QUASI-CONFORMAL CURVATURE TENSOR ON GENERALIZED $(\kappa, \mu)$-CONTACT METRIC MANIFOLDS

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ABSTRACT. The object of the present paper is to characterize 3-dimensional generalized $(\kappa, \mu)$-contact metric manifolds satisfying certain curvature conditions on quasi-conformal curvature tensor.

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1. Introduction

In 1995, Blair, Koufogiorgos and Papantoniou [9] introduced the notion of $(\kappa, \mu)$-contact metric manifolds where $\kappa$, $\mu$ are real constants. Assuming $\kappa$, $\mu$ smooth functions, Koufogiorgos and Tsichlias [18] introduced the notion of generalized $(\kappa, \mu)$-contact metric manifolds and gave several examples. Again they also show that such manifold does not exist in dimension greater than three. In a recent paper [2], Yildiz, De and Cetinkaya study concircular curvature tensor in 3-dimensional generalized $(\kappa, \mu)$-contact metric manifolds. Generalized $(\kappa, \mu)$-contact metric manifolds have been studied by several authors ([17], [11], [19], [1]) and many others. In [6], the authors studied extended pseudo projective curvature tensor on contact metric manifolds. Quasi-conformal curvature tensor on Sasakian manifolds has been studied by De, Jun and Gazi [23]. After the Reimannian curvature tensor, Weyl conformal curvature tensor plays an important role in differential geometry as well as in theory of relativity. In [16], Yano and Sawaki defined the notion of the quasi-conformal curvature tensor which is extended form of conformal curvature tensor. According to them a quasi-conformal curvature is defined by

\[
\bar{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right]g(Y,Z)X - g(X,Z)Y,
\]

(1)
for all $X, Y \in TM$, where $a$ and $b$ are constants, $S$ is the Ricci tensor, $Q$ is the Ricci operator and $r$ is the scalar curvature of the $n$-dimensional manifold $M^n (n \geq 3)$. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1) takes the form

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z,$$

(2)

where $C$ is conformal curvature tensor [15]. Thus $C$ is a particular case of the tensor $\tilde{C}$. In a recent paper [25], De and Matsuyama studied quasi-conformally flat manifolds satisfying certain curvature condition on the Ricci tensor. They proved that a quasi-conformally flat manifold satisfying

$$S(X, Y) = rT(X)T(Y),$$

(3)

where $S$ is the Ricci tensor, $r$ is the scalar curvature and $T$ is a nonzero 1-form defined by $T(X) = g(X, \rho)$, $\rho$ is a unit vector field, can be expressed as a locally wrapped product $I \times_{\rho} M^*$, where $M^*$ is an Einstein manifold. From this result, it easily follows that a quasi-conformal flat space time satisfying (3) is a Robertson-Walker space time [4].

Let $M$ be an almost contact metric manifold equipped with an almost contact metric structure $(\varphi, \xi, \eta, \varphi)$. Since at each point $p \in M$ the tangent space $T_p M$ can be decomposed into direct sum $T_p M = \varphi(T_p M) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of $T_p M$ generated by $\{\xi_p\}$, the conformal curvature tensor $C$ is a map

$$C : T_p M \times T_p M \times T_p M \rightarrow \varphi(T_p) \oplus \{\xi_p\} \quad p \in M$$

. It may be natural to consider to consider the following particular cases: (1) the projection of the image of $C$ in $\varphi(T_p M)$ is zero; (2) the projection of the image of $C$ in $\{\xi_p\}$ is zero; (3) the projection of image of $C|_{\varphi(T_p M) \times \varphi(T_p M) \times \varphi(T_p M)}$ in $\varphi(T_p M)$ is zero. An almost contact metric manifold satisfying the case (1), (2) and (3) is said to be conformally symmetric [12], $\xi$-conformally flat [13] and $\varphi$-conformally flat [14] respectively. In an analogous way, we define $\xi$-quasi-conformally flat generalized $(\kappa, \mu)$-contact metric manifolds.

In [24], the authors studied $\xi$-conformally flat $N(\kappa)$-contact metric manifolds. In [5], quasi-conformal curvature tensor on Kenmotsu manifolds was studied by Özgür
and De. In a recent paper [22], De and Sarkar studied quasi-conformally flat and extended quasi-conformally flat \((\kappa, \mu)\)-contact metric manifolds. Motivated by the above studies, we characterize a 3-dimensional generalized \((\kappa, \mu)\)-contact metric manifolds satisfying certain curvature conditions on the quasi-conformal curvature tensor. The present paper is organized as follows:

After preliminaries in section 3, we characterize quasi-conformally flat generalized \((\kappa, \mu)\)-contact metric manifolds. In the next section, we prove that a generalized \((\kappa, \mu)\)-contact metric manifold is locally \(\phi\)-quasiconformally symmetric if and only if the generalized \((\kappa, \mu)\)-contact metric manifold is a \((\kappa, \mu)\)-contact metric manifold provided \(a + b \neq 0\). Besides these, we prove that a \(\xi\)-quasiconformally flat generalized \((\kappa, \mu)\)-contact metric manifold is an \(N(\kappa)\)-contact metric manifold provided \((a + b) \neq 0\). Finally, it is shown that generalized \((\kappa, \mu)\)-contact metric manifold satisfying \(\tilde{C} \cdot S = 0\) is \(\eta\)-Einstein provided \((a + b) \neq 0\).

2. Preliminaries

An odd dimensional differentiable manifold \(M^n\) is called almost contact manifold if there is an almost contact structure \((\varphi, \xi, \eta)\) consisting of a \((1, 1)\) tensor field \(\varphi\), a vector field \(\xi\), a 1-form \(\eta\) satisfying

\[
\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \quad (4)
\]

From (4) it follows that

\[
\varphi_\xi = 0, \quad \eta \circ \varphi = 0.
\]

Let \(g\) be a compatible Reimannian metric with \((\varphi, \xi, \eta)\), that is,

\[
g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \quad \text{for all } X, Y \in TM. \quad (5)
\]

An almost contact metric structure becomes a contact metric structure if

\[
g(X, \varphi Y) = d\eta(X, Y), \quad \text{for all } X, Y \in TM. \quad (6)
\]

Given a contact metric manifold \(M^n(\varphi, \xi, \eta, g)\) we define a \((1, 1)\) tensor field \(h\) by

\[
h = \frac{1}{2}L_\xi \varphi \quad \text{where } L \text{ denotes the Lie differentiation.}
\]

Then \(h\) is symmetric and satisfies

\[
h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \nabla \xi = -\varphi - \varphi h, \quad trace(h) = trace(\varphi h) = 0, \quad (7)
\]

where \(\nabla\) is the Levi-Civita connection.

A contact metric manifold is said to be an \(\eta\)-Einstein manifold if

\[
S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y), \quad (8)
\]
where $a, b$ are smooth functions and $X, Y \in TM$, $S$ is the Ricci tensor.

Blair, Koufogiorgos and Papantoniou [9] considered the $(\kappa, \mu)$-nullity condition and gave several reasons for studying it. The $(\kappa, \mu)$-nullity distribution $N(\kappa, \mu)$ ([9], [3]) of a contact metric manifold $M$ is defined by

$$N(\kappa, \mu) : p \mapsto N_p(\kappa, \mu) = \{U \in T_pM \mid R(X, Y)U = (\kappa I + \mu h)(g(Y, U)X - g(X, U)Y)\}$$

for all $X, Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$.

A contact metric manifold $M^n$ with $\xi \in N(\kappa, \mu)$ is called a $(\kappa, \mu)$-contact metric manifold. Then we have

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \quad (9)$$

For all $X, Y \in TM$. If $\mu = 0$, then the $(\kappa, \mu)$-nullity distribution $N(\kappa, \mu)$ is reduced to $\kappa$-nullity distribution $N(\kappa)$ [21]. If $\xi \in N(\kappa)$, then we call contact metric manifold $M$ an $N(\kappa)$-contact metric manifold.

In a $(\kappa, \mu)$-contact metric manifold the following relations hold:

$$h^2 = (\kappa - 1)\varphi^2, \quad (10)$$

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (11)$$

$$R(\xi, X)Y = \kappa[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \quad (12)$$

$$S(X, \xi) = (n - 1)\kappa\eta(X), \quad (13)$$

$$S(X, Y) = [(n - 3) - \frac{n - 1}{2} \mu]g(X, Y) +$$

$$[(n - 3) + \mu]g(hX, Y) + [(3 - n) + \frac{n - 1}{2}(2\kappa + \mu)]\eta(X)\eta(Y), \quad (14)$$

$$r = (n - 1)(n - 3 + \kappa - \frac{n - 1}{2} \mu), \quad (15)$$

A $(\kappa, \mu)$-contact metric manifold is called a generalized $(\kappa, \mu)$-contact metric manifold if $\kappa, \mu$ are smooth functions. In [18], Koufogiorgos and Tsichlias proved its existence for 3-dimensional case, whereas greater than 3-dimensional, such manifold does not exist. In generalized $(\kappa, \mu)$-contact metric manifold $M^3(\varphi, \xi, \eta, g)$ the following relations hold ([18], [3]):

$$\xi \kappa = 0, \quad (16)$$

$$\xi r = 0, \quad (17)$$

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\begin{align}
\tag{18}
h \text{grad} \mu &= \text{grad} \mu,
\end{align}

\begin{align}
S(X,Y) &= -\mu g(X,Y) + \mu g(hX,Y) + (2\kappa + \mu)\eta(X)\eta(Y), \\
S(X,hY) &= -\mu g(X,hY) - (\kappa - 1)\mu g(X,Y) + (\kappa - 1)\mu \eta(X)\eta(Y), \\
S(X,\xi) &= 2\kappa \eta(X), \\
QX &= \mu (hX - X) + (2\kappa + \mu)\eta(X)\xi, \\
r &= 2(\kappa - \mu). \\
(\nabla_X h)Y &= \{(1 - \kappa)g(X,\varphi Y) \\
&- g(X,\varphi hY)\}\xi - \eta(Y)\{(1 - \kappa)\varphi X \\
&+ \varphi hX\} - \mu \eta(X)\varphi hY, \\
(\nabla_X \varphi)Y &= \{g(X,Y) + g(X,hY)\}\xi - \eta(Y)(X + hX). \\
\end{align}

3. Quasi-conformally flat generalized (\kappa, \mu)-contact metric manifolds

**Definition 1.** A generalized (\kappa, \mu)-contact metric manifold \(M^3\) is called quasi-conformally flat if the quasi-conformal curvature tensor \(\tilde{C} = 0\).

It is known that conformal curvature tensor vanishes identically in a 3-dimensional Riemannian manifold. Hence, from (2) we obtain

\begin{align}
\tag{26}
R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \\
&\quad \frac{r}{2}[g(Y,Z)X - g(X,Z)Y].
\end{align}

Substituting \(Y = Z = \xi\) in (26) we have

\begin{align}
\tag{27}
QX &= \frac{1}{2}(r - 2\kappa)X + \frac{1}{2}(6\kappa - r)\eta(X)\xi + \mu hX.
\end{align}

Taking inner product with \(Y\) of (27) we get

\begin{align}
\tag{28}
S(X,Y) &= \frac{1}{2}(r - 2\kappa)g(X,Y) \\
&\quad + \frac{1}{2}(6\kappa - r)\eta(X)\eta(Y) + \mu g(hX,Y).
\end{align}
From (1) we have
\[ \tilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{3} [\frac{a}{2} + 2b][g(Y,Z)X - g(X,Z)Y]. \] (29)

Putting (26), (27) and (28) in (29) we have
\[ \tilde{C}(X,Y)Z = (a + b)\left\{ \frac{4\kappa + 2\mu}{3}[g(X,Z)Y - g(Y,Z)X] + (\kappa + \mu) \right. \]
\[ \left. \left[ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\eta(X)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \right] + \mu[g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y] \right\}. \] (30)

Thus we have
\[ \text{Lemma 3.1.} \] Let \( M \) be a 3-dimensional generalized \((\kappa, \mu)\) contact metric manifold. Then the quasi-conformal curvature tensor vanishes identically provided \( a + b = 0 \).

Next we assume that \( a + b \neq 0 \) and \( M \) is Quasi-conformally flat. Then from (30) we have
\[ \frac{4\kappa + 2\mu}{3} \left[ g(X,Z)Y - g(Y,Z)X \right] + (2\kappa + \mu) \]
\[ \left[ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\eta(X)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \right] + \mu[g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y] = 0. \] (31)

Taking inner product with \( W \) of (31) we get
\[ \frac{4\kappa + 2\mu}{3} \left[ g(X,Z)g(Y,W) - g(Y,Z)g(X,W) \right] + (2\kappa + \mu) \]
\[ \left[ g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) + \eta(Y)\eta(Z)g(X,W) - \eta(Z)\eta(X)g(Y,W) \right] + \mu[g(Y,Z)g(hX,W) - g(X,Z)g(hY,W) + g(hY,Z)g(X,W) - g(hX,Z)g(Y,W)] = 0. \] (32)

Putting \( Y = Z = \xi \) we have
\[ \mu g(hX,W) = -\frac{2\kappa + \mu}{3} g(X,W) + \frac{2\kappa + \mu}{3} \eta(X)\eta(W). \] (33)

From (19) and (33) we obtain
\[ S(X,W) = ag(X,W) + b\eta(X)\eta(W), \] (34)

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where
\[ a = -\mu - \frac{2\kappa + \mu}{3} \]
and
\[ b = (2\kappa + \mu) + \frac{2\kappa + \mu}{3}. \]

Hence from (34) we conclude the following:

**Theorem 3.1.** A 3-dimensional quasi-conformally flat generalized \((\kappa, \mu)\) contact metric manifold is an \(\eta\)-Einstein manifold if \(a + b \neq 0\).

### 4. Locally \(\varphi\)-Quasi-conformally Symmetric Generalized \((\kappa, \mu)\)-Contact Metric Manifolds

**Definition 2.** A contact metric manifold is said to be locally \(\varphi\)-symmetric if the manifold satisfy the following:

\[ \varphi^2((\nabla_X R)(Y, Z)W) = 0, \]

for all vector fields \(X, Y, Z, W\) orthogonal to \(\xi\). This notion was introduced for Sasakian manifolds by Takahashi [20].

In this paper, we study locally \(\varphi\)-quasi-conformally symmetric 3-dimensional generalized \((\kappa, \mu)\)-contact metric manifolds. A generalized \((\kappa, \mu)\)-contact manifold is called \(\varphi\)-quasi-conformally symmetric if the condition

\[ \varphi^2((\nabla_X \tilde{C})(Y, Z)W) = 0, \]

holds on the manifold, where \(X, Y, Z, W\) are orthogonal to \(\xi\).

Let us consider \(M\) be a 3-dimensional generalized \((\kappa, \mu)\)-contact metric manifold. Taking covariant differentiation of (30) we have

\[
((\nabla_W \tilde{C})(X, Y)Z) = (a + b)\left\{-\left(\frac{4W\kappa + 2W\mu}{3}\right)[g(Y, Z)X - g(X, Z)Y] + (2\kappa + \mu)\right. \\
\left. [g(Y, Z)g(W + hW, \varphi X) - g(X, Z)g(W + hW, \varphi Y)]\xi + \right. \\
(W\mu)[g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y] + \\
\mu[(1 - \kappa)g(W, \varphi X) + g(W, h\varphi X)]g(Y, Z)\xi - \mu[(1 - \kappa) \\
g(W, \varphi Y) + g(W, h\varphi Y)]g(X, Z)\xi\right\},
\]
for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Operating $\varphi^2$ to the above equation, we obtain

$$\varphi^2((\nabla_W \tilde{C})(X, Y)Z) = (a + b)\{ - \left( \frac{4W\kappa + 2W\mu}{3} \right) + (W\mu)[g(Y, Z)[g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y]], \quad (38)$$

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Thus from (38) we conclude that if $\kappa$ and $\mu$ are constants, then $M$ is locally $\varphi$-quasiconformally symmetric. Conversely, let us consider that $M$ is locally $\varphi$-quasiconformally symmetric.

From (36) and (38) we have if $(a + b) \neq 0$

$$-(\frac{4W\kappa + 2W\mu}{3}) \left[ g(Y, Z)X - g(X, Z)Y \right] + (W\mu)[g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y] = 0. \quad (39)$$

Taking inner product with $U$ of (39) we get

$$(\frac{4W\kappa + 2W\mu}{3}) \left[ g(Y, Z)g(X, U) - g(X, Z)g(Y, U) \right] - (W\mu)[g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y] = 0. \quad (40)$$

Contracting $X$ and $Z$ we obtain

$$2\left( \frac{4W\kappa + 2W\mu}{3} \right) Y - (W\mu)hY = 0. \quad (41)$$

Applying $h$ on both sides of (41) we have

$$2\left( \frac{4W\kappa + 2W\mu}{3} \right) hY - (W\mu)h^2Y = 0. \quad (42)$$

Taking trace on both sides of (42) and using $\text{trace}(h) = 0$ we obtain $\mu$ is constant. Thus $\kappa$ is also constant. Therefore, we can state the following:

**Theorem 4.1.** Let $M$ be a 3-dimensional generalized $(\kappa, \mu)$-contact metric manifold. $M$ is locally $\varphi$-quasiconformally symmetric if and only if $M$ is a $(\kappa, \mu)$-contact metric manifold provided $a + b \neq 0$. 

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5. \(\xi\)-QUASICONFORMALLY FLAT GENERALIZED \((\kappa, \mu)\)-CONTACT METRIC
MANIFOLDS

Assume that \(M^3\) is a \(\xi\)-quasi-conformally flat \((\kappa, \mu)\)-contact metric manifold. So we have

\[\tilde{C}(X,Y)\xi = 0.\] (43)

From (1) we have

\[
\tilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{3} (a^2 + 2b)
\]

\[
\left[ g(Y,Z)X - g(X,Z)Y \right].
\] (44)

Using (26) in (44) we obtain

\[
\tilde{C}(X,Y)Z = (a + b) \left\{ [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - 2r3 \left[ g(Y,Z)X - g(X,Z)Y \right] \right\}.
\] (45)

Putting \(Z = \xi\) and using (21), (22) and (43) we have

\[
(a + b) \left[ (2\kappa - \mu - \frac{2r}{3}) (\eta(Y)X - \eta(X)Y) + \mu (\eta(Y)hX - \eta(X)hY) \right] = 0.
\] (46)

Putting \(Y = \xi\) in (46) we obtain

\[
(a + b) \left[ (2\kappa - \mu - \frac{2r}{3}) (X - \eta(X)\xi) + \mu hX \right] = 0.
\] (47)

Applying \(h\) on both sides of (47) we get

\[
(a + b) \left[ (2\kappa - \mu - \frac{2r}{3}) hX + \mu h^2 \right] = 0.
\] (48)

Taking trace on both sides of (48) and using \(\text{trace}(h) = 0\) we have

\[
(a + b) \mu \text{trace}(h^2) = 0.
\] (49)

As \(\text{trace}(h^2) \neq 0\) we can conclude that

if \((a + b) \neq 0\), then \(\mu = 0\).

If \(\mu = 0\), then \(M^3\) is an \(N(\kappa)\)-contact metric manifold.

From the above discussion we can state the following:

**Theorem 5.1.** Let \(M\) be a 3-dimensional \(\xi\)-quasi-conformally flat generalized \((\kappa, \mu)\)-contact metric manifold. Then \(M\) is an \(N(\kappa)\)-contact metric manifold provided \((a + b) \neq 0\).
6. Generalized \((\kappa, \mu)\)-contact metric manifold satisfying \(\overline{C}\cdot S = 0\)

Let \(M^3\) be a generalized \((\kappa, \mu)\)-contact metric manifold satisfying \(\overline{C}\cdot S = 0\), which implies that

\[
S(\overline{C}(X, Y)U, V) + S(U, \overline{C}(X, Y)V) = 0. \tag{50}
\]

Putting \(X = U = \xi\) in (50) and using (21) we have

\[
S(\overline{C}(\xi, Y)\xi, V) = 2\kappa \eta(\overline{C}(\xi, Y)V). \tag{51}
\]

Putting \(X = \xi\) in (37) and using (21) we obtain

\[
\overline{C}(\xi, Y)V = (a + b) \left\{ [S(Y, V)\xi + 2\kappa \eta(V)Y + g(Y, V)2\kappa \xi - \eta(V)QY] - \frac{2r}{3} [g(Y, V)\xi - \eta(V)Y] \right\}. \tag{52}
\]

Taking inner product with \(\xi\) of (52) we get

\[
\eta(\overline{C}(\xi, Y)V) = (a + b) \left\{ [S(Y, V) + 2\kappa g(Y, V)] - \frac{2r}{3} [g(Y, V) - \eta(V)\eta(Y)] \right\}. \tag{53}
\]

Putting \(V = \xi\) in (52) and using (21) and (22) we have

\[
\overline{C}(\xi, Y)\xi = (a + b) \left\{ (2\kappa - \mu - \frac{2r}{3}) (\eta(Y)\xi - Y) - \mu hY \right\}, \tag{54}
\]

which implies

\[
S(\overline{C}(\xi, Y)\xi, V) = (a + b) \left\{ -(2\kappa - \mu - \frac{2r}{3}) 2\kappa \eta(Y)\eta(V) - (2\kappa - \mu - \frac{2r}{3}) S(Y, V) - \mu S(hY, V) \right\}. \tag{55}
\]

Putting (53) and (55) in (51) we obtain

\[
(a + b) \left\{ (4\kappa - \mu - \frac{2r}{3}) S(Y, V) + \mu S(hY, V) + (4\kappa^2 - \frac{4kr}{3}) g(Y, V) + (2\kappa - \mu - \frac{2r}{3} + \frac{4kr}{3}) \eta(V)\eta(Y) \right\} = 0,
\]

Thus if \((a + b) \neq 0\)

\[
\left\{ (4\kappa - \mu - \frac{2r}{3}) S(Y, V) + \mu S(hY, V) + (4\kappa^2 - \frac{4kr}{3}) g(Y, V) + (2\kappa - \mu - \frac{2r}{3} + \frac{4kr}{3}) \eta(V)\eta(Y) \right\} = 0. \tag{56}
\]
Using (19) and (20) in (56) we have
\[ \mu g(hY, V) = a_1 g(Y, V) + b_1 \eta(Y) \eta(V), \]
where
\[ a_1 = \frac{[3\mu^2 \kappa - 4\mu^2 - 8\kappa^2]}{[8\kappa + 4\mu]}, \]
and
\[ b_1 = -\frac{[(8\kappa - 2\mu)(3\kappa + \mu) + 3\mu^2 \kappa + 2\kappa + \mu]}{8\kappa + \mu}. \]
From (57) and (19) we obtain
\[ S(Y, V) = a g(Y, V) + b \eta(Y) \eta(V), \]
where
\[ a = -\mu + \frac{[3\mu^2 \kappa - 4\mu^2 - 8\kappa^2]}{[8\kappa + 4\mu]}, \]
and
\[ b = (2\kappa + \mu) - \frac{[(8\kappa - 2\mu)(3\kappa + \mu) + 3\mu^2 \kappa + 2\kappa + \mu]}{8\kappa + \mu}. \]
From (58) we can state the following:

**Theorem 6.1.** Let \( M \) be a 3-dimensional generalized \((\kappa, \mu)\)-contact metric manifold satisfying \( \tilde{C} \cdot S = 0 \). Then \( M \) is an \( \eta \)-Einstein manifold provided \((a + b) \neq 0\).

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