ON SPACELIKE PARALLEL $p_{1}$-EQUIDISTANT RULED SURFACES IN THE MINKOWSKI 3-SPACE $R^{3}_{1}$

M. Masal, N. Kuruoğlu

Abstract. In this paper, radii and curvature axes of osculator Lorentz spheres and arc lengths of indicatrix curves of base curves of spacelike parallel $p_{1}$-Equidistant ruled surfaces in the Minkowski 3-space $R^{3}_{1}$ are given.


Keywords: ruled surface, Minkowski, spacelike, parallel $p_{1}$-equidistant.

1. Introduction

I. E. Valeontis, (see [3]), defined parallel $p$-equidistant ruled surfaces in $E^{3}$ and gave some results related with striction curves of ruled surfaces. Then he also studied on existence theorem related with homothety of parallel $p$-equidistant ruled surfaces.

M. Masal, N. Kuruoğlu, (see [1]) obtained arc lengths, curvature radii, curvature axes, spherical involute and areas of real closed spherical indicatrix curves of base curves (leading curves) of parallel $p$-equidistant ruled surfaces in $E^{3}$.

And also, M. Masal, N. Kuruoğlu, (see [2]) defined spacelike parallel $p_{1}$-equidistant ruled surfaces in the Minkowski 3-space $R^{3}_{1}$ and obtained dralls, the shape operators, Gaussian curvatures, mean curvatures, shape tensor, $q^{th}$ fundamental forms of these surfaces.

This paper is organized as follows: in Section 3 we find radii and curvature axes of osculator Lorentz spheres of spacelike parallel $p_{1}$-equidistant ruled surfaces in the Minkowski 3-space.

And later in Section 4 we give arc lengths of indicatrice curves of spacelike parallel $p_{1}$-equidistant ruled surfaces.

2. Preliminaries

Let $\alpha: I \rightarrow R^{3}_{1}$, $\alpha(t) = (\alpha_{1}(t), \alpha_{2}(t), \alpha_{3}(t))$ be a differentiable spacelike curve with arc-length in the Minkowski 3-space, where $I$ is an open interval in $R$ containing the
origin. Let $V_i$ be the tangent vector field of $\alpha$, $D$ be the Levi-Civita connection on $R^3_1$ and $D_{V_i}V_1$ be a spacelike vector. If $V_1$ moves along $\alpha$, then we obtain a spacelike ruled surface which is given by the parametrization

$$M : \varphi(t, v) = \alpha(t) + vV_1(t).$$

(1)

$\{V_1, V_2, V_3\}$ is an Frenet frame field along $\alpha$ in $R^3_1$, where $V_1$ and $V_2$ are spacelike vectors and $V_3$ is a timelike vector, (see [2]). If $k_1$ and $k_2$ are the naturel curvature and torsion of $\alpha(t)$, respectively, then the Frenet formulas are, (see [4])

$$V'_1 = k_1V_2, \quad V'_2 = -k_1V_1 + k_2V_3, \quad V'_3 = k_2V_2.$$  

(2)

Using $V_1 = \alpha'$ and $V_2 = \frac{\alpha''}{\|\alpha''\|}$, we have $k_1 = \|\alpha''\| > 0$, where "\" means derivate with respect to time $t$, (see [2]).

**Definition 1.** The planes which are corresponding to the subspaces $Sp\{V_1, V_2\}$, $Sp\{V_2, V_3\}$ and $Sp\{V_3, V_1\}$ are called asymptotic plane, polar plane and central plane, respectively, (see [2]).

**Definition 2.** Let $M$ and $M^*$ be two spacelike ruled surfaces in $R^3_1$; and $p_1, p_2$ and $p_3$ be the distances between the polar planes, central planes and asymptotic planes, respectively.

If

**i)** The generator vectors of $M$ and $M^*$ are parallel,

**ii)** The distances $p_i$, $1 \leq i \leq 3$, at the corresponding points of $\alpha$ and $\alpha^*$ are constant, then the pair of ruled surfaces $M$ and $M^*$ are called the spacelike parallel $p_i$-equidistant ruled surfaces in $R^3_1$. If $p_i = 0$, then the pair of $M$ and $M^*$ are called the spacelike parallel $p_i$-equivalent ruled surfaces in $R^3_1$.

From the definition 2, the spacelike parallel $p_i$-equidistant ruled surfaces have the following parametric representations, (see [2]).

$$M : \varphi(t, v) = \alpha(t) + vV_1(t), \quad (t, v) \in I \times R,$$

$$M^* : \varphi^*(t^*, v^*) = \alpha^*(t^*) + v^*V_1(t^*), \quad (t^*, v^*) \in I \times R.$$ 

where, $t$ and $t^*$ are the arc parameters of curves $\alpha$ and $\alpha^*$, respectively. From now on $M$ and $M^*$ will be assumed the spacelike parallel $p_i$-equidistant ruled surfaces.

**Theorem 1.** 

**i)** The Frenet frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ are equivalent at the corresponding points in $M$ and $M^*$, respectively. (For $\frac{dt}{dt^*} > 0$.)

**ii)** If $k_1$ and $k_1^*$ are the naturel curvatures and $k_2, k_2^*$ are the torsions of base curves of $M$ and $M^*$, respectively, then we have, (see [2]).

$$k_i^* = k_i \frac{dt}{dt^*}, \quad 1 \leq i \leq 2.$$ 

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3. The Osculator Lorentz Spheres of Spacelike Parallel $p_i$-Equidistant Ruled Surfaces

In this Section, we will investigate radii and curvature axes of osculator Lorentz spheres of the spacelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$. We compute the locus of center of the osculator sphere $S^1_2$ which is fourth order contact with the base curve $\alpha$ of $M$. Let us consider the function $f$ defined by

$$f: I \rightarrow \mathbb{R},$$

where $a$ and $R$ are the center and radius of $S^1_2$, respectively. Since $S^1_2$ is fourth order contact with the curve $\alpha$, we get

$$f(t) = f'(t) = f''(t) = f'''(t) = 0.$$

From $f(t) = 0$ we have

$$\langle \alpha(t) - a, \alpha(t) - a \rangle = R^2,$$

Then $f'(t) = 0$, we obtain

$$\langle V_1(t), \alpha(t) - a \rangle = 0,$$

Using $f''(t) = 0$ and equation (2) we get

$$\langle V_2(t), \alpha(t) - a \rangle = -\frac{1}{k_1(t)}.$$ (6)

For the vector $\alpha(t) - a$, we can write

$$\alpha(t) - a = m_1(t)V_1(t) + m_2(t)V_2(t) + m_3(t)V_3(t), \quad m_i(t) \in \mathbb{R},$$

where $\{V_1, V_2, V_3\}$ is the Frenet frame field of $M$. From equation (7), we obtain

$$\langle \alpha(t) - a, V_1(t) \rangle = m_1(t), \langle \alpha(t) - a, V_2(t) \rangle = m_2(t), \langle \alpha(t) - a, V_3(t) \rangle = -m_3(t).$$ (8)

From equations (5) and (6), we get

$$m_1(t) = 0, \quad m_2(t) = -\frac{1}{k_1(t)}.$$ (9)

Using equations (4), (7) and (9) we find
\[ R = \sqrt{m_2^2 - m_3^2} \quad (10) \]

or
\[ m_3 = \pm \sqrt{m_2^2 - R^2}. \quad (11) \]

Substituting equation (9) to equation (7), we have the center \( a \) of \( S_1^2 \) as follows
\[ a = \alpha(t) + \frac{1}{k_1} V_2(t) - \lambda V_3(t), \quad \lambda = m_3(t) \in R. \quad (12) \]

Using \( f'''(t) = 0 \)
\[ k_1' \langle V_2(t), \alpha(t) - a \rangle + k_1 \langle V'_2(t), \alpha(t) - a \rangle + k_1 \langle V_2(t), V_1(t) \rangle = 0 \]

is obtained. Hence, from (2), (8) and (9) we get
\[ m_3 = - \frac{k_1'}{k_1^2 k_2} = - \frac{m'_2}{k_2}. \quad (13) \]

Similarly, we find the locus of center of osculator sphere \( S^{*2} \) which is fourth order contact with the base curve \( \alpha^* \) of \( M^* \). Let us consider the function \( f^* \) defined by
\[ f^* : I \to R \]
\[ t^* \to f^* (t^*) = \langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^* \rangle - R^{*2}, \quad (14) \]

where \( a^* \) and \( R^* \) are the center and the radius of \( S^{*2} \). Since \( S^{*2} \) is fourth order contact with the curve \( \alpha^* \), we can write
\[ f^*(t^*) = f''^*(t^*) = f'''^*(t^*) = f''''^*(t^*) = 0. \]

From \( f^*(t^*) = f''^*(t^*) = f'''^*(t^*) = 0 \) and (2), we get
\[ \langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^* \rangle = R^{*2}, \quad (15) \]
\[ \langle V_1^*(t^*), \alpha^*(t^*) - a^* \rangle = 0, \quad (16) \]
\[ \langle V_2^*(t^*), \alpha^*(t^*) - a^* \rangle = -\frac{1}{k_1^*(t^*)}. \quad (17) \]

Furthermore, for the vector \( \alpha^*(t^*) - a^* \),
\[ \alpha^*(t^*) - a^* = m_1^*(t^*)V_1^*(t^*) + m_2^*(t^*)V_2^*(t^*) + m_3^*(t^*)V_3^*(t^*), \quad m_i^*(t^*) \in R, \quad (18) \]
can be written, where \( \{V_1^*, V_2^*, V_3^*\} \) is Frenet frame field of \( M^* \). Using (18), we find
\[
\begin{align*}
\langle \alpha^*(t^*) - a^*, V_1^*(t^*) \rangle &= m_1^*(t^*), \\
\langle \alpha^*(t^*) - a^*, V_2^*(t^*) \rangle &= m_2^*(t^*), \\
\langle \alpha^*(t^*) - a^*, V_3^*(t^*) \rangle &= -m_3^*(t^*).
\end{align*}
\]
(19)

Considering equations (16) and (17), we have
\[
m_1^*(t^*) = 0, \quad m_2^*(t^*) = -\frac{1}{k_1^*(t^*)}.
\]
(20)

From (15), (18) and (20), we get
\[
R^* = \sqrt{m_2^{*2} - m_3^{*2}}
\]
(21)
or
\[
m_3^* = \pm \sqrt{m_2^{*2} - R^*2}.
\]
(22)

Using (18), for the center \( a^* \) of \( S_1^{*2} \), we can write
\[
a^* = \alpha^*(t^*) + \frac{1}{k_1^*}V_2^*(t^*) - \lambda^*V_3^*(t^*), \quad \lambda^* = m_3^*(t^*) \in R.
\]
(23)

Then \( f^{*''''}(t^*) = 0 \) we find
\[
k_1^* \langle V_2^*(t^*), \alpha^*(t^*) - a^* \rangle + k_1^* \left( V_2^*(t^*), \alpha^*(t^*) - a^* \right) + k_1^* \langle V_2^*(t^*), V_1^*(t^*) \rangle = 0.
\]

Thus from (2), (19) and (20), we have
\[
m_3^* = \frac{-k_1^*}{k_1^*k_2^*} = -\frac{m_2^*}{k_2^*}.
\]
(24)

Now, let us find the relations between the radii of osculator Lorentz spheres and curvature axes of the base curves of \( M \) and \( M^* \):

Using Theorem 1, (ii) equations (9) and (20) we obtain
\[
m_1^*(t^*) = m_1(t) = 0, \quad m_2^*(t^*) = \frac{dt^*}{dt}m_2(t).
\]
(25)

If \( \frac{dt^*}{dt} \) is constant, then considering the Theorem 1, (ii), we get
\[
k_1^* = k_1^*\left( \frac{dt}{dt^*} \right)^2.
\]
(26)
So, from equations (24), (26) and (13), we have

$$m_3^* = \frac{dt^*}{dt} m_3.$$  \hspace{1cm} (27)

Combining (7), (18), (25), (27) and Theorem 1, we find

$$\alpha^* - a^* = \frac{dt^*}{dt} (\alpha - a).$$  \hspace{1cm} (28)

Similarly, thinking (10), (21), (25) and (27), we obtain

$$R^* = \left| \frac{dt^*}{dt} \right| R.$$  \hspace{1cm} (29)

Hence, we can give the following theorem without proof:

**Theorem 2.**

**i)** If $q_{\alpha}$ and $q_{\alpha^*}$ are the curvature axes (the locus of center of osculator Lorentz spheres) of the base curves $\alpha$ and $\alpha^*$ of $M$ and $M^*$, then we have

$$q_{\alpha^*} - a^* = \frac{dt^*}{dt} (q_{\alpha} - \alpha).$$

**ii)** If $R$ and $R^*$ are the radii of osculator Lorentz spheres of base curves $\alpha$ and $\alpha^*$ of $M$ and $M^*$, then we get

$$R^* = \left| \frac{dt^*}{dt} \right| R.$$
From the Frenet formulas and Theorem 1, (ii), we obtain

\[ S_{V_i} = \int \| V_i' \| dt \quad \text{and} \quad S_{V_i^*} = \int \| V_i'^* \| dt^*, \quad 1 \leq i \leq 3. \]

Similarly, for the arc lengths \( S_\alpha \) and \( S_{\alpha^*} \) of the indicatrix curves \((\alpha)\) and \((\alpha^*)\) generated by the spacelike curves \( \alpha \) and \( \alpha^* \) on the pseudosphere \( S^2_1 \), we find \( S_\alpha = \int \| \alpha' \| dt = \int dt \) and \( S_{\alpha^*} = \int \| \alpha'^* \| dt^* = \int dt^* \), respectively. If \( \frac{k_1}{k_1^*} \) is constant, using Theorem 1, (ii), we get

\[ S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha. \]

Thus, we can give the following theorems without proofs:

**Theorem 3.** If \( S_{V_i} \) and \( S_{V_i^*} \), \( 1 \leq i \leq 3 \), are the arc lengths of indicatrix curves of Frenet vectors \( V_i \) and \( V_i^* \) of base curves \( \alpha \) and \( \alpha^* \) of \( M \) and \( M^* \), respectively, then we have

\[ S_{V_i^*} = S_{V_i}, \quad 1 \leq i \leq 3. \]

**Theorem 4.** Let \( S_\alpha \) and \( S_{\alpha^*} \) be the arc lengths of indicatrix curves of base curves \( \alpha \) and \( \alpha^* \) of \( M \) and \( M^* \), respectively. If \( \frac{k_1}{k_1^*} \) is constant, then we get \( S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha. \)

**References**


Melek Masal
Department of Elementary Education, Faculty of Education
Sakarya University,
Hendek, Sakarya, Turkey
email: mmasal@sakarya.edu.tr

Nuri Kuruoğlu
Faculty of Arts and Sciences,
Department of Mathematics and Computer Sciences,
Bahcesehir University,
Istanbul, Turkey
email: kuruoglu@bahcesehir.edu.tr