INTEGRALS INVOLVING $\tilde{H}$ - FUNCTION

D. KUMAR SINGH, O. MISHRA

Abstract. The present paper deals with various integral formulas involving $\tilde{H}$-function due to Inayat-Hussain multiplied with algebraic functions and special functions.

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1. Introduction

In an attempt to evaluate certain Feynman integral in two different ways which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions. Inayat-Hussain ([4], p.4126) introduced a generalization of Fox’s $H$-function in the form:

$$\tilde{H}(z) = H^{m,n}_{p,q} = \hat{H}^{m,n}_{p,q} \left[ z \left( \frac{(\alpha_j, A_j; a_j)_1, n, (\alpha_j, A_j)}{(\beta_j, B_j)_1, m, (\beta_j, B_j; b_j)} \right)_{m+1, q} \right] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \psi(s) z^s ds,$$

where

$$\psi(s) = \frac{\prod_{j=1}^{m} \Gamma (\beta_j - B_j s) \prod_{j=1}^{n} \Gamma (1 - \alpha_j + A_j s)^{a_j}}{\prod_{j=m+1}^{p} \Gamma (1 - \beta_j + B_j s)^{b_j} \prod_{j=n+1}^{q} \Gamma (\alpha_j - A_j s)}$$

which contains fractional powers of some of the $\Gamma$-functions. Here $z$ may be real or complex but is not equal to zero and an empty product is interpreted as unity; $p$, $q$, $m$, and $n$ are integers such that $1 \leq m \leq q, 0 \leq n \leq p; A_j > 0 (j = 1, ..., p), B_j > 0 (j = 1, ..., q)$ and $a_j (j = 1, ..., n)$ and $b_j (j = m + 1, ..., q)$ can be taken on non-integer values. The poles of the integrand of (1) are assumed to be simple. The contour in (1) is presumed to be the imaginary axis $\Re(s) = 0$, which is suitable indented in order to avoid the singularities of the gamma functions and to keep these singularities at appropriate sides. It has been shown by Buschman and Srivastava ([13], p. 61).
that the sufficient condition for absolute convergence of the contour integral (1) is given by
\[
\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_jA_j| - \sum_{j=m+1}^{q} |b_jB_j| - \sum_{j=n+1}^{p} |A_j| > 0. \tag{3}
\]

This condition provides exponential decay of the integrand in (1), and region of absolute convergence of (1) is
\[
|\arg z| < \frac{1}{2}\pi \Omega. \tag{4}
\]

For further details about the $\overline{H}$ function the reader is referred to the original paper of Bushman and Srivastava [13] and Inayat-Hussain [4]. When the exponents $a_j = b_j = 1$, ∀i,j, then $\overline{H}$ function reduces to the familiar Fox's H-function defined by Fox [8]; see also Mathai and Saxena [6].
\[
\overline{H}^{m,n}_{p,q}(z) = H^{m,n}_{p,q} \left[ z \left( \alpha_1, A_1 \right), \ldots, \left( \alpha_p, A_p \right) \right] = \frac{1}{2\pi i} \int_{L} \chi(s) z^s ds, \tag{5}
\]
where
\[
\chi(s) = \prod_{j=1}^{m} \Gamma (\beta_j - B_j s) \prod_{j=1}^{n} \Gamma (1 - \alpha_j + A_j s) \prod_{j=m+1}^{q} \Gamma (1 - \beta_j + B_j s) \prod_{j=n+1}^{p} \Gamma (\alpha_j - A_j s) \tag{6}
\]
an empty product is interpreted as unity; the integer m, n, p, q satisfy the inequalities $0 \leq n \leq p$ and $1 \leq m \leq q$, the coefficients $A_j > 0$($j = 1, \ldots, p$) and $B_j > 0$($j = 1, \ldots, q$) and the complex parameters $\alpha_j$ and $\beta_j$ are such that the poles of the integrand are simple and $L$ is a suitable contour of Mellin-Barnes type in complex s-plane separating the poles of $\Gamma (1 - \alpha_j + A_j s)$ for $j = 1, \ldots, n$. The integral in (1) converge absolutely and defines the H-function, analytic in the sector
\[
|\arg z| < \frac{1}{2}\pi \lambda^*, \tag{7}
\]
where
\[
\lambda^* = \sum_{j=1}^{m} B_j - \sum_{j=1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0 \tag{8}
\]
the point $z = 0$ being tacitly excluded.

A detailed account of the $H$ function is available from the monograph of Mathai and Saxena [6]. Existence conditions, analytic continuation and asymptotic expansions of the $H$ function have been discussed by Braaksma [7].
A relation connecting $L^\nu(z)$, the polylogarithm of complex order $\nu$ and the $\bar{H}$-function is the following:

$$L^\nu(z) = H_{1,2\nu-1}^{1,1\nu} \left[-z\right]_{(0,1),(0,1: \nu-1)},$$ (9)

which readily follows on comparing their contour integral definitions. An account of $L^\nu(z)$, the polylogarithm of complex order $\nu$ is available from the book by Marichev [12].

Existence condition for the $\bar{H}$-function can be established by following the procedure adopted by Braaksma ([7], pp. 278-279), that the function $\bar{H}(z)$ makes sense and defines an analytic function of $z$ in the following two cases.

I. $\mu > 0$, and $0 < |z| < \infty$ where

$$\mu = \sum_{j=1}^{m} |B_j| + \sum_{j=m+1}^{q} |B_j\beta_j| = \sum_{j=1}^{n} |A_j\alpha_j| - \sum_{j=n+1}^{p} |A_j|.$$ (10)

II. $\mu = 0$ and $0 < |z| < \tau^{-1}$ holds.

$$\tau = \left\{ \prod_{j=1}^{m} (B_j)^{-B_j} \right\} \left\{ \prod_{j=1}^{n} (A_j)^{A_j} \right\} \left\{ \prod_{j=n+1}^{p} (A_j) \right\} \left\{ \prod_{j=n+1}^{q} (B_j)^{-B_j} \right\}.$$ (11)

By calculating the residue at the poles of $\Gamma(\beta_j - B_j s)$ for $j = 1, ..., m$ in (1) we obtain the following representation of the $\bar{H}$-function in a computable from as

$$\bar{H}(z) = \bar{H}_{p,q}^{m,n}[z] = \sum_{h=1}^{\infty} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \psi(\zeta)z^\zeta}{\nu!B_h},$$ (12)

where $\zeta = \frac{(\beta_h + \nu)}{B_h}$ (3) exist for $0 < |z| < \infty$, if $\mu < 0$ or $\mu = 0$ and $0 < |z| < \tau^{-1}$, where $\tau^{-1}$ is defined in (11), $\mu$ in (10), $\psi(.)$ in (2); $B_h(\beta_j + v_1) \neq B_j(\beta_h + v_2)$ for $j \neq h$, $h = 1, ..., m; v_1, v_2 = 0, 1, 2, ...$.

The behaviour of the $\bar{H}$-function for small value of $|z|$ follows easily from a result given by Rathai [5]

$$\bar{H}_{p,q}^{m,n}[z] = O(|z|^\alpha)$$ (13)

$$\alpha = \min_{0 \leq j \leq M} \Re \left( \frac{b_j}{\beta_j} \right), |z| \to 0.$$ (14)

Rathie [5, pg. 303, Eq. 5.4] is also represent $\bar{H}$ in the following form

$$\bar{H}_{p,q}^{m,n}[z] = \bar{H}_{p,q}^{m,n} \left[ \left( a_1, \alpha_1, A_1 \right) ... \left( a_n, \alpha_n, A_n \right), \left( a_{n+1}, \alpha_{n+1}, A_{n+1} \right), \left( a_p, \alpha_p, 1 \right), \left( b_1, \beta_1, 1 \right) ... \left( b_m, \beta_m, 1 \right), \left( b_{m+1}, \beta_{m+1}, B_{m+1} \right) ... \left( b_q, \beta_q, B_q \right) \right]$$ (15)
In this section we will be calculate the $\bar{H}_{p,q}^{m,n}$ function. Here $G_{p,q}^{m,n}$ in the G-function, see Luke[16].

2. Integral with Algebraic Function

In this section we will be calculate the $\bar{H}$ function with some algebraic function.

$$I_1 = \int_0^1 y^{-\rho}(1-y)^{\rho-\sigma-1}\bar{H}_{p,q}^{m,n}[zy]dy$$

$$= \frac{1}{2\pi i} \int_L \psi(s)z^s\left\{ \int_0^1 y^{-s+1} - 1 \right\} dy ds$$

$$= \frac{1}{2\pi i} \int_L \psi(s)z^s\{B(1 - s, \rho - \sigma)\} ds$$

$$= \frac{1}{2\pi i} \int_L \prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^p \{\Gamma(1 - a_j + A_j s)\}^{\alpha_j} \Gamma(1 - s + \sigma) z^s ds$$

$$= \Gamma(\rho - \sigma) \bar{H}_{p+1,q+1}^{m,n+1}\left[ z \left| \begin{array}{c} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1 - \rho, 1; 1) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (\sigma, 1; 1) \end{array} \right. \right]. \quad (18)$$

$$I_2 = \int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\bar{H}_{p,q}^{m,n}(zx)dx$$

$$= \frac{1}{2\pi i} \int_L \psi(s)z^s\left\{ \int_0^1 x^{s+1} - 1 \right\} dy ds$$

$$= \frac{1}{2\pi i} \int_L \prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^p \{\Gamma(1 - a_j + A_j s)\}^{\alpha_j} \Gamma(1 + s + \rho) z^s ds$$

$$= \Gamma(\rho) \bar{H}_{p+1,q+1}^{m,n+1}\left[ z \left| \begin{array}{c} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1 - \rho, 1; 1) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (\rho, 1; 1) \end{array} \right. \right]. \quad (19)$$

$$I_3 = \int_0^\infty x^{-\rho}(x - 1)^{\sigma-1}\bar{H}_{p,q}^{m,n}(zx)dx$$

$$= \frac{1}{2\pi i} \int_L \psi(s)z^s\left\{ \int_0^\infty x^{s+1} - 1 \right\} dy ds.$$
Putting $x = t + 1$ and $dx = dt$ we get

$$=rac{1}{2\pi i} \int_L \psi(s) z^s \left\{ \int_0^\infty t^{\sigma-1}(t+1)^{-\rho-s-\sigma} dt \right\} ds$$

$$=rac{1}{2\pi i} \int_L \prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{i=1}^n \{\Gamma(1 - a_j + A_j s)\}^{\alpha_i} \Gamma(\rho) \Gamma(\rho - s - \sigma) \frac{\Gamma(\rho - s)}{\Gamma(\rho)} z^s ds$$

$$= \Gamma(\sigma) \bar{H}^{m+1,n}_{\rho+1,q+1} \left[ \int \frac{(a_j, A_j, \alpha_j)_{1,n} \cdot (a_j, A_j)_{n+1,p}, (\rho, 1)}{(b_j, B_j)_{1,m} \cdot (b_j, B_j; \beta_j)_{m+1,q}, (\rho - s, 1)} \right]. \quad (20)$$

$$I_4 = \int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma} \bar{H}^{m,n}_{p,q}(zx) dx$$

$$= \frac{1}{2\pi i} \int_L \psi(s) z^s \beta^{-\sigma} \left\{ \int_0^\infty x^{\rho+s-1}\left(\frac{x}{\beta} + 1\right)^{-\sigma} \right\} ds.$$

Putting $x = t\beta$ and $dx = \beta dt$ we get

$$=rac{1}{2\pi i} \beta^{-\sigma} \int_L \psi(s)(z\beta)^s \left\{ \int_0^\infty t^{\rho+s-1}(t+1)^{-\rho+s-\rho-s} dt \right\} ds$$

$$=rac{1}{2\pi i} \beta^{-\sigma} \int_L \prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{i=1}^n \{\Gamma(1 - a_j + A_j s)\}^{\alpha_i} \Gamma(\rho) \Gamma(\rho - s - \sigma) \frac{\Gamma(\rho - s)}{\Gamma(\rho)} z^s ds.$$
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\[
\times \frac{\Gamma(1+\rho+\mu s)}{\Gamma(2+\rho+\mu s+\sigma)} (2^\mu z)^s ds
\]

\[
= 2^{\rho+\sigma+1} \Gamma(1+\sigma) \overline{H}_{p+1,q+1}^{m,n+1} \left[ 2^\mu z \right] \left( a_j, A_j; \alpha_j \right)_{1,n+1} \left( b_j, B_j; \beta_j \right)_{m+1,q+1} \left( -\rho, \mu; 1 \right)
\]

(24)

3. Integral with Jacobi Polynomials

The Jacobi polynomials \( P_{n}^{(\alpha,\beta)}(x) \) ([9], p. 254) may be defined by

\[
P_{n}^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[ \begin{array}{c} -n, 1+\alpha+\beta+n; \\ \frac{1-x}{2} \end{array} ; 1+\alpha \right]
\]

when \( \alpha = \beta = 0 \). The polynomial in (25) becomes the Legendre polynomial ([9], p. 157).

From (25) it follows that \( P_{n}^{(\alpha,\beta)}(x) \) is a polynomial degree precisely \( n \) and that

\[
P_{n}^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}.
\]

In dealing with the Jacobi polynomial it is natural to make much use of our knowledge of the \( {}_2F_1 \) function ([9], p. 45).

\[
I_6 = \int_{-1}^{+1} x^\lambda (1-x)^\alpha (1+x)^\beta P_{n}^{(\alpha,\beta)}(x) \overline{H}_{p,q}^{m,n} \left( z (1+x)^h \right) dx
\]

\[
= \frac{1}{2\pi i} \int_L \Psi(s) z^s \left\{ \int_{-1}^{+1} x^\lambda (1-x)^\alpha (1+x)^\delta+hs P_{n}^{(\alpha,\beta)}(x) dx \right\} ds.
\]

But by the formula ([14], p. 52)

\[
\int_{-1}^{1} x^\lambda (1-x)^\alpha (1+x)^\beta P_{n}^{(\alpha,\beta)}(x) dx
\]

\[
= (-1)^n 2^{\alpha+\beta+1} \Gamma(\delta+1) \Gamma(n+\alpha+1) \Gamma(\delta+\beta+1) \Gamma(\delta+\alpha+n+2)
\]

\[\times {}_3F_2 \left[ \begin{array}{c} -\lambda, \delta+\beta+1, \delta+1; \\ \delta+\beta+n+1, \delta+\alpha+n+2 \end{array} ; 1 \right].\]

Provided: \( \alpha > -1 \) and \( \beta > -1 \). We have

\[
= \frac{1}{2\pi i} \int_L \Psi(s) z^s (-1)^n 2^{\alpha+\beta+hs+1}
\]
Provided:

(i) \( \text{Re}(\lambda) > -1 \) and \( |\arg z| < \frac{1}{2}\pi\Omega \)

(ii) \( \alpha > -1, \beta > -1 \).

\[
I_7 = \int_{-1}^{+1} (1 - x)^\delta (1 + x)^\nu P_n^{(\mu,\nu)}(x) P_m^{(\rho,\sigma)}(x) \mathcal{H}_{p,q}^{m,n} \left( z (1 - x)^h \right) dx
\]
\[
= \frac{1}{2\pi i} \int_L \psi(s) z^s \left\{ \int_{-1}^{+1} (1 - x)^{\delta + h s} (1 + x)^\nu P_n^{(\mu,\nu)}(x) P_m^{(\rho,\sigma)}(x) dx \right\} ds.
\]

Now using the definition Jacboi polynomial (25)

\[
= \frac{(1 + \rho) m}{m!} \sum_{k=0}^{\infty} \frac{(-m)_k (1 + \rho + \sigma + m)_k}{(1 + \rho)_k 2^k k!}
\]
\[
\times \frac{1}{2\pi i} \int_L \psi(s) z^s \left\{ \int_{-1}^{+1} (1 - x)^{\delta + h s + k} (1 + x)^\nu P_n^{(\mu,\nu)}(x) dx \right\} ds.
\]

(27)

Again using (25) in (27) we get

\[
= \frac{\Gamma (1 + \rho + m) \Gamma (1 + \mu + n)}{\Gamma (1 + \mu) m! n!}
\]
By using the formula (23), equation (28) becomes

\[
\begin{align*}
\times \sum_{k=0}^{\infty} \frac{(-m)_k (-n)_k (1 + \rho + \sigma + m)_k (1 + \mu + \nu + n)_k}{2^{2k} (k!)^2 \Gamma (1 + \rho + k) \Gamma (1 + \mu + k)} \\
\times \frac{1}{2\pi i} \int_L \psi(s) 2^h z^s \left\{ \int_{-1}^{+1} (1 - x)^{\delta + hs + 2k} (1 + x)^\nu dx \right\} ds. \quad (28)
\end{align*}
\]

Provided: \( \text{Re}(\nu) > -1 , \ | \text{arg} \ z | < \frac{1}{2} \pi \Omega \) and \( \delta \) are positive.

\[
\begin{align*}
I_8 = \int_{-1}^{+1} (1 - x)^\rho (1 + x)^{\sigma} P_n^{(\mu, \nu)}(x) H_p^{m,n}(z (1 - x)^h (1 + x)^t) dx \\
= \frac{1}{2\pi i} \int_L \psi(s) z^s \left\{ \int_{-1}^{+1} (1 - x)^\rho (1 + x)^{\sigma} P_n^{(\mu, \nu)}(x) (1 - x)^{hs} (1 + x)^{ts} dx \right\} ds. \\
= \frac{1}{2\pi i} \int_L \psi(s) z^s \left\{ \int_{-1}^{+1} (1 - x)^{\rho + hs} (1 + x)^{\sigma + ts} (1 + \mu)_{n+1} \frac{1}{n!} \frac{1}{2\pi i} \int_L \psi(s) z^s \right\} ds.
\end{align*}
\]

\[
\begin{align*}
= \frac{(1 + \mu)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \mu + \nu + n)_k}{(1 + \mu)_k k! 2^k} \frac{1}{2\pi i} \int_L \psi(s) z^s \left\{ \int_{-1}^{+1} (1 - x)^{\rho + hs + k - 1 + 1} (1 + x)^{\sigma + ts - 1 + 1} dx \right\} ds. \quad (30)
\end{align*}
\]
Now using (23) in (30) we get

\[\frac{1}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \mu + \nu + n)_k}{(1 + \mu)_k k!} \]

\[\times \frac{1}{2\pi i} \int_L \prod_{j=m+1}^{n} \Gamma (b_j - B_j s) \prod_{j=1}^{p} \Gamma (1 - a_j + A_j s)^{\alpha_j} \prod_{j=m+1}^{n} \Gamma (a_j - A_j s) \]

\[\times \frac{\Gamma (1 + k + \rho + hs) \Gamma (1 + \sigma + ts)}{\Gamma (k + \rho + hs + \sigma + ts + 2)} \left(2^{h+1} \right)^s ds.\]

\[= \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \mu + \nu + n)_k}{k! (1 + \mu)_k} \]

\[\times \tilde{H}_{p+2,q+1}^{m,n+2} \left[ \phi^{h+1} \right] \]

\[= (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1}, (-\rho - k, h; 1), (-\sigma, t; 1) \]

\[= (b_j, B_j)_1, (b_j, B_j)_{n+1} \] (31)

Provided: \(|\arg z| < \frac{1}{2} \pi \Omega\) and \(\Re(\mu) > -1\), and \(\Re(\nu) > -1\).

\[I_9 = \int_{-1}^{+1} (1 - x)^{\rho} (1 + x)^{\sigma} P_\omega (x) \tilde{H}_{p,q}^{m,n} \left( z (1 + x)^{-h} \right) dx \]

\[= \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \alpha + \beta + n)_k}{(1 + \alpha)_k k! 2^k} \frac{1}{2\pi i} \int_L \psi(s) z^s \]

\[\times \left\{ \int_{-1}^{1} (1 - x)^{\rho+k-1+1} (1 + x)^{\sigma-hs-1+1} dx \right\} ds. \quad (32)\]

Now using (23) in (31) we get

\[= \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \alpha + \beta + n)_k}{k! (1 + \alpha)_k} \]

\[\times \frac{1}{2\pi i} \int_L \prod_{j=1}^{n} \Gamma (b_j - B_j s) \prod_{j=1}^{n} \Gamma (1 - a_j + A_j s)^{\alpha_j} \prod_{j=m+1}^{n} \Gamma (a_j - A_j s) \]

\[\times \frac{\Gamma (1 + \sigma - hs) \Gamma (1 + \rho + k)}{\Gamma (2 + \rho + k + \sigma - hs)} \left(2^{-h} z^s \right) ds.\]

\[= \frac{2^{\rho+1} (1 + \alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \alpha + \beta + n)_k \Gamma (1 + \rho + k)}{k! (1 + \alpha)_k} \]

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Using (23) in (34), we get

\[ \times \tilde{H}^{m+1,n}_{p+1,q+1} \left[ 2^{-h} |z|^{(a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1 + \rho, h) \times} \right. \\
\left. (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (2 + \rho + k + \sigma, h) \right]. \quad (33) \]

Provided:

(i) $| \arg z | < \frac{1}{2} \pi \Omega$ and $\Re(\alpha) > -1$, and $\Re(\beta) > -1$

(ii) $\Re \left[ \rho + h \min \left( \frac{b_j}{\beta_j} \right) \right] > -1$ where $j = 1, 2, 3, \ldots, m.$

\[ I_{10} = \int_{-1}^{1} (1 - x)^{\rho} (1 + x)^{\sigma} P_n^{(\mu, \nu)} (x) \tilde{H}^{m,n}_{p,q} \left( z (1 - x)^h (1 + x)^{-t} \right) dx. \]

\[ = \frac{1}{2 \pi i} \int_L \psi(s) z^s \left\{ \int_{-1}^{1} (1 - x)^{\rho + hs} (1 + x)^{\sigma - ts} \frac{(1 + \mu)^n}{n!} {}_2 F_1 \left[ -n, 1 + \mu + \nu + n; \frac{1 - x}{2} \right] dx ds. \right\} \]  

\[ \times \frac{1}{2 \pi i} \int_L \psi(s) z^s \left\{ \int_{-1}^{1} (1 - x)^{\rho + hs + k} (1 + x)^{\sigma - ts} dx ds. \right\} \quad (34) \]

Using (33) in (34), we get

\[ = \frac{(1 + \mu)^n}{n!} 2^{\rho + \sigma + 1} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \mu + \nu + n)_k}{(1 + \mu)_k k!} \]

\[ \times \frac{1}{2 \pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma (b_j - B_j s) \prod_{j=1}^{n} \Gamma (1 - a_j + A_j s)^{\alpha_j}}{\prod_{j=m+1}^{p} \Gamma (1 - b_j + B_j s)^{\beta_j} \prod_{j=n+1}^{q} \Gamma (a_j - A_j s)} \]

\[ \times \frac{\Gamma (1 + \rho + k + hs) \Gamma (1 + \sigma - ts)}{\Gamma (k + \rho + hs + \sigma - ts + 2)} \left( 2^{h-t} z \right)^{s} ds. \]

\[ = 2^{\rho + \sigma + 1} \frac{(1 + \mu)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \mu + \nu + n)_k}{k! (1 + \mu)_k} \]

\[ \times \tilde{H}^{m+1,n+1}_{p+1,q+2} \left[ 2^{h-t} |z|^{(a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (-\rho - k; h, 1)} \times \right. \\
\left. (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1 + \sigma, t) (-1 + \rho - k - \sigma, h - t; 1) \right]. \quad (35) \]

Provided:

(i) $| \arg z | < \frac{1}{2} \pi \Omega$

(ii) $\Re \left[ \rho + h \min \left( \frac{b_j}{\beta_j} \right) \right] > -1$ and $\Re \left[ \sigma + t \min \left( \frac{b_j}{\beta_j} \right) \right] > -1$ where $j = 1, 2, 3, \ldots, m.$
4. Special Cases

(i) If we replace \( \delta \) by \( \lambda - 1 \) and put \( \mu = \nu = \rho = \sigma = 0 \) the integral \( I_7 \) transform to the following integral involving product of Legendre Polynomials

\[
I_{11} = 2^3 \sum_{k=0}^{\infty} \frac{(-m)_k (-n)_k (1 + m)_k (1 + n)_k}{(kl)^2 \Gamma (1 + k)} \int_0^1 \bar{H}_{p+1,q+1}^{m,n+1} \left[ 2^h z \right] \left( a_j, A_j, \alpha_j \right)_{1,p}, (-\lambda + 1 - 2k, h; 1) \left( b_j, B_j, \beta_j \right)_{1,q}, (-\lambda - 2k, h; 1) \],
\]

(36)

Provided: \(| \arg z | < \frac{1}{2} \pi \Omega \).

(ii) If \( \mu = \nu = 0, \rho \) is replaced by \( \rho - 1 \) and \( \sigma \) by \( \sigma - 1 \) then the integral \( I_8 \) transforms into the following integral involving Legendre polynomials

\[
I_{12} = \int_{-1}^{+1} (1 - x)^{\rho - 1} (1 + x)^{\sigma - 1} P_n(x) \bar{H}_{p,q}^{m,n} \left( z (1 - x)^h (1 + x)^{t} \right) dx
= 2^{\rho + \sigma - 1} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + n)_k}{k!} \bar{H}_{p+2,q+1}^{m,n+2} \left[ 2^{h+1} z \right] \left( a_j, A_j, \alpha_j \right)_{1,n}, (-\rho + 1 - k, h; 1), (-\sigma + 1, t; 1) \left( b_j, B_j, \beta_j \right)_{1,m}, (1 - \rho - k - \sigma, h + t; 1) \],
\]

(37)

Provided: \(| \arg z | < \frac{1}{2} \pi \Omega \).

(iii) Replacing \( \rho \) by \( \rho - 1 \), \( \sigma \) by \( \sigma - 1 \) and putting \( \mu = \nu = 0 \) the integral \( I_{10} \) takes the forms of the following integral

\[
I_{13} = \int_{-1}^{+1} (1 - x)^{\rho - 1} (1 + x)^{\sigma - 1} P_n(x) \bar{H}_{p,q}^{m,n} \left( z (1 - x)^h (1 + x)^{-t} \right) dx
= 2^{\rho + \sigma - 1} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + n)_k}{(k!)^2} \bar{H}_{p+1,q+2}^{m+1,n+1} \left[ 2^{h-t} z \right] \left( a_j, A_j, \alpha_j \right)_{1,n}, (-\rho + 1 - k, h; 1) \left( b_j, B_j, \beta_j \right)_{1,m}, (1 - \rho - k - \sigma, h - t; 1) \].
\]

(38)

Provided:
\(| \arg z | < \frac{1}{2} \pi \Omega \) \( \Re \left[ \rho + h \min \left( \frac{b_j}{\pi_j} \right) \right] > -1 \) and \( \Re \left[ \sigma + t \min \left( \frac{b_j}{\pi_j} \right) \right] > -1 \) where \( j = 1, 2, 3, \ldots, m \).
5. INTEGRAL WITH BESSEL MAITLAND FUNCTION

The special cases of the Wright function ([2], vol. 3, section 18.1) and ([10], [11]) in the form

\[
\phi(B, b; z) \equiv a \psi_1 \begin{bmatrix} - & (b, B) \mid z \\
\end{bmatrix} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Bk + b)} \frac{z^k}{k!}
\]  

(39)

with complex \( z, b \in \mathbb{C} \) and real \( B \in \mathbb{R} \).

When \( B = \delta, b = \nu + 1 \) and \( z \) is replaced by \(-z\), the function \( \phi(\delta, \nu + 1; -z) \) is defined by

\[
J^\delta_\nu(z) \equiv \phi(\delta, \nu + 1 : z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k + \nu + 1)} \frac{(-z)^k}{k!}
\]  

(40)

and such a function is known as the Bessel Maitland function or the Wright generalized Bessel function see ([15], p. 352).

\[
I_{14} = \int_0^{\infty} x^l J^\tau_\nu (\gamma x) dx
\]

\[
= \frac{1}{2\pi i} \int_L \Psi(s) z^s \left\{ \int_0^{\infty} x^{l+\gamma s} J^\tau_\nu(x) dx \right\} ds.
\]

Now using the formula ([14], p. 55)

\[
\int_0^{\infty} x^l J^\tau_\nu(x) dx = \frac{\Gamma(l+1)}{\Gamma(1 + \nu - \tau - \tau l)}
\]

\( \Re(l) > -1, 0 < \tau < 1 \).

\[
= \frac{1}{2\pi i} \int_L \prod_{j=m+1}^{p} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j s) \prod_{j=n+1}^{p} \Gamma(a_j - A_j s)
\]

\[
\times \frac{\Gamma(l + \gamma s + 1)}{\Gamma(1 + \nu - \tau - \tau l - \tau \gamma s)} z^s ds
\]

\[
= \bar{H}^{m,n+1}_{p+2,q} \left[ z \begin{bmatrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (-l, \gamma; 1), (1 + \nu - \tau - \tau l, \tau \gamma) \\
(b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \\
\end{bmatrix} 
\]

(41)

Provided:

(i) \( |\arg z| < \frac{1}{2} \pi \Omega \)

(ii) \( \gamma - \tau \gamma > 0, \gamma > 0 \)

(iii) \( 0 < \tau < 1 \) and \( \Re(l + 1) > 0 \).
The Legendre functions are solution of Legendre’s differential equation ([3], sec. 3.1, vol. 1)

\[(1 - z^2) \frac{d^2 \omega}{dz^2} - 2z \frac{d\omega}{dz} + \left[ \nu (\nu + 1) - \mu^2 (1 - z^2)^{-1} \right] \omega = 0, \tag{42}\]

where \(z, \nu, \mu\) unrestricted.

Under the substitution \(\omega = \left( z^2 - 1 \right) \frac{1}{\nu} \), equation (42) becomes

\[(1 - z^2) \frac{d^2 \nu}{dz^2} - 2 (\mu + 1) z \frac{d\nu}{dz} + (\mu - \nu) (\nu + \mu + 1) \nu = 0, \tag{43}\]

and with \(\xi = \frac{1}{2} - \frac{1}{2}z\) as the independent variable this differential equation becomes

\[\xi (1 - \xi) \frac{d^2 \nu}{d\xi^2} + (\mu + 1) (1 - 2\xi) \frac{d\nu}{d\xi} + (\nu - \mu) (\nu + \mu + 1) \nu = 0. \tag{44}\]

This is the Gauss hypergeometric type equation with \(a = \mu - \nu, b = \nu + \mu + 1\) and \(c = \mu + 1\).

Hence it follows that the function

\[\omega = P_{\nu}^\mu(z) = \frac{1}{\Gamma(1 - \mu)} \left( \frac{z + 1}{z - 1} \right)^{\frac{1}{2}\mu} \cdot F \left[ -\nu, \nu + 1; 1 - \mu; 1 - \frac{1}{2}z \right], |1 - z| < 2\]

is a solution of (42).

The function \(P_{\nu}^\mu(z)\) is known as the Legendre function of first kind ([3], vol. 1). It is one valued and regular in \(z\)-plane supposed cut along the real axis form 1 to \(-\infty\).

\[I_{15} = \int_0^1 x^{\sigma - 1} (1 - x^2)^{\frac{3}{2}} P_{\nu}^\mu(x) H_{m,n}^{p,q}(zx^\gamma) dx = \frac{1}{2\pi i} \int_{L} \psi(s) z^s \left\{ \int_0^1 x^{\sigma - 1 + \gamma s} (1 - x^2)^{\frac{3}{2}} P_{\nu}^\mu(x) dx \right\} ds. \tag{45}\]

Now using the formula ([3], sec 3.12, vol. 1)

\[\int_0^1 x^{\sigma - 1} (1 - x^2)^{\frac{3}{2}} P_{\nu}^\mu(x) dx = \frac{(-1)^\delta \pi^{\frac{1}{2}} 2^{-\sigma - \delta} \Gamma(\sigma) \Gamma(1 + \delta + \nu)}{\Gamma \left( \frac{1}{2} + \frac{\sigma}{2} + \frac{\delta}{2} - \frac{\nu}{2} \right) \Gamma \left( 1 + \frac{\sigma}{2} + \frac{\delta}{2} + \frac{\nu}{2} \right) \Gamma(1 - \delta + \nu)} \tag{46}\]

Provided: \(\Re(\sigma) > 0, \delta = 1, 2, 3, \ldots\).

Now integral (45) becomes

\[= 2^{-\sigma - \delta} (-1)^\delta \left( \pi \right)^{\frac{1}{2}} \frac{\Gamma(1 + \delta + \nu)}{\Gamma(1 - \delta + \nu)}\]
Now using the formula ([3], sec 3.12, vol. 1)

\[
\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma (b_j) \prod_{j=1}^{p} \Gamma (a_j)}{\prod_{j=m+1}^{p} \Gamma (1 - b_j + A_j s)} \frac{1}{\prod_{j=m+1}^{p} \Gamma (1 - a_j + A_j s)} \, ds
\]

\[
\times \frac{\Gamma (\sigma + \gamma s)}{\Gamma \left( \frac{1}{2} + \frac{\sigma + \gamma s}{2} + \frac{\delta - \nu}{2} - \frac{\sigma - \nu}{2} \right) \Gamma \left( 1 + \frac{\sigma + \gamma s}{2} + \frac{\delta - \nu}{2} - \frac{\sigma - \nu}{2} \right)} \, z^{s} 2^{-\gamma s} \, ds
\]

\[
= 2^{-\sigma - \delta} (-1)^{\delta} \frac{1}{2} \frac{\Gamma (1 + \delta + \nu)}{\Gamma (1 - \delta + \nu)}
\]

\[
\times \tilde{H}_{p+1,q+2}^{m,n+1} \left[ \frac{z}{2z} \right]_{(a_j, \beta_j)_{l+1,m}, (a_j, \beta_j)_{n+1,p}, (1 - \sigma, \gamma; 1)} \left( (a_j, \beta_j)_{l+1,m}, (a_j, \beta_j)_{n+1,p}, (1 - \sigma, \gamma; 1) \right)
\]

Provided: \( |\arg(z)| < \frac{1}{4} \pi \Omega, \sigma > 0 \) and \( \delta \) is non negative integer.

\[
I_{16} = \frac{1}{2\pi i} \int_L \psi(s) z^{s} \left\{ \int_{0}^{1} x^{\sigma-1} (1 - x^{2})^{\frac{\sigma - \nu}{2}} P_{\nu}^{\delta}(x) \tilde{H}_{m,n}^{p,q}(zx^{\gamma}) \, dx \right\}
\]

\[
= \frac{1}{2\pi i} \int_L \psi(s) z^{s} \left\{ \int_{0}^{1} x^{\sigma-1 + \gamma s} (1 - x^{2})^{\frac{\sigma - \nu}{2}} P_{\nu}^{\delta}(x) \, dx \right\}
\]

Now using the formula ([3], sec 3.12, vol. 1)

\[
\int_{0}^{1} x^{\sigma-1} (1 - x^{2})^{\frac{\sigma - \nu}{2}} P_{\nu}^{\delta}(x) \, dx = \frac{\pi^{\frac{1}{2}} 2^{-\sigma + \delta} \Gamma (\sigma)}{\Gamma \left( \frac{1}{2} + \sigma - \frac{\sigma - \nu}{2} \right) \Gamma \left( 1 + \sigma - \frac{\sigma - \nu}{2} \right)}
\]

Provided: \( \Re(\sigma) > 0, \delta = 1, 2, 3, \ldots \)

\[
= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma (b_j) \prod_{j=1}^{p} \Gamma (a_j)}{\prod_{j=m+1}^{p} \Gamma (1 - b_j + A_j s)} \frac{1}{\prod_{j=m+1}^{p} \Gamma (1 - a_j + A_j s)} \, ds
\]

\[
\times \frac{2^{-\sigma - \delta} \pi^{\frac{1}{2}} \Gamma (\sigma + \gamma s)}{\Gamma \left( \frac{1}{2} + \frac{\sigma + \gamma s}{2} - \frac{\sigma - \nu}{2} - \frac{\sigma - \nu}{2} \right) \Gamma \left( 1 + \frac{\sigma + \gamma s}{2} - \frac{\sigma - \nu}{2} - \frac{\sigma - \nu}{2} \right)} \, z^{s} 2^{-\gamma s} \, ds
\]

\[
= 2^{-\sigma - \delta} \pi^{\frac{1}{2}}
\]

\[
\times \tilde{H}_{p+1,q+2}^{m,n+1} \left[ \frac{z}{2z} \right]_{(a_j, \beta_j)_{l+1,m}, (a_j, \beta_j)_{n+1,p}, (1 - \sigma, \gamma; 1)} \left( (a_j, \beta_j)_{l+1,m}, (a_j, \beta_j)_{n+1,p}, (1 - \sigma, \gamma; 1) \right)
\]

Provided: \( |\arg(z)| < \frac{1}{4} \pi \Omega, \Re(\sigma) > 0 \) and \( \Re(\delta) > 0 \) is non negative integer.
7. Integral Involving Hypergeometric Function

In the study of second order linear differential equation with three regular singular points, there arise the function

\[ F(a, b; c; z) = 1 + \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \]  

(51)

For \( c \) neither zero nor a negative integer in (51) the notation

\[ (\alpha)_n = \alpha (\alpha + 1) (\alpha + 2) \ldots (\alpha + n - 1), n \geq 0. \]

is called the factorial function and the function in (51) is called the hypergeometric function ([9], p. 45).

\[ I_{17} = \int_{1}^{\infty} x^{-\rho} (x - 1)^{\sigma-1} \text{ } \text{ } \frac{\Gamma (\nu + \sigma - \rho, \lambda + \sigma - \rho)}{\sigma} ; (1 - x) \text{ } \bar{H}^{m,n}_p q(z) \text{ } dx \]

\[ = \frac{1}{2\pi i} \int_{L} \psi(x) z^s \{ \int_{1}^{\infty} x^{-\rho + s} (x - 1)^{\sigma-1} \text{ } \text{ } \frac{\Gamma (\nu + \sigma - \rho, \lambda + \sigma - \rho)}{\sigma} ; (1 - x) \} \text{ } dx \} ds. \]

Putting \( x = t + 1 \) and \( dx = dt \)

\[ = \sum_{k=0}^{\infty} \frac{(\nu + \sigma - \rho)_k (\lambda + \sigma - \rho)_k (-1)^k}{(\sigma)_k k!} \]

\[ \times 1 \frac{1}{2\pi i} \int_{L} \prod_{j=1}^{m} \Gamma (b_j - B_j s) \prod_{j=1}^{n} \Gamma (1 - a_j + A_j s)_{\alpha_j}^j \prod_{j=m+1}^{p} \Gamma (a_j - A_j s)_{\beta_j}^j \prod_{j=n+1}^{q} \Gamma (a_j - A_j s) \]

\[ \times \frac{\Gamma (\sigma + k) \Gamma (\rho - s - k - \sigma)}{\Gamma (\rho - s)} z^s ds \]

\[ = \Gamma (\sigma + k) \sum_{k=0}^{\infty} \frac{(\nu + \sigma - \rho)_k (\lambda + \sigma - \rho)_k (-1)^k}{(\sigma)_k k!} \]

\[ \times \bar{H}^{m+1,n+1}_{p+1,q+1} \left[ z| \frac{(a_j, A_j; \alpha_j)_{1,n}}{(b_j, B_j)_{1,m}}, \frac{(a_j, A_j)_{n+1,p}}{(b_j, B_j; \beta_j)_{m+1,q}}, (\rho, 1)} \right]. \]  

(52)

Provided: \( |\text{arg } z| < \frac{1}{2} \pi \Omega \)

Remark 1. If we put \( a_j = b_j = 1, \forall i, j \) in various results then with the help of (5) it can be reduced in the form of the \( H \)-function and help of the equation (15) all finding results can be written in the form of \( \bar{I} \)-functions.
References


Dhamendra Kumar Singh
Department of Mathematics, University of Institute Engineering and Technology,
CSJM University, Kanpur-208024,  
(U.P.), India  
email: drdksinghabp@gmail.com

Omendra Mishra  
Department of Mathematics, University of Institute Engineering and Technology,  
CSJM University, Kanpur-208024,  
(U.P.), India  
email: mishraomendra@gmail.com