SOME PROPERTIES OF CERTAIN SUBCLASSES OF
MEROMORPHIC MULTIVALENT FUNCTIONS OF COMPLEX
ORDER DEFINED BY CERTAIN LINEAR OPERATOR

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Abstract. In this paper we introduce two subclasses \(H^n_p(q, s, \alpha_1; A, B, b)\) and \(H^{n,*}_p(q, s, \alpha_1; A, B, b)\) of meromorphic \(p\)-valent functions of complex order defined by certain linear operator. We study the various important properties and characteristics of these two subclasses such as, coefficients estimate, radii of starlikeness and convexity and closure theorems. We also extend the familiar concept of \(\delta\)-neighborhoods of analytic functions to these subclasses.

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1. Introduction

Let \(\sum_p\) be the class of functions of the form:

\[
f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \ldots\}),
\]

which are analytic and \(p\)-valent in the punctured unit disc \(U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \setminus \{0\}\). Let \(f\) and \(F\) two analytic functions in the unit disc \(U\), we say that \(f\) is subordinate to \(F\) if there exists an analytic function \(w(z)\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in U)\) such that \(f = F(w(z))\). We denote by \(f \prec F\) this subordination. For functions \(f(z) \in \sum_p\) given by (1.1) and \(g(z) \in \sum_p\) defined by

\[
g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (p \in N),
\]

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then the Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is given by
\[
(f * g)(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k b_k z^k = (g * f)(z).
\] (1.3)

For complex numbers \( \alpha_1, ..., \alpha_q \) and \( \beta_1, ..., \beta_s \) \( (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s) \), we now define the generalized hypergeometric function \( {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \) by (see, for example, [18, p.19])
\[
qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!}.
\] (1.4)

\( (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U) \),

where \((\theta)_\nu\) is the Pochhammer symbol defined, in terms of the Gamma function \( \Gamma \), by
\[
(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 
1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}) \\
\theta(\theta + 1) \cdots (\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}).
\end{cases}
\] (1.5)

Corresponding to the function \( h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \), defined by
\[
h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p} {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),
\] (1.6)

we consider a linear operator
\[
H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \sum_p \rightarrow \sum_p,
\]
which is defined by the following Hadamard product:
\[
H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z)
\] (1.7)

We observe that, for a function \( f(z) \) of the form (1.1), we have
\[
H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = z^{-p} + \sum_{k=1-p}^{\infty} \Gamma_{k+p}(\alpha_1) a_k z^k.
\] (1.8)

where, for convenience
\[
\Gamma_m(\alpha_1) = \frac{(\alpha_1)_m \cdots (\alpha_q)_m}{(\beta_1)_m \cdots (\beta_s)_m} \frac{1}{(m)!}
\] (1.9)
If, for convenience, we write
\[ H_{p,q,s}(\alpha_1) = H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s), \] (1.10)
then one can easily verify from the definition (1.8) that (see [16])
\[ z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1)f(z) - (\alpha_1 + p)H_{p,q,s}(\alpha_1)f(z). \] (1.11)

The linear operator \( H_{p,q,s}(\alpha_1) \) was investigated recently by Liu and Srivastava [16] and Aouf [5]. Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by (for example) Dziok and Srivastava ([7] and [8]), Gangadharan et al. [9], Liu [14].

In particular, for \( q = 2, s = 1 \) and \( \alpha_2 = 1 \), we obtain the linear operator
\[ H_{p,2,1}(\alpha_1, 1; \beta_1)f(z) = L_p(\alpha_1, \beta_1)f(z) \quad (f(z) \in \sum_p; \quad \alpha_1 > 0; \quad \beta_1 > 0), \]
which was introduced and studied by Liu and Srivastava [15]. Also we note that, for any integer \( n > -p \) and \( f(z) \in \sum_p \), we have
\[ H_{p,2,1}(n + p, 1; 1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1 - z)^{n+p}} \ast f(z), \]
where \( D^{n+p-1}f(z) \) is the differential operator studied by Urallegaddi and Somanatha [19] and Aouf [4].

For a function \( f \in \sum_p \), we define
\[ I^0 (H_{p,q,s}(\alpha_1)f(z)) = H_{p,q,s}(\alpha_1)f(z), \]
\[ I^1 (H_{p,q,s}(\alpha_1)f(z)) = z(H_{p,q,s}(\alpha_1)f(z))' + \frac{p + 1}{z^p}, \]
\[ I^2 (H_{p,q,s}(\alpha_1)f(z)) = z(I^1 (H_{p,q,s}(\alpha_1)f(z)))' + \frac{p + 1}{z^p}, \]
and (in general)
\[ I^n (H_{p,q,s}(\alpha_1)f(z)) = z(I^{n-1} (H_{p,q,s}(\alpha_1)f(z)))' + \frac{p + 1}{z^p} \]
\[ = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} k^n \Gamma_{k+p} a_k z^k \quad (p \in \mathbb{N}; \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \]
where \( \Gamma_{k+p} \) is given by (1.9).

We note that \( I^n (H_{p,2,1}(\alpha, 1; \beta)f(z)) = I^n (L_p(\alpha, \beta)f(z)) \) (see Ghanim and Darus [10]).
Making use of the operator $I^n (H_{p,q,s} (\alpha_1) f(z))$, we say that a function $f(z) \in \sum_p$ is in the class $H^n_p(q, s, \alpha_1; A, B, b)$ if it satisfies the following inequality:

$$p = \frac{1}{b} \left\{ \frac{z (I^n (H_{p,q,s} (\alpha_1) f(z)))'}{I^n (H_{p,q,s} (\alpha_1) f(z))} + p \right\} < \frac{1 + A z}{1 + B z} \quad (1.12)$$

or, equivalently, to

$$\left| \frac{z (I^n (H_{p,q,s} (\alpha_1) f(z)))'}{I^n (H_{p,q,s} (\alpha_1) f(z))} + p \right| < \left| \frac{z (I^n (H_{p,q,s} (\alpha_1) f(z)))'}{I^n (H_{p,q,s} (\alpha_1) f(z))} + p [(A - B) b + B] \right| \quad (-1 \leq B < A \leq 1; \alpha_1, ..., \alpha_q \in \mathbb{C} \text{ and } \beta_1, ..., \beta_s \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad p \in \mathbb{N}; q, s, n \in \mathbb{N}_0; q \leq s + 1; b \in \mathbb{C}^*; z \in U). \quad (1.13)$$

Let $\sum_p^*$ denote the subclass of $\sum_p$ consisting of functions of the form:

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \quad (p \in \mathbb{N}). \quad (1.14)$$

We now write

$$H^n_{p}^*(q, s; \alpha_1; A, B, b) = H^n_{p}(q, s; \alpha_1; A, B, b) \cap \sum_p^*.$$

We note the following interesting relationship with some of the special function classes which were investigated recently:

(i) $H^0_{p}^*(q, s; \alpha_1; A, B, 1) = \sum_{p,q}^* (A, B)$ (see Goyal et al. [12]);

(ii) $H^0_{p}^*(2, 1; a, 1, c; A, B, b) = H^+_{a,c} (p; A, B, b, 1) (a \in \mathbb{R}; c \in \mathbb{R} \setminus \{0, -1, -2, ...\})(see Aqlan and Kulkarni [6]).$

Also, we note that:

(i) $H^n_{p}^*(q, s; \alpha_1; \beta, -\beta, b) = H^n_{p}^*(q, s; \alpha_1; \beta, b)$

where

$$f(z) \in \sum_p^* : \left\{ \begin{array}{l} \left| \frac{z (I^n (H_{p,q,s} (\alpha_1) f(z)))'}{I^n (H_{p,q,s} (\alpha_1) f(z))} + p \right| < \beta \\
\left| \frac{z (I^n (H_{p,q,s} (\alpha_1) f(z)))'}{I^n (H_{p,q,s} (\alpha_1) f(z))} + p (1 - 2b) \right| \end{array} \right\} \quad (0 < \beta \leq 1; b \in \mathbb{C}^*; \alpha_1, ..., \alpha_q \in \mathbb{C} \text{ and } \beta_1, ..., \beta_s \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ q \leq s + 1; p \in \mathbb{N}; q, s, n \in \mathbb{N}_0; z \in U). \quad (1.15)$$
(ii) \( H_p^{n,*}(q, s; \alpha_1; \beta, -(2\gamma - 1) \beta, b) = H_p^{n,*}(q, s; \alpha_1; \beta, \gamma, b) \)

\[
= \begin{cases} 
    f(z) \in \sum_p^* : & \left| \frac{z (I^n (H_{p,q,s}(\alpha_1)f(z)))'}{I^n (H_{p,q,s}(\alpha_1)f(z))} + p}{(2\gamma - 1) \frac{z (I^n (H_{p,q,s}(\alpha_1)f(z)))'}{I^n (H_{p,q,s}(\alpha_1)f(z))} + p [2\gamma (1 - b) - 1]} \right| < \beta \\
\end{cases}
\]

\((0 < \beta \leq 1; \frac{1}{2} \leq \gamma \leq 1; b \in \mathbb{C}^*; \alpha_1, ..., \alpha_q \in \mathbb{C} \text{ and } \beta_1, ..., \beta_s \in \mathbb{C} \setminus \mathbb{Z}_0^*; \quad q \leq s + 1; \quad p \in \mathbb{N}; \quad q, s, n \in \mathbb{N}_0; \quad z \in U)\) \quad \text{(1.16)}

(iii) \( H_p^{n,*}(2, 1; \delta + p, 1, 1; A, B, b) = D_p^{n,*}(\delta; A, B, b) \)

\[
= \begin{cases} 
    f(z) \in \sum_p^* : & \left| \frac{z (I^n (D^{\delta+p-1}f(z)))'}{I^n (D^{\delta+p-1}f(z))} + p}{B z (I^n (D^{\delta+p-1}f(z)))'} I^n (D^{\delta+p-1}f(z)) + p [(A - B) b + B]} \right|< 1 \\
\end{cases}
\]

\((-1 \leq B < A \leq 1; \delta > -p; \quad p \in \mathbb{N}; \quad n \in \mathbb{N}_0; \quad b \in \mathbb{C}^*; \quad z \in U)\) \quad \text{(1.17)}

(iv) \( H_p^{n,*}(2, 1; a, 1, c; A, B, b) = L_p^{n,*}(a, c; A, B, b) \)

\[
= \begin{cases} 
    f(z) \in \sum_p^* : & \left| \frac{z (I^n (L_p(a,c)f(z)))'}{I^n (L_p(a,c)f(z))} + p}{B z (I^n (L_p(a,c)f(z)))'} I^n (L_p(a,c)f(z)) + p [(A - B) b + B]} \right|< 1 \\
\end{cases}
\]

\((-1 \leq B < A \leq 1; \quad a \in \mathbb{R}; \quad c \in \mathbb{R} \setminus \{0, -1, -2, ...\}; \quad p \in \mathbb{N}; \quad n \in \mathbb{N}_0; \quad b \in \mathbb{C}^*; \quad z \in U)\) \quad \text{(1.18)}

2. SOME BASIC PROPERTIES OF THE CLASS \( H_p^{n,*}(q, s, \alpha_1; A, B, b) \)

We begin by proving the necessary and sufficient condition (involving coefficient bounds) for the class \( H_p^{n,*}(q, s, \alpha_1; A, B, b) \).
Theorem 1. Let the function \( f(z) \) defined by (1.14) be in the class \( \sum_{p}^{*} \). Then the function \( f(z) \) belongs to the class \( H_{p}^{n,*}(q, s, \alpha_{1}; A, B, b) \) if and only if

\[
\sum_{k=p}^{\infty} k^{n} [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p} |a_{k}| \leq p |b| (A - B).
\]  

(2.1)

Proof. Assuming that the inequality (2.1) holds true then from (2.1), we find that

\[
\frac{z (I^{n} (H_{p,q,s}(\alpha_{1})f(z)))' + pI^{n} (H_{p,q,s}(\alpha_{1})f(z))}{Bz (I^{n} (H_{p,q,s}(\alpha_{1})f(z)))' + [Bp(1 - b) + Apb] I^{n} (H_{p,q,s}(\alpha_{1})f(z))} = 1 \quad (z \in U^{*}),
\]

(2.2)

\[ z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1 \}. \] Hence, by the maximum modulus theorem, we have \( f(z) \in H_{p}^{n,*}(q, s, \alpha_{1}; A, B, b) \).

Conversely, suppose that \( f(z) \) is in the class \( H_{p}^{n,*}(q, s, \alpha_{1}; A, B, b) \) with \( f(z) \) of the form (1.14), then we find from (1.13), that

\[
\frac{z (I^{n} (H_{p,q,s}(\alpha_{1})f(z)))' + pI^{n} (H_{p,q,s}(\alpha_{1})f(z))}{Bz (I^{n} (H_{p,q,s}(\alpha_{1})f(z)))' + [Bp(1 - b) + Apb] I^{n} (H_{p,q,s}(\alpha_{1})f(z))} = 1.
\]

(2.3)

If we choose \( z \) to be real and \( z \to 1^{+} \), we get

\[
\sum_{k=p}^{\infty} k^{n} [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p} |a_{k}| \leq p |b| (A - B),
\]

(2.4)

which is precisely the assertion (2.1) of Theorem 1.

Corollary 2. Let the function \( f(z) \) be defined by (1.14) be in the class \( H_{p}^{n,*}(q, s, \alpha_{1}; A, B, b) \). Then

\[
|a_{k}| \leq \frac{p |b| (A - B)}{k^{n} [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p}} \quad (k \geq p).
\]

(2.5)
The result is sharp for the function \( f(z) \) given by

\[
f(z) = z^{-p} + \frac{p|b|(A-B)}{k^n[(k+p)(1-B)-p|b|(A-B)]\Gamma_{k+p}}z^k \quad (k \geq p).
\] (2.6)

Next we prove the following growth and distortion properties for the class \( H_p^{n,s}(q,s,\alpha_1;A,B,b) \).

**Theorem 3.** If a function \( f(z) \) defined by (1.14) is in the class \( H_p^{n,s}(q,s,\alpha_1;A,B,b) \), then for \( |z| = r < 1 \), we have

\[
\left\{ \frac{(p+m-1)!}{(p-1)!} - \frac{p!|b|(A-B)}{[2(1-B)-|b|(A-B)]p^n(p-m)!\Gamma_{2p}}r^{2p} \right\}r^{-(p+m)}
\leq |f^{(m)}(z)| \leq \left\{ \frac{(p+m-1)!}{(p-1)!} - \frac{p!|b|(A-B)}{[2(1-B)-|b|(A-B)]p^n(p-m)!\Gamma_{2p}}r^{2p} \right\}r^{-(p+m)}.
\] (2.7)

The result is sharp for the function \( f(z) \) given by

\[
f(z) = z^{-p} + \frac{|b|(A-B)}{p^n[2(1-B)-|b|(A-B)]\Gamma_{2p}}z^p.
\] (2.8)

**Proof.** Let \( f(z) \in H_p^{n,s}(q,s,\alpha_1;A,B,b) \). Then we find from Theorem 1 that

\[
p\frac{[2(1-B)-|b|(A-B)]p^n(p-m)!\Gamma_{2p}}{p!} \sum_{k=p}^{\infty} \frac{k!}{(k-m)!}|a_k| \leq \sum_{k=p}^{\infty} [(k+p)(1-B)-p|b|(A-B)]^{k+p} \Gamma_{k+p} |a_k| \leq p|b|(A-B),
\]

which yields

\[
\sum_{k=p}^{\infty} \frac{k!}{(k-m)!}|a_k| \leq \frac{|b|(A-B)}{p^n[2(1-B)-|b|(A-B)]\Gamma_{2p}(p-m)!}p!.
\] (2.9)

Now, by differentiating both sides of (1.14) \( m \) times with respect to \( z \), we have

\[
f^{(m)}(z) = (-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-(p+m)} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!}|a_k|z^{k-m} \quad (p \in \mathbb{N}; 0 \leq m < p)
\] (2.10)

and Theorem 3 follows easily from (2.9) and (2.10), respectively.

Finally, it is easy to see that the bounds in (2.7) are attained for the function \( f(z) \) given by (2.8).
Next, we determine the radii of meromorphically $p$-valent starlikeness and convexity of order $\varphi$ ($0 \leq \varphi < p$) for functions in the class $H_{p}^{n,*}(q, s, \alpha_1; A, B, b)$.

**Theorem 4.** Let the function $f(z)$ defined by (1.14) be in the class $H_{p}^{n,*}(q, s, \alpha_1; A, B, b)$, then we have:

(i) $f(z)$ is meromorphically $p$-valent starlike of order $\varphi$ ($0 \leq \varphi < p$) in the disc $|z| < r_1$, that is,

$$\Re \left\{ -\frac{zf'(z)}{f(z)} \right\} > \varphi \quad (|z| < r_1; \ 0 \leq \varphi < p; \ p \in \mathbb{N}), \quad (2.11)$$

where

$$r_1 = \inf_{k \geq p} \left\{ \left( k^n [(k+p)(1-B)-p|b| (A-B)] \Gamma_{k+p} (p-\varphi) \right) \left( k+\varphi \right) \right\}^{1/k+p}, \quad (2.12)$$

(ii) $f(z)$ is meromorphically $p$-valent convex of order $\varphi$ ($0 \leq \varphi < p$) in the disc $|z| < r_2$, that is,

$$\Re \left\{ -\left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \varphi \quad (|z| < r_2; \ 0 \leq \varphi < p; \ p \in \mathbb{N}), \quad (2.13)$$

where

$$r_2 = \inf_{k \geq p} \left\{ \left( k^{n-1} [(k+p)(1-B)-p|b| (A-B)] \Gamma_{k+p} p(p-\varphi) \right) \left( k+\varphi \right) \right\}^{1/k+p}. \quad (2.14)$$

Each of these results is sharp for the function $f(z)$ given by (2.6).

**Proof.** (i) From the definition (1.14), we easily get

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq 2(p-\varphi) - \sum_{k=p}^{\infty} (k-p+2\varphi) |a_k| |z|^{k+p}. \quad (2.15)$$

Thus, we have the desired inequality

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq 1 \quad (0 \leq \varphi < p; \ p \in \mathbb{N}), \quad (2.16)$$

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if
\[ \sum_{k=p}^{\infty} \left( \frac{k + \varphi}{p - \varphi} \right) |a_k||z|^{k+p} \leq 1. \]  
(2.17)

Hence, by Theorem 1, (2.17) will be true
\[ \left( \frac{k + \varphi}{p - \varphi} \right) |z|^{k+p} \leq \frac{k^n [(k + p)(1 - B) - p |b| (A - B)] \Gamma_{k+p}}{p |b| (A - B)} \]
(2.18)

Thus, we have the desired inequality
\[ \frac{1 + z f''(z) + p}{1 + z f''(z) - p + 2\varphi} \leq \frac{\sum_{k=p}^{\infty} k(k + p) |a_k||z|^{k+p}}{2p(\varphi - p) - \sum_{k=p}^{\infty} k(k - p + 2\varphi) |a_k||z|^{k+p}}. \]  
(2.19)

(ii) In order to prove the second assertion of Theorem 3, we find from the definition (1.14) that
\[ |z| \leq \left\{ \frac{k^n [(k + p)(1 - B) - p |b| (A - B)] \Gamma_{k+p}}{p |b| (A - B)} \left( \frac{k + p}{(k + \varphi)} \right) \frac{1}{k + p} \right\}. \]  
(2.20)

The last inequality (2.20) readily yields the disc \(|z| < r_2\), where \(r_2\) defined by (2.14), and the proof of Theorem 4 is completed by merely verifying that each assertion is sharp for the function \(f(z)\) given by (2.6).
3. Closure theorems

In this section we first prove:

**Theorem 5.** The class \(H_p^{n,s}(q, s, \alpha_1; A, B, b)\) is closed under convex linear combinations.

**Proof.** Let each of the functions

\[
f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}| z^k \quad (j = 1, 2; \ p \in \mathbb{N}) \tag{3.1}
\]

be in the class \(H_p^{n,s}(q, s, \alpha_1; A, B, b)\). It is sufficient to show that the function \(F(z)\) defined by

\[
F(z) = (1 - t)f_1(z) + tf_2(z) \quad (0 \leq t \leq 1) \tag{3.2}
\]

is also in the class \(H_p^{n,s}(q, s, \alpha_1; A, B, b)\). Since

\[
F(z) = z^{-p} + \sum_{k=p}^{\infty} [(1 - t)|a_{k,1}| + t|a_{k,2}|] z^k \quad (0 \leq t \leq 1). \tag{3.3}
\]

With the aid of Theorem 1, we have

\[
\sum_{k=p}^{\infty} k^n [(k + p) (1 - B) - p|b|(A - B)] \Gamma_{k+p} \cdot \{(1 - t)|a_{k,1}| + t|a_{k,2}| \} \\
= (1 - t) \sum_{k=p}^{\infty} k^n [(k + p) (1 - B) - p|b|(A - B)] \Gamma_{k+p} |a_{k,1}| + \\
t \sum_{k=p}^{\infty} k^n [(k + p) (1 - B) - p|b|(A - B)] \Gamma_{k+p} |a_{k,2}| \\
\leq (1 - t)p|b|(A - B) + tp|b|(A - B) = p|b|(A - B),
\]

which shows that \(F(z) \in H_p^{n,s}(q, s, \alpha_1; A, B, b)\).

**Theorem 6.** Let \(f_{p-1}(z) = \frac{1}{z^p}\) and

\[
f_k(z) = \frac{1}{z^p} + \frac{p|b|(A - B)}{k^n [(k + p) (1 - B) - p|b|(A - B)] \Gamma_{k+p}} z^k \quad (k \geq p; \ p \in \mathbb{N}). \tag{3.4}
\]

Then \(f(z) \in H_p^{n,s}(q, s, \alpha_1; A, B, b)\) if and only if it can be expressed in the form:

\[
f(z) = \sum_{k=p-1}^{\infty} \mu_k f_k(z), \tag{3.5}
\]
where \( \mu_k \geq 0 \ (k \geq p - 1; \ p \in \mathbb{N}) \) and \( \sum_{k=p-1}^{\infty} \mu_k = 1 \).

Proof. Let the function \( f(z) \) expressed in the form given by (3.5), then

\[
f(z) = z^{-p} + \sum_{k=p}^{\infty} \mu_k \frac{p |b| (A - B)}{k^n [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p}} z^k
\]

and for this function, we have

\[
\sum_{k=p}^{\infty} k^n [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p} \mu_k \frac{p |b| (A - B)}{k^n [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p}} = \sum_{k=p}^{\infty} \mu_k p |b| (A - B) = p |b| (A - B)(1 - \mu_{p-1}) \leq p |b| (A - B),
\]

which shows that \( f(z) \in H^{n,*}_{p}(q, s, \alpha_1; A, B, b) \) by Theorem 1.

Conversely, suppose that the function \( f(z) \) defined by (1.14) belongs to the class \( H^{n,*}_{p}(q, s, \alpha_1; A, B, b) \). Since

\[
|a_k| \leq \frac{p |b| (A - B)}{k^n [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p}} \quad (k \geq p; \ p \in \mathbb{N})
\]

by Corollary 2, setting

\[
\mu_k = \frac{k^n [(k + p) (1 - B) - p |b| (A - B)] \Gamma_{k+p}}{p |b| (A - B)} |a_k| \quad (k \geq p; \ p \in \mathbb{N})
\]

and

\[
\mu_{p-1} = 1 - \sum_{k=p}^{\infty} \mu_k,
\]

it follows that

\[
f(z) = \sum_{k=p-1}^{\infty} \mu_k f_k(z).
\]

This completes the proof of Theorem 6.

4. Neighborhoods and partial sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [11] and Ruscheweyh [17] and (more recently) by
Altintas et al. ([1], [2] and [3]), Liu [13] and Liu and Srivastava [15], we begin by introducing here the $\delta$-neighborhoods of a function $f(z) \in \Sigma_p$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \left\{ g : g \in \Sigma_p, g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \text{ and } \right.$$

$$\sum_{k=1-p}^{\infty} \frac{k^n [(k + p) (1 + |B|) - p |b| (A - B)]}{p |b| (A - B)} \Gamma_{k+p} b_k - a_k \leq \delta$$

$$\right\}$$. \hspace{1cm} (4.1)

Making use of the definition (4.1), we now prove:

**Theorem 7.** Let the function $f(z)$ defined by (1.1) be in the class $H^n_p(q, s, \alpha_1; A, B, b)$. If $f(z)$ satisfies the condition:

$$\frac{f(z) + \epsilon z^{-p}}{1 + \epsilon} \in H^n_p(q, s, \alpha_1; A, B, b) \quad (\epsilon \in \mathbb{C}; \ |\epsilon| < \delta; \ \delta > 0)$$ \hspace{1cm} (4.2)

then

$$N_\delta(f) \subset H^n_p(q, s, \alpha_1; A, B, b).$$ \hspace{1cm} (4.3)

**Proof.** It is easily from (1.13) that $g(z) \in H^n_p(q, s, \alpha_1; A, B, b)$ if and only if, for any complex number $\sigma$ with $|\sigma| = 1$, we have

$$z \left( \left( I^n (H_{p,q,s}(\alpha_1) f(z)) \right)' + p I^n (H_{p,q,s}(\alpha_1) f(z)) \right) \neq \sigma \quad (z \in U),$$ \hspace{1cm} (4.4)

which is equivalent to

$$\frac{(g * h)(z)}{z^{-p}} \neq 0 \quad (z \in U),$$ \hspace{1cm} (4.5)

where, for convenience,

$$h(z) = z^{-p} + \sum_{k=1-p}^{\infty} c_k z^k$$

$$= z^{-p} + \sum_{k=1-p}^{\infty} \frac{k^n [(k + p) (1 - B\sigma) - pb\sigma (A - B)]}{pb\sigma (A - B)} \Gamma_{k+p} z^k.$$ \hspace{1cm} (4.6)
From (4.6), we have
\[ |c_k| = \left| \frac{k^n [(k + p) (1 - B\sigma) - pb\sigma(A - B)]}{pb\sigma(A - B)} \Gamma_{k+p} \right| \leq \frac{k^n [(k + p) (1 + |B|) - p|b|(A - B)]}{p|b|(A - B)} \Gamma_{k+p} \text{ (} k \geq p; p \in \mathbb{N} \text{).} \] (4.7)

If \( f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \in \Sigma_p \) satisfies the condition (4.2), then (4.5) yields
\[ \left| \frac{(f * h)(z)}{z^{-p}} \right| > \delta \text{ (} z \in U^*; \delta > 0 \text{).} \] (4.8)

Next, if we suppose that
\[ \Phi(z) = z^{-p} + \sum_{k=1-p}^{\infty} d_k z^k \in N_{\delta}(f) \] (4.9)
we easily see that
\[ \left| \frac{[\Phi(z) - f(z)] * h(z)}{z^{-p}} \right| = \left| \sum_{k=1-p}^{\infty} (d_k - a_k) c_k z^{k+p} \right| \leq |z| \sum_{k=1-p}^{\infty} \frac{k^n [(k + p) (1 + |B|) - p|b|(A - B)]}{p|b|(A - B)} \Gamma_{k+p} |d_k - a_k| \]
\[ < \delta \text{ (} z \in U; \delta > 0 \text{).} \] (4.10)

Thus we have (4.5), and hence also (4.4) for any \( \sigma \in \mathbb{C} \) such that \(|\sigma| = 1\), which implies that \( \Phi(z) \in H^\alpha_p(q, s, a_1; A, B, b) \). This evidently proves the assertion (4.3) of Theorem 7.

We now define the \( \delta \)-neighborhoods of a function \( f(z) \in \Sigma^*_p \) of the form (1.14) as follows:
\[
N_{\delta}^+(f) = \begin{cases}
g \in \Sigma_{p}^*, g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \text{ and } \\
\sum_{k=p}^{\infty} k^n [(k + p) (1 + |B|) - p |b| (A - B)] \frac{\Gamma_{k+p}}{p |b| (A - B)} \Gamma_{k+p} |b_k| - |a_k| \leq \delta
\end{cases}
\]

(4.11)

**Theorem 8.** Let \(-1 \leq B < A \leq 1; \alpha_1, \ldots, \alpha_q \in \mathbb{C} \text{ and } \beta_1, \ldots, \beta_s \in \mathbb{C} \setminus \mathbb{Z}_0^*; p \in \mathbb{N}; q, s, n \in \mathbb{N}_0; q \leq s + 1; b \in \mathbb{C}^*) \) then

\[
N_{\delta}^+(f) \subset H_n^{*, p}(q, s, \alpha_1 + 1; A, B, b) \quad (\delta = \frac{2p}{\alpha_1 + 2p}).
\]

(4.12)

The result is sharp in the sense that \(\delta\) cannot be increased.

**Proof.** Making use of the same method as in the proof of Theorem 7, we can show that [cf. Equation (4.6)]

\[
h(z) = z^{-p} + \sum_{k=p}^{\infty} c_k z^k
\]

\[= z^{-p} + \sum_{k=p}^{\infty} \frac{k^n [(k + p) (1 - |B\sigma|) - pb\sigma(A - B)]}{pb\sigma(A - B)} \Gamma_{k+p} z^k.\]

(4.13)

Thus, under the hypothesis \(-1 \leq B < A \leq 1; -1 \leq B < 0; q \leq s + 1; p \in \mathbb{N}; q, s, n \in \mathbb{N}_0; b \in \mathbb{C}^*\), if \(f(z) \in H_n^{*, p}(q, s, \alpha_1 + 1; A, B, b)\) is given by (1.14), we obtain

\[
\left| \frac{(f + h)(z)}{z^{-p}} \right| = \left| 1 + \sum_{k=p}^{\infty} c_k |a_k| z^{k+p} \right|
\]

\[\geq 1 - \sum_{k=p}^{\infty} \frac{k^n [(k + p) (1 + |B|) - p |b| (A - B)]}{p |b| (A - B)} \Gamma_{k+p} |a_k|
\]

\[\geq 1 - \frac{\alpha_1}{\alpha_1 + 2p} \sum_{k=p}^{\infty} \frac{k^n [(k + p) (1 + |B|) - p |b| (A - B)]}{p |b| (A - B)} \Gamma_{k+p} |a_k|
\]

\[\geq 1 - \frac{\alpha_1}{\alpha_1 + 2p} = 2p
\]

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The remaining part of the proof of Theorem 8 is similar to that of Theorem 8 and
we skip the details involved.

**Theorem 9.** Let $f(z) \in \Sigma_p$ defined by (1.1) and $-1 \leq B \leq 0$, we define the partial
sums $S_1(z)$ and $S_q(z)$ as follows

$$S_1(z) = z^{-p} \quad \text{and} \quad S_q(z) = z^{-p} + \sum_{k=1-p}^{q-1} a_k z^k \quad (q \in \mathbb{N}\{1\}), \quad \text{(4.14)}$$

it being understood that an empty sum is (as usual) nil. Suppose also that

$$\sum_{k=1-p}^{\infty} c_k + |a_k| \leq 1 \quad \left( c_{k+p} = \frac{k^n [(k+p)(1+|B|) - p |b|(A-B)]}{p |b|(A-B)} \right) \quad \Gamma_{k+p}. \quad \text{(4.15)}$$

Then

(i) $f(z) \in H_n^p(q,s,\alpha_1; A, B, b)$

(ii) $\Re \left\{ \frac{f(z)}{S_q(z)} \right\} > 1 - \frac{1}{c_q} \quad (z \in U; q \in \mathbb{N}) \quad \text{(4.16)}$

and

(iii) $\Re \left\{ \frac{S_q(z)}{f(z)} \right\} > \frac{c_q}{1 + c_q} \quad (z \in U; q \in \mathbb{N}). \quad \text{(4.17)}$

The estimates in (4.16) and (4.17) are sharp.

**Proof.** Since $z^{-p} + \varepsilon z^{-p} = z^{-p} \in H_n^p(q,s,\alpha_1; A, B, b)$, $|\varepsilon| < 1$, then by Theo-
rem 7, we have $N_\delta(f) \subset H_n^p(q,s,\alpha_1; A, B, b)$, $p \in \mathbb{N}$ $(N_1(z^{-p})$ denoting the 1-
neighbourhood). Now since

$$\sum_{k=1-p}^{\infty} c_k |a_k| \leq 1, \quad \text{(4.18)}$$

then $f \in N_1(z^{-p})$ and $f \in H_n^p(q,s,\alpha_1; A, B, b)$.(ii) Since $\{c_k\}$ is an increasing se-
quence, we obtain

$$\sum_{k=1-p}^{q-p-1} |a_k| + c_q \sum_{k=q-p}^{\infty} |a_k| \leq \sum_{k=1-p}^{\infty} c_{k+p} |a_k| \leq 1, \quad \text{(4.19)}$$
where we have used the hypothesis (4.15) again. By setting
\[ h_1(z) = c_q \left\{ \frac{f(z)}{S_q(z)} - (1 - \frac{1}{c_q}) \right\} = 1 + \frac{c_q \sum_{k=q-p}^{\infty} |a_k| z^{k+p}}{1 + \sum_{k=1-p}^{q-p-1} |a_k| z^{k+p}} \]
and applying (4.19), we find that
\[ \left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| \leq \frac{c_q \sum_{k=q-p}^{\infty} |a_k|}{2 - 2 \sum_{k=1-p}^{q-p-1} |a_k| - c_q \sum_{k=q-p}^{\infty} |a_k|} \leq 1 \quad (z \in U), \quad (4.20) \]
which readily yields the assertion (4.16) of Theorem 9. If we take
\[ f(z) = z^{-p} - \frac{z^q}{c_q}, \quad (4.21) \]
then
\[ \frac{f(z)}{S_q(z)} = 1 - \frac{z^{p+q}}{c_q} \to 1 - \frac{1}{c_q}, \quad \text{as} \quad z \to 1^-, \]
which shows that the bound in (4.16) is the best possible for each \( q \in N \).

(iii) Just as in part (ii) above, if we put
\[ h_2(z) = (1 + c_q) \left\{ \frac{S_q(z)}{f(z)} - \frac{c_q}{1 + c_q} \right\} = 1 - \frac{(1 + c_q) \sum_{k=q-p}^{\infty} |a_k| z^{k+p}}{1 + \sum_{k=1-p}^{q-p-1} |a_k| z^{k+p}}, \quad (4.22) \]
and make use of (4.19), we can deduce that
\[ \left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| \leq \frac{(1 + c_q) \sum_{k=q-p}^{\infty} |a_k|}{2 - 2 \sum_{k=1-p}^{q-p-1} |a_k| - (1 - c_q) \sum_{k=q-p}^{\infty} |a_k|} \leq 1, \]
which leads us immediately to the assertion (4.17) of Theorem 9.

The bound in (4.17) is sharp for each \( n \in N \), with the extremal function \( f(z) \) given by (4.21). The proof of Theorem 9 is thus completed.

**Remark 1.** Taking \( n = 0, q = 2, s = 1, \alpha_1 = a \quad (a > 0), \alpha_2 = 1 \) and \( \beta_1 = c \quad (c > 0), \) in all our results, we obtain the results obtained by Aqlan and Kulkarni [6].
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