\). If \(f \in ST_{\Sigma} \left( b, \frac{1 + A z}{1 + B z} \right)\), then

\[ |a_2| \leq \frac{|b| (A - B)}{\sqrt{2 \left| (3^{m+1} - 2^{2m+1}) (A - B) b + 2 (1 + B) 2^{2m} \right|}} \]

and

\[ |a_3| \leq \frac{|A - B| (1 + |1 + B|) |b|}{2 \left( (3^{m+1} - 2^{2m+1}) \right)}. \]

221
If we let $\phi(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots$, $0 < \alpha \leq 1$, in the above theorem, we get the following:

**Corollary 7.** Let $0 < \alpha \leq 1$. If $f \in ST_\Sigma(b, \alpha)$, then

$$|a_2| \leq \frac{|b| \alpha}{\sqrt{\left(3^{m+1} - 2^{2m+1}\right) ab + (1 - \alpha) 2^{2m}}}$$

and

$$|a_3| \leq \frac{(1 + |\alpha - 1|) \alpha |b|}{(3^{m+1} - 2^{2m+1})}.$$ 

**Remark 1.** If we let $b = 1, m = 0$, Theorem 2.2 and Theorem 2.5 reduce to the result of R.M.Ali et.al [1], corollary 2.1 and corollary 2.2.

**References**


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