DISCRETE CHARACTERIZATION OF EXPONENTIAL STABILITY
OF EVOLUTION FAMILY OVER HILBERT SPACE

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Abstract. In this article we prove that if $U = \{U(m,n)\}_{m \geq n \geq 0}$ is a positive $q$-periodic discrete evolution family of bounded linear operators acting on a complex Hilbert space $H$ then $U$ is uniformly exponentially stable if for each unit vector $x$ in $H$ the series $\sum_{m=0}^{\infty} \phi(|\langle U(m,0)x,x \rangle|)$ is bounded, where $\phi : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$ is a non decreasing function such that $\phi(0) = 0$ and $\phi(t) > 0$ for all $t \in (0, \infty)$. We also prove the converse of the above result by putting an extra condition i.e. if $U$ is uniformly exponentially stable and $\sum_{i=0}^{\infty} \phi(x_i) = \phi(\sum_{i=0}^{\infty} (x_i))$ for any $x_i \in \mathbb{R}_+$ then the series $\sum_{m=0}^{\infty} \phi(|\langle U(m,0)x,x \rangle|)$ is bounded.

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1. Introduction

In 1970, Datko [7] brought forth one of the remarkable result in the stability of strongly continuous semigroup which argues that a strongly continuous semigroup $T = \{T(t)\}_{t \geq 0}$ of bounded linear operators acting on complex or real Banach space is uniformly exponentially stable if and only if

$$\int_{0}^{\infty} \|T(t)x\|dt < \infty.$$ 

In 1972, Pazy [13] had a research on the results of Datko and further improved his attempt by stating that a strongly continuous semigroup of bounded linear operators acting on real or complex Banach space is uniformly exponentially stable if and only if

$$\int_{0}^{\infty} \|T(t)x\|^pdt < \infty, \text{ for any } p \geq 1.$$ 

Rolewicz [17] generalizes the Pazy theorem a step ahead. He state that if

$$\int_{0}^{\infty} \phi\|T(t)x\|dt < \infty,$$
then the semigroup $T$ is uniformly exponentially stable, where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non decreasing function such that $\phi(0) = 0$ and $\phi(t) > 0$ for all $t \in (0, \infty)$, onward we will call this function as an $\mathcal{R}$-function. Later on, special cases were proved by Zabczyk and Przyluski, details can be found in [21] and [16] respectively. Zheng [23] and W. Littman [11] obtained the new proofs of Rolewicz from which they discard the condition of continuity on $\phi$.

Let $X$ be a Banach space and $X^*$ be its dual space, then $T = \{T(t)\}_{t \geq 0}$ is called weak $L^p$ stable for $p \geq 1$, if

$$\int_0^\infty \|\langle T(t)x, x^* \rangle\|^p dt < \infty.$$ \hspace{1cm} (1)

The weak $L^p$ stability of a semi group does not imply uniform exponential stability, counter examples can be found in [8, 9, 12]. For further results on this topic we recommend, [1, 2, 3, 18, 20].

Recently C. Buse and G. Rahmat [5] tried to extend the result given in (1) to evolution family by Weak Rolewicz type approach i.e. they proved that if

$$\int_0^\infty \phi(\|\langle U(t,0)x, x \rangle\|)dt < \infty,$$

where $\phi$ is a function as defined above then $U$ is uniformly exponentially stable.

Our paper is the continuation of the last coated paper in discrete form. Different simultaneous results concerning discrete semigroups and discrete evolution family can be found in [4, 6, 10, 14, 15, 19, 22].

In the first section of this article we will give some preliminaries and in second section we will present our main results.

2. Notations and Preliminaries

We denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_+$ the sets of real numbers, complex numbers and positive integers respectively. $\sigma(A)$ denotes the spectral radius of $A$. By $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators acting on $X$. As usual $\langle \cdot, \cdot \rangle$ denotes the scalar product on a Hilbert space $H$. The norms in $X, H, \mathcal{L}(X)$ and $\mathcal{L}(H)$ will be denoted by the same symbol, namely $\| \cdot \|$.

The family $U = \{U(m,n) : m,n \in \mathbb{Z}_+, m \geq n\}$ is called $q$-periodic discrete evolution family if it satisfies the following properties.

- (i) $U(m,m) = I$.
- (ii) $U(m,n)U(n,r) = U(m,r)$.
\[ (iii) \quad U(m + q, n + q) = U(m, n). \]

It is well known that \( U \) is exponentially bounded, that is, there exist \( \omega \in \mathbb{R} \) and \( M_\omega \geq 0 \) such that
\[
\| U(m, n) \| \leq M_\omega e^{\omega (m-n)}, \quad \text{for all } m \geq n. \tag{2}
\]
The growth bound of exponentially bounded evolution family \( U \) is defined by
\[
\omega_0(U) := \inf \{ \omega \in \mathbb{R} : \text{there is } M_\omega \geq 0 \text{ such that (2) holds} \}.
\]
A bounded linear operator \( A \), acting on a Hilbert space \( H \), is positive if \( \langle Ax, x \rangle \geq 0 \) for every \( x \in H \). An evolution family \( \{U(m, n) : m \geq n\} \) is called self-adjoint (positive) if each operator \( U(m, n) \) with \( m \geq n \), is self-adjoint (respectively positive).

The family \( U \) is uniformly exponentially stable if its growth bound is negative. An evolution family is self-adjoint if each member of the family is self-adjoint.

Here we will recall few lemmas from [5], without proof, so that the paper will be self-contained.

**Lemma 1** ([5]). Let \( X \) be a complex Banach space and let \( V \in \mathcal{L}(X) \). If the spectral radius of \( V \) is greater or equal to 1, then for all \( 0 < \varepsilon < 1 \) and any sequence \( (a_n) \) with \( a_n \to 0 \) (as \( n \to \infty \)) and \( \| (a_n) \|_\infty \leq 1 \), there exists a unit vector \( u_0 \in X \), such that
\[
\| V^n u_0 \| \geq (1 - \varepsilon) |a_n|, \quad \text{for all } n \in \mathbb{Z}_+.
\]

Throughout this article, \( (t_n) \) will be a sequence of nonnegative real numbers, such that \( 1 \leq q \leq t_{n+1} - t_n \leq \alpha \) for every \( n \in \mathbb{Z}_+ \) and some positive real number \( \alpha \).

**Lemma 2** ([5]). Let \( U = \{U(t, s)\}_{t \geq s \geq 0} \) be a strongly continuous \( q \)-periodic \( (q \geq 1) \) evolution family of bounded linear operators acting on a Banach space \( X \) and let \( (t_n) \) be a sequence as given before. If the evolution family is not uniformly exponentially stable, then there exists a positive constant \( C \), having the properties: for every \( \mathbb{C} \)-valued sequence \( (b_n) \) with \( b_n \to 0 \) (as \( n \to \infty \)) and \( \| (b_n) \|_\infty \leq 1 \), there exists a unit vector \( u_0 \in X \), such that
\[
\| U(t_n, 0) u_0 \| \geq C |b_{n+1}|, \quad \text{for all } n \in \mathbb{Z}_+. \tag{3}
\]

An evolution family \( U \) is said to satisfy the weak discrete Rolewicz condition, equation (2.4) in [5], if
\[
\sum_{n=0}^{\infty} \phi([|U(t_n, 0)x, y|]) < \infty. \tag{4}
\]
Lemma 3 ([5]). Let \( \phi \) be an \( \mathcal{R} \)-function and let \( \mathcal{U} = \{ U(t, s) \}_{t \geq s \geq 0} \) be a strongly continuous \( q \)-periodic \( (q \geq 1) \) evolution family acting on a Banach space \( X \). If the family \( \mathcal{U} \) satisfies
\[
\sum_{n=0}^{\infty} \phi(\| U(t_n, 0)x \|) < \infty,
\]
then it is uniformly exponentially stable.

3. Main Results

Let \( T \) be a discrete semigroup of bounded linear operators acting on complex Hilbert space \( H \). When \( T \) is self-adjoint i.e. \( T(m) = T^*(m) \), for every \( m \in \mathbb{Z}_+ \) then
\[
\langle T(m)x, x \rangle = \langle T(m/2)T(m/2)x, x \rangle = \| T(m/2)x \|^2.
\]

Let \( K \) and \( L \) be two self-adjoint operators then we have the following inequality
\[
\| \langle KLx, y \rangle \| \leq \langle K^2y, y \rangle \langle L^2x, x \rangle \text{ for all } x, y \in H. \tag{5}
\]

We are in the position to state our first theorem.

Theorem 4. Let \( \phi \) be an \( \mathcal{R} \)-function and let \( \mathcal{T} = \{ T(m) \}_{m \in \mathbb{Z}_+} \) be a self-adjoint discrete semigroup of bounded linear operators acting on a complex Hilbert space \( H \). Then

1. If the series \( \sum_{m=0}^{\infty} \phi(\| T(m)x \|) \) is bounded for all \( x \in H \) with \( \| x \| = 1 \) then \( \mathcal{T} \) is uniformly exponentially stable.

2. If the semigroup \( \mathcal{T} \) is uniformly exponentially stable and \( \sum_{i=0}^{\infty} \phi(x_i) = \phi(\sum_{i=0}^{\infty} x_i) \) for any \( x_i \in \mathbb{R}_+ \) then the series \( \sum_{m=0}^{\infty} \phi(\| T(m)x \|) \) is bounded.

Proof. Case 1. Using inequality (5), we can write
\[
\| \langle T(2m)x, y \rangle \|^2 = \| \langle T(2m - n)T(n)x, y \rangle \|^2 \leq \langle T(4m - 2n)y, y \rangle \langle T(2n)x, x \rangle \leq Me^{8\omega} \langle T(2n)x, x \rangle.
\]

Hence, for any unit vector \( x \in H \), one has
\[
\| \langle T(2m)x, x \rangle \|^2 \leq Me^{8\omega} \langle T(2n)x, x \rangle.
\]

As \( \phi \) is an increasing function, so we can write
\[
\phi(1/Me^{8\omega}\| T(m)x \|)^4 \leq \phi(\| T(2n)x, x \|).
\]

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Taking summation on both sides
\[ \sum_{n=0}^{\infty} \phi\left(\frac{1}{Me^{8w}}\|T(m)x\|^4\right) \leq \sum_{n=0}^{\infty} \phi\left(|\langle T(2n)x, x \rangle|\right). \]

Since
\[ \sum_{n=0}^{\infty} \phi\left(|\langle T(2n)x, x \rangle|\right) < \infty, \]
so
\[ \sum_{n=0}^{\infty} \phi\left(\frac{1}{Me^{8w}}\|T(m)x\|^4\right) < \infty. \]

Hence by using Lemma 3, we can say that \( T \) is uniformly exponentially stable.

Case 2. Let \( T \) is uniformly exponentially stable, then there exists \( v > 0 \) and \( M \geq 0 \) such that
\[ \|T(m)\| \leq Me^{-vm}, \]
replacing \( m \) by \( m/2 \)
\[ \|T(m/2)x\| \leq Me^{-vm/2} \]
or
\[ \sqrt{\langle T(m)x, x \rangle} \leq Me^{-vm/2}. \]

applying \( \phi \)
\[ \phi(\langle T(m)x, x \rangle) \leq \phi(M^2e^{-vm}). \]

Taking summation on both sides
\[ \sum_{m=0}^{\infty} \phi(\langle T(m)x, x \rangle) \leq \sum_{m=0}^{\infty} \phi(M^2e^{-vm}) \]
\[ = \phi\left(\sum_{m=0}^{\infty} M^2e^{-vm}\right) \]
\[ \leq \phi\left(\frac{M^2e^v}{e^v - 1}\right). \]

By definition of \( \phi \), \( \phi\left(\frac{M^2e^v}{e^v - 1}\right) < \infty. \) Hence
\[ \sum_{m=0}^{\infty} \phi(\langle T(m)x, x \rangle) < \infty. \]

Thus proof is complete.
We also extend the above result to self-adjoint q-discrete evolution family \( U \) as follows.

**Theorem 5.** Let \( U = \{ U(m, n) : m, n \in \mathbb{Z}_+, m \geq n \} \) is a self-adjoint q-periodic discrete evolution family acting on a complex Hilbert space \( H \) then the following two statement holds true

1. If the series \( \sum_{m=0}^{\infty} \phi(\|U(m,0)x\|) \) is bounded for all \( x \in H \) with \( \|x\| = 1 \) then \( U \) is uniformly exponentially stable.

2. If the evolution family \( U \) is uniformly exponentially stable and \( \sum_{i=0}^{\infty} \phi(x_i) = \phi(\sum_{i=0}^{\infty} x_i) \) for any \( x_i \in \mathbb{R}_+ \) then the series \( \sum_{m=0}^{\infty} \phi(\|U(m,0)x\|) \) is bounded.

**Proof.** Case 1. Using inequality (5) we can write

\[
|\langle U(m,0)x, y \rangle|^2 = |\langle U(m,n)U(n,0)x, y \rangle|^2 
\leq \langle U^2(m,n)y, y \rangle \langle U^2(n,0)x, x \rangle
\leq \|U(m,n)y\|\|U(n,0)x\|^2 
\leq M^2 e^{4qw}\|U(n,0)x\|^2.
\]

Hence, for any unit vector \( x \in H \), one has

\[
1/Me^{2qw}|\langle U(m,0)x, x \rangle| \leq \|U(n,0)x\|.
\]

Since \( \phi \) is an increasing function, so we can write

\[
\phi(1/Me^{2qw}|\langle U(m,0)x, x \rangle|) \leq \phi(\|U(n,0)x\|).
\]

Taking summation on both sides

\[
\sum_{m=0}^{\infty} \phi(1/Me^{2qw}|\langle U(m,0)x, x \rangle|) \leq \sum_{n=0}^{\infty} \phi(\|U(n,0)x\|).
\]

Since

\[
\sum_{n=0}^{\infty} \phi(\|U(n,0)x\|) < \infty,
\]

so

\[
\sum_{m=0}^{\infty} \phi(1/Me^{2qw}|\langle U(m,0)x, x \rangle|) < \infty.
\]

Equivalently, we can write

\[
\sum_{m=0}^{\infty} \phi(1/Me^{2qw}\|U(m,0)x\|^2) < \infty.
\]
So again using Lemma 3, we can say that $\mathcal{U}$ is uniformly exponentially stable.

*Case 2.* Let $\mathcal{U}$ is uniformly exponentially stable, then there exists two positive constants $v$ and $M$ such that

$$
\|U(m,n)\| \leq Me^{-v(m-n)}.
$$

Putting $n = 0$, we get

$$
\|U(m,0)\| \leq Me^{-vm}.
$$

Since $\phi$ is an increasing function, so we can write

$$
\phi(\|U(m,0)x\|) \leq \phi(Me^{-vm}).
$$

Taking summation on both sides

$$
\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|) \leq \sum_{m=0}^{\infty} \phi(Me^{-vm}) = \phi(\sum_{m=0}^{\infty} Me^{-vm}) \leq \phi\left(\frac{Me^v}{e^v - 1}\right).
$$

By definition of $\phi$ we have

$$
\phi\left(\frac{Me^v}{e^v - 1}\right) < \infty,
$$

hence

$$
\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|) < \infty.
$$

Thus the series $\sum_{m=0}^{\infty} \phi(\|U(m,0)x\|)$ is bounded.

We also extend similar idea to positive discrete evolution family.

**Theorem 6.** Let $\mathcal{U} = \{U(m,n) : m,n \in \mathbb{Z}_+\}$ is a positive $q$-periodic discrete evolution family acting on a complex Hilbert space $H$ then the following two statement holds true

1. If the series $\sum_{m=0}^{\infty} \phi((U(m,0)x,x))$ is bounded for all $x \in H$ with $\|x\| = 1$ then $\mathcal{U}$ is uniformly exponentially stable.

2. If the evolution family $\mathcal{U}$ is uniformly exponentially stable and $\sum_{i=0}^{\infty} \phi(x_i) = \phi(\sum_{i=0}^{\infty} (x_i))$ for any $x_i \in \mathbb{R}_+$ then the series $\sum_{m=0}^{\infty} \phi((U(m,0)x,x))$ is bounded.
Proof. Case 1. Using inequality (5) we can write
\[
|\langle U^{1/2}(m,0)x,y \rangle|^2 = |\langle U^{1/2}(m,n)U^{1/2}(n,0)x,y \rangle|^2 \\
\leq \langle U(m,n)y,y \rangle \langle U(n,0)x,x \rangle \\
|\langle U^{1/2}(m,0)x,y \rangle|^2 \leq \langle U(m,n)y,y \rangle \langle U(n,0)x,x \rangle \\
\leq Me^{2qw} \langle U(n,0)x,x \rangle.
\]
Hence, for any unit vector \(x \in H\), one has
\[
1/Me^{2qw}|\langle U(m,0)x,x \rangle|^2 \leq \langle U(n,0)x,x \rangle.
\]
Since \(\phi\) is an increasing function, so we can write
\[
\phi(1/Me^{2qw}|\langle U(m,0)x,x \rangle|^2) \leq \phi(\langle U(n,0)x,x \rangle).
\]
Taking summation on both sides
\[
\sum_{m=0}^{\infty} \phi(1/Me^{2qw}|\langle U(m,0)x,x \rangle|^2) \leq \sum_{n=0}^{\infty} \phi(\langle U(n,0)x,x \rangle).
\]
Since
\[
\sum_{n=0}^{\infty} \phi(\langle U(n,0)x,x \rangle) < \infty,
\]
so
\[
\sum_{m=0}^{\infty} \phi(1/Me^{2qw}|\langle U(m,0)x,x \rangle|^2) < \infty.
\]
Hence using Lemma 3, we can say that \(U\) is uniformly exponentially stable.

Case 2. Let \(U\) is uniformly exponentially stable, then there exists \(v \in \mathbb{R}\) and \(M \geq 0\) such that
\[
\|U(m,n)\| \leq Me^{-v(m-n)},
\]
Putting \(n = 0\) we get
\[
\|U(m,0)\| \leq Me^{-vm},
\]
replacing \(m\) by \(m/2\)
\[
\|U(m/2,0)x\| \leq Me^{-vm/2} \\
\sqrt{\langle U(m,0)x,x \rangle} \leq Me^{-vm/2}.
\]
Since \(\phi\) is increasing function so we can write
\[
\phi(\langle U(m,0)x,x \rangle) \leq \phi(M^2e^{-vm}).
\]
Taking summation on both sides
\[
\sum_{m=0}^{\infty} \phi(\langle (U(m,0)x,x \rangle) \leq \sum_{m=0}^{\infty} \phi(M^2 e^{-vm})
\]
\[
= \phi(\sum_{m=0}^{\infty} M^2 e^{-vm})
\]
\[
= \phi(M^2 \sum_{m=0}^{\infty} e^{-vm})
\]
\[
\leq \phi(M^2 \frac{e^v}{e^v-1}) < \infty.
\]

Hence
\[
\sum_{m=0}^{\infty} \phi(\langle (U(m,0)x,x \rangle) < \infty.
\]

Which completes the proof.

References

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