APPLICATION OF GENERALIZED HADAMARD PRODUCT ON SPECIAL CLASSES OF ANALYTIC P-VALENT FUNCTIONS

R.M. EL-ASHWAH

ABSTRACT. In this paper the author established certain results concerning the quasi-Hadamard product for generalized subclasses of p-valent functions with positive coefficients.

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1. Introduction

Let $A(p)$ denote the class of analytic $p$-valent functions in the unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}).$$

(1)

A function $f(z) \in A(p)$ is called $p$-valent starlike of order $\alpha$ if $f(z)$ satisfies

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (2)$$

for $0 \leq \alpha < p$ and $z \in U$. We denote by $S_p^*(\alpha)$ the class of all starlike $p$-valent functions of order $\alpha$. Also a function $f(z) \in A(p)$ is called $p$-valent convex of order $\alpha$ if $f(z)$ satisfies

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (3)$$

for $0 \leq \alpha < p$ and $z \in U$. We denote by $C_p(\alpha)$ the class of convex $p$-valent functions of order $\alpha$. 

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For $p < \beta < p + \frac{1}{2}$ and $z \in U$, let $M_p(\beta)$ denote the subclass of $A(p)$ consisting of functions $f(z)$ of the form (1) and satisfying the condition
\[ \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < \beta \] (4)
and let $N_p(\beta)$ denote the subclass of $A(p)$ consisting of functions $f(z)$ of the form (1) and satisfying the condition
\[ \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta, \] (5)
it follows from (4) and (5) that
\[ f(z) \in N_p(\beta) \iff \frac{zf'(z)}{p} \in M_p(\beta) \] (6)
The subclasses $M_p(\beta)$ and $N_p(\beta)$ and some related classes have been studied by several authors (e.g. [5], [8], [10] and [11]).

Furthermore, let $V(p)$ denote the subclass of analytic $p$-valent functions of the form:
\[ f(z) = a_p z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_p > 0; a_{n+p} \geq 0). \] (7)
Also, let
\[ f_i(z) = a_{p,i} z^p + \sum_{n=1}^{\infty} a_{n+p,i} z^{n+p} \quad (a_{p,i} > 0; a_{n+p,i} \geq 0), \] (8)
and
\[ g_j(z) = b_{p,j} z^p + \sum_{n=1}^{\infty} b_{n+p,j} z^{n+p} \quad (b_{p,i} > 0; b_{n+p,i} \geq 0), \] (9)
the quasi-Hadamard product $(f_i * g_j)(z)$ of the functions $f_i(z)$ and $g_j(z)$ by
\[ (f_i * g_j)(z) = a_{p,i} b_{p,j} z + \sum_{n=2}^{\infty} a_{n+p,i} b_{n+p,j} z^{n+p} \quad (i, j = 1, 2, 3, \ldots). \]
Similarly, we can define the quasi-Hadamard product of more than two functions.

Also, let $V_p(\beta) = M_p(\beta) \cap V(p)$ and $U_p(\beta) = N_p(\beta) \cap V(p)$, following the technique of Uralegaddi et al. [12], we can obtain the following lemmas.
**Lemma 1.** Let the function \( f(z) \in V(p) \), then \( f(z) \in V_p(\beta) \) \((p < \beta < p + \frac{1}{2})\) if and only if
\[
\sum_{n=1}^{\infty} (n + p - \beta)a_{n+p} \leq (\beta - p)a_p. \tag{10}
\]

**Lemma 2.** Let the function \( f(z) \in V(p) \), then \( f(z) \in U_p(\beta) \) \((p < \beta < p + \frac{1}{2})\) if and only if
\[
\sum_{n=1}^{\infty} \left(\frac{n + p}{p}\right)(n + p - \beta)a_{n+p} \leq (\beta - p)a_p. \tag{11}
\]

Let \( \varphi(z) \) be a fixed function of the form:
\[
\varphi(z) = c_p z + \sum_{n=2}^{\infty} c_{n+p} z^{n+p} \quad (c_p, c_{n+p} \geq 0). \tag{12}
\]

Using the function defined by (12), we now define the following new classes.

**Definition 1.** A function \( f(z) \in V_{p,\varphi}(c_{n+p}, \delta) \) \((c_n \geq c_2 > 0)\) if and only if
\[
\sum_{n=1}^{\infty} c_{n+p} a_{n+p} \leq \delta a_p \quad (\delta > 0). \tag{13}
\]

**Definition 2.** A function \( f(z) \in U_{p,\varphi}(c_{n+p}, \delta) \) \((c_{n+p} \geq c_{p+1} > 0)\) if and only if
\[
\sum_{n=1}^{\infty} \left(\frac{n + p}{p}\right) c_{n+p} a_{n+p} \leq \delta a_p \quad (\delta > 0). \tag{14}
\]

Also, we introduce the following class of analytic \( p \)-valent functions which plays an important role in the discussion that follows.

**Definition 3.** A function \( f(z) \in V_{p,\varphi}(c_{n+p}, \delta) \) \((c_{n+p} \geq c_{p+1} > 0)\) if and only if
\[
\sum_{n=1}^{\infty} \left(\frac{n + p}{p}\right)^k c_{n+p} a_{n+p} \leq \delta a_p \quad (\delta > 0), \tag{15}
\]
where \( k \) is any fixed nonnegative real number.

For suitable choices of \( c_n, \delta, k \) and \( a_0 = 1 \) we obtain :

1. \( V_{p,\varphi}^1 \left( \frac{n + p}{p} \right)(n + p - \gamma)\theta(n, p), \gamma - p = A_p^h(z)q, \gamma \) \((h(z) = \frac{z^p}{1-z}, \theta(n, p) = (\alpha_1)_{n=0}^{(a_0)}(\beta_1)_{n=0}^{(b_0)}(\gamma)_{n}^{(1)}; q \leq s + 1(\alpha_i > 0 \text{ for } i = 1, 2, ..., q; \beta_j > 0 \text{ for } j = 1, 2, ..., s), p < \gamma < p + \frac{1}{2}) \) (Najafzadeh et al. [5]);
(ii) \( V^0_{p,\varphi}((n + p - \lambda + |n + p - 2\alpha + \lambda|), 2(\alpha - p)) = \mathfrak{M}_p(\alpha, \lambda) \) (0 < \( \lambda < p, \alpha > p \)) (Sun et al.[11]);

(iii) \( V^1_{p,\varphi}(\frac{n+p}{p}) (n + p - \lambda + |n + p - 2\alpha + \lambda|), 2(\alpha - p)) = \mathfrak{M}_p(\alpha, \lambda) \) (0 < \( \lambda < p, \alpha > p \)) (Sun et al.[11]);

(iv) \( V^0_{1,\varphi}((n - \beta), (\beta - 1)) = V(\beta) \) (1 < \( \beta < \frac{4}{3} \)) (Uralegaddi et al.[12]);

(v) \( V^1_{1,\varphi}(n(n - \beta), (\beta - 1)) = U(\beta) \) (1 < \( \beta < \frac{4}{3} \)) (Uralegaddi et al. [12]);

(vi) \( V^0_{1,\varphi}((n - 1) + |n - 2\beta + 1|, 2(\beta - 1)) = M(\beta) \) (\( \beta > 1, a_0 = 1 \)) (Niswaki and Owa [3] and Owa and Niswaki [6]);

(vii) \( V^1_{1,\varphi}(n (n - 1) + |n - 2\beta + 1|), 2(\beta - 1) = N(\beta) \) (\( \beta > 1, a_0 = 1 \)) (Niswaki and Owa [3] and Owa and Niswaki [6]).

Evidently, \( V^0_{p,\varphi}(c_n, \delta) = V_{p,\varphi}(c_n, \delta) \) and \( V^1_{p,\varphi}(c_n, \delta) = U_{p,\varphi}(c_n, \delta) \). Further \( V^2_{p,\varphi}(c_n, \delta) \subset \bigcap V^0_{p,\varphi}(c_n, \delta) \), if \( \gamma_1 > \gamma_2 \geq 0 \), the containment being proper. Moreover for any positive integer \( k \), we have the following inclusion relation

\[
V^k_{p,\varphi}(c_n, \delta) \subset \bigcap V^{k-1}_{p,\varphi}(c_n, \delta) \subset \ldots \subset V^2_{p,\varphi}(c_n, \delta) \subset U_p(c_n, \delta) \subset V_p(c_n, \delta).
\]

We also note that for nonnegative real number \( k \), the class \( V^k_{p,\varphi}(c_n, \delta) \) is nonempty as the function

\[
f(z) = a_p z^p + \sum_{n=1}^{\infty} \left( \frac{n+p}{p} \right)^{-k} \frac{\delta a_p}{c_n+p} \lambda_{n+p} a_{n+p} z^{n+p}, \tag{16}\]

where \( a_p > 0, \lambda_{n+p} \geq 0 \) and \( \sum_{n=1}^{\infty} \lambda_{n+p} \leq 1 \), satisfy the inequality (15).

The quasi-Hadamard product of two or more \( p \)-valent functions has recently been defined and used by Aouf et al. [1], Hossen [3] and Sekine [9].

The object of this paper is to establish a result concerning the quasi-Hadamard product of functions in the classes \( V^k_{p,\varphi}(c_n, \delta), U_{p,\varphi}(c_n, \delta) \) and \( V_{p,\varphi}(c_n, \delta) \).

2. The Main Results

**Theorem 3.** Let the functions \( f_i(z) \) defined by (8) belong to the class \( U_{p,\varphi}(c_n, \delta) \) for every \( i = 1, 2, \ldots, m \); and let the functions \( g_j(z) \) defined by (9) belong to the class \( V_{p,\varphi}(c_n, \delta) \) for every \( i = 1, 2, \ldots, q \). If \( c_n \geq \left( \frac{n+p}{p} \right) \delta \)

\((n \in \mathbb{N})\). Then the quasi-Hadamard product \( f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_q(z) \) belongs to the class \( V^{2m+q-1}_{p,\varphi}(c_n, \delta) \).
Proof. It is sufficient to show that
\[
\sum_{n=1}^{\infty} \left[ \left( \frac{n+p}{p} \right)^{2m+q-1} c_{n+p} \left( \prod_{i=1}^{m} |a_{n+p,i}| \cdot \prod_{j=1}^{q} |b_{n+p,j}| \right) \right] \leq \delta \left( \prod_{i=1}^{m} a_{p,i} \cdot \prod_{j=1}^{q} b_{p,j} \right).
\]

Since \( f_i(z) \in U_{p,\varphi}(c_n, \delta) \), we have
\[
\sum_{n=1}^{\infty} \left( \frac{n+p}{p} \right) c_{n+p} a_{n+p,i} \leq \delta a_{p,i}
\]
for every \( i = 1, 2, ..., m \). Therefore
\[
a_{n+p,i} \leq \left( \frac{n+p}{p} \right)^{-1} \left( \frac{\delta}{c_{n+p}} \right) a_{p,i},
\]
and hence
\[
a_{n+p,i} \leq \left( \frac{n+p}{p} \right)^{-2} a_{p,i},
\]
the inequalities (17) and (18) hold for every \( i = 1, 2, ..., m \). Further, since \( g_j(z) \in V_{\varphi}(c_n, \delta) \), we have
\[
\sum_{n=1}^{\infty} c_{n+p} b_{n+p,j} \leq \delta b_{p,j},
\]
for every \( j = 1, 2, ..., q \). Hence we obtain
\[
|b_{n+p,j}| \leq \left( \frac{n+p}{p} \right)^{-1} b_{0,j},
\]
for every \( j = 1, 2, ..., q \).

Using (18) for \( i = 1, 2, ..., m \), (20) for \( j = 1, 2, ..., q - 1 \) and (19) for \( j = q \), we have
\[
\sum_{n=1}^{\infty} \left[ \left( \frac{n+p}{p} \right)^{2m+q-1} c_{n+p} \left( \prod_{i=1}^{m} |a_{n+p,i}| \cdot \prod_{j=1}^{q} |b_{n+p,j}| \right) \right] \leq \sum_{n=1}^{\infty} \left[ \left( \frac{n+p}{p} \right)^{2m+q-1} c_n \left( \left( \frac{n+p}{p} \right)^{-2m} \left( \frac{n+p}{p} \right)^{-2(q-1)} \prod_{i=1}^{m} a_{p,i} \cdot \prod_{j=1}^{q-1} b_{p,j} \right) |b_{n+p,q}| \right]
\]
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Hence \( f_1 \ast f_2 \ast \ldots \ast f_m \ast g_1 \ast g_2 \ast \ldots \ast g_q \in V_{\varphi}^{2m+q-1}(c_n, \delta). \)

We note that the required estimate can also be obtained by using (18) for \( i = 1, 2, \ldots, m - 1, \) (20) for \( j = 1, 2, \ldots, q, \) and (17) for \( i = m. \)

Taking into account the quasi-Hadamard product functions \( f_1(z), f_2(z), \ldots, f_m(z) \) only, in the proof of Theorem 1 and using (18) for \( i = 1, 2, \ldots, m - 1, \) and (17) for \( i = m, \) we obtain

**Corollary 4.** Let the functions \( f_i(z) \) defined by (8) belong to the class \( U_{\varphi}(c_n, \delta) \) for every \( i = 1, 2, \ldots, m. \) If \( c_n \geq n\delta, \) \( (n \in \mathbb{N}), \) then the quasi-Hadamard product \( f_1 \ast f_2 \ast \ldots \ast f_m(z) \) belongs to the class \( V_{\varphi}^{2m-1}(c_n, \delta). \)

Also taking into account the quasi-Hadamard product functions \( g_1(z), g_2(z), \ldots, g_q(z) \) only, in the proof of Theorem 1 and using (20) for \( j = 1, 2, \ldots, q - 1, \) and (19) for \( j = q, \) we obtain

**Corollary 5.** Let the functions \( g_i(z) \) defined by (9) belong to the class \( V_{\varphi}(c_n, \delta) \) for every \( i = 1, 2, \ldots, q. \) If \( c_n \geq n\delta, \) \( (n \in \mathbb{N}). \) Then the quasi-Hadamard product \( g_1 \ast g_2 \ast \ldots \ast g_q \) belongs to the class \( V_{\varphi}^{q-1}(c_n, \delta). \)

**Remark 1.** (i) Putting \( p = 1 \) in the above results, we obtain the results obtained by El-Ashwah [2];

(ii) Putting \( c_{n+p} = (k + p - \gamma)\theta(k, p) \) and \( \delta = \gamma - p \) \( (p < \gamma < p + \frac{1}{2}) \) in the above results we obtain results corresponding to the class \( A_{\varphi}^{h(z)}(m, n, \gamma) \) \( (h(z) = \frac{z^p}{1 - z^p}, \theta(k, p) = \frac{\alpha_1 \ldots (\alpha_m)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{1}{(1)_k}, n \leq m + 1 \)

\( (\alpha_i > 0 \text{ for } i = 1, 2, \ldots, q; \beta_j > 0 \text{ for } j = 1, 2, \ldots, s, p < \gamma < p + \frac{1}{2}); \)

(iii) Putting \( c_{n+p} = (n + p - \lambda - |n + p - 2\alpha + \lambda|) \) and \( \delta = 2(\gamma - p) \) in the above results we obtain results corresponding to the classes \( \mathfrak{M}_p(\alpha, \lambda) \) and \( \mathfrak{N}_p(\alpha, \lambda)(0 < \lambda < p, \alpha > p). \)

**References**


R.M. El-Ashwah
Department of Mathematics
Faculty of Science
Damietta University
NewDamietta 34517, Egypt
email: r_elashwah@yahoo.com