I-ASYMPTOTICALLY LACUNARY EQUIVALENT SET SEQUENCES DEFINED BY A MODULUS FUNCTION

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Abstract. Let $I \subseteq 2^\mathbb{N}$ be a non-trivial ideal, $\theta = (k_r)$ be a lacunary sequence and $f$ be a modulus function. Our aim in this study is to introduce some new notions such that $I_W(f)$—asymptotic equivalence, $I_W(w_f)$—asymptotic equivalence and $I_W(N^f_\theta)$—asymptotic equivalence for set sequences. We also prove some inclusion theorems.

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1. Introduction

Asymptotic equivalence was introduced by Pobyvanets [24] and Marouf extended Pobyvanets’s work [20]. Patterson, Savaş and some other authors studied on this concept and they extended asymptotic equivalence to asymptotic statistical equivalence and asymptotic lacunary statistical equivalence. [21, 22]

Das, Savaş and Ghosal in [7] introduced $I$-statistical convergence and $I$-lacunary statistical convergence with ideal. Also in [25], $I$-asymptotically statistical equivalent and $I$-asymptotically lacunary statistical equivalent sequences were studied.

Wijsman statistical convergence which is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades [18]. After this definition, Ulusu and Nuray [28] introduced Wijsman lacunary statistical convergence of set sequences. In [29] they also defined asymptotically lacunary statistical equivalent set sequences and presented theorems about asymptotic equivalence Wijsman sense. In addition, they also presented asymptotically equivalent (Wijsman sense) analogs of theorems in [29].

Recently, Kiriş, Savaş and Nuray [12] introduced $I$-asymptotically statistical equivalent and $I$—asymptotically lacunary statistical equivalent set sequences.

In this paper we introduce the concepts of $I_W(f)$-asymptotically equivalent, $I_W(w_f)$-asymptotically equivalent and $I_W(N^f_\theta)$-asymptotically equivalent set sequences and we present some natural inclusion theorems.
2. Definitions and Notations

First we recall the basic definitions and concepts (see [20], [30]). Two nonnegative sequences \(x = (x_k)\) and \(y = (y_k)\) are said to be asymptotically equivalent if
\[
\lim_{k \to \infty} \frac{x_k}{y_k} = 1
\]
(denoted by \(x \sim y\)).

A family of sets \(I \subseteq 2^\mathbb{N}\) is called an ideal if and only if
(i) \(\emptyset \in I\)
(ii) For each \(A, B \in I\) we have \(A \cup B \in I\)
(iii) For each \(A \in I\) and each \(B \subseteq A\) we have \(B \in I\)

An ideal is called non-trivial if \(\mathbb{N} \notin I\) and non-trivial ideal is called admissible if \(\{n\} \in I\) for each \(n \in \mathbb{N}\).

A family of sets \(F \subseteq 2^\mathbb{N}\) is a filter in \(\mathbb{N}\) if and only if
(i) \(\emptyset \notin F\)
(ii) For each \(A, B \in F\) we have \(A \cap B \in F\)
(iii) For each \(A \in F\) and each \(B \supseteq A\) we have \(B \in F\)

If \(I\) is a non-trivial ideal of \(\mathbb{N}\), then the family of sets
\[
F(I) = \{M \subset \mathbb{N} : \exists A \in I : M = \mathbb{N} \setminus A\}
\]
is a filter of \(\mathbb{N}\) and it is called the filter associated with the ideal.

Let \(I \subset 2^\mathbb{N}\) be an admissible ideal in \(\mathbb{N}\). The sequence \((x_n)\) of elements of \(\mathbb{R}\) is said to be \(I\)-convergent to \(L \in \mathbb{R}\) if for each \(\varepsilon > 0\) the set
\[
A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I.
\]

Now we have some easy but important examples about \(I\)-convergence.

**Example 1.** Take for \(I\) class the \(I_f\) of all finite subsets of \(\mathbb{N}\). Then \(I_f\) is an admissible ideal and \(I_f\)-convergence coincides with the usual convergence.

**Example 2.** Denote by \(I_0\) the class of all \(A \subset \mathbb{N}\) which has natural density zero. Then \(I_0\) is an admissible ideal and \(I_0\)-convergence coincides with the statistical convergence.

Let \((X, \rho)\) be a metric space. For any point \(x \in X\) and any non-empty subset \(A\) of \(X\), we define the distance from \(x\) to \(A\) by
\[
d(x, A) = \inf_{a \in A} \rho(x, A).
\]
Let \((X, \rho)\) be a metric space. For any non-empty closed subsets \(A, A_k \subseteq X\), we say that the sequence \(\{A_k\}\) is Wijsman convergent to \(A\) if
\[
\lim_{k \to \infty} d(x, A_k) = d(x, A)
\]
for each \(x \in X\). In this case we write \(W - \lim A_k = A\).

Let \((X, \rho)\) a metric space. For any non-empty closed subsets \(A_k\) of \(X\), we say that the sequence \(\{A_k\}\) is bounded if
\[
\sup_k d(x, A_k) < \infty
\]
for each \(x \in X\). In this case we write \(\{A_k\} \in L_\infty\).

Let \((X, \rho)\) a metric space. For any non-empty closed subsets \(A, A_k \subseteq X\), we say that the sequence \(\{A_k\}\) is Wijsman statistical convergent to \(A\) if \(\{d(x, A_k)\}\) is statistically convergent to \(d(x, A)\); i.e., for \(\varepsilon > 0\) and for each \(x \in X\),
\[
\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.
\]
In this case we write \(st - \lim_{W} A_k = A\) or \(A_k \rightarrow A(WS)\).

In [17] Nakano introduced the notion of a modulus function as follows: By a modulus function, we mean a function \(f\) from \([0, \infty)\) to \([0, \infty)\) such that
\begin{enumerate}
  \item \(f(x) = 0\) if and only if \(x = 0\);
  \item \(f(x + y) \leq f(x) + f(y)\) for all \(x \geq 0, y \geq 0\);
  \item \(f\) is increasing;
  \item \(f\) is continuous from the right at 0.
\end{enumerate}
It follows from that \(f\) must be continuous on \([0, 1)\). A modulus may be bounded or unbounded. Başarır [3], Maddox [19], Pehlivan [23] and many others used a modulus function \(f\) to define some new sequence spaces.

3. Main Results

For non-empty closed subsets \(A_k\) and \(B_k\) of \(X\), define \(d(x; A_k, B_k)\) as follows:
\[
d(x; A_k, B_k) = \begin{cases} 
d(x, A_k) & \text{if, } x \notin A_k \cup B_k \\
\frac{d(x, A_k)}{d(x, B_k)} & \text{if, } x \notin A_k \cup B_k \\
L & \text{if, } x \in A_k \cup B_k.
\end{cases}
\]

We begin with the following definitions.
Definition 1. Let $(X, \rho)$ be a metric space, $\mathcal{I} \subset 2^\mathbb{N}$ be a non-trivial in $\mathbb{N}$. For any non-empty closed subsets $A_k, B_k \subseteq X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be strongly asymptotically equivalent of multiple $L$ (Wijsman sense) with respect to the ideal $\mathcal{I}$ provided that every $\varepsilon > 0$, for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} |d(x; A_k, B_k) - L| \geq \varepsilon \right\} \in \mathcal{I},$$

(denoted by $A_k \mathcal{I}_{W} \sim B_k$) and simply strongly asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 2. Let $(X, d)$ be a metric space, $\mathcal{I} \subset 2^\mathbb{N}$ be a non-trivial in $\mathbb{N}$ and $f$ be a modulus function. For non-empty closed subsets $A_k, B_k \subset X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be $f$-asymptotically equivalent of multiple $L$ (Wijsman sense) with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ k \in \mathbb{N} : f(|d(x; A_k, B_k) - L|) \geq \varepsilon \right\} \in \mathcal{I},$$

(denoted by $A_k \mathcal{I}_{W}^{(f)} \sim B_k$) and simply $f$-asymptotically equivalent (Wijsman sense) with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 3. Let $(X, d)$ be a metric space, $\mathcal{I} \subset 2^\mathbb{N}$ be a non-trivial in $\mathbb{N}$ and $f$ be a modulus function. For any non-empty closed subsets $A_k, B_k \subseteq X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be strongly $f$-asymptotically equivalent of multiple $L$ (Wijsman sense) with respect to the ideal $\mathcal{I}$ provided that every $\varepsilon > 0$, for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \varepsilon \right\} \in \mathcal{I},$$

(denoted by $A_k \mathcal{I}_{W}^{(w)} \sim B_k$) and simply strongly $f$-asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 4. Let $(X, d)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be strongly $f$-asymptotically lacunary equivalent of multiple $L$ (Wijsman sense) with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) \geq \varepsilon \right\} \in \mathcal{I},$$
\( \left( \text{denoted by } A_k \overset{N}{\sim} (I^W) B_k \right) \) and simply strongly \( f \)-asymptotically lacunary equivalent with respect to the ideal \( I \), if \( L = 1 \).

**Lemma 1.** Let \( f \) be a modulus function and let \( 0 < \delta < 1 \). Then for \( y \neq 0 \) and each \( \left( \frac{x}{y} \right) > \delta \), we have \( f \left( \frac{x}{y} \right) \leq \frac{2f(1)}{\delta} \left( \frac{x}{y} \right) \).

**Theorem 2.** Let \( I \subset 2^\mathbb{N} \) be a non-trivial in \( \mathbb{N} \) and \( f \) be a modulus function. Then,

\( \text{(i) If } A_k \overset{I}{\sim} B_k \text{ then } A_k \overset{I^W}{\sim} B_k \text{ and } \)

\( \text{(ii) } \lim_{t \to \infty} \frac{f(t)}{t} = \alpha > 0, \text{ then } A_k \overset{I^W}{\sim} B_k \iff A_k \overset{I^W(f)}{\sim} B_k. \)

**Proof.** (i)—Let \( A_k \overset{I^W}{\sim} B_k \) and \( \varepsilon > 0 \) be given. Choose \( 0 < \delta < 1 \) such that \( f(t) < \varepsilon \) for \( 0 \leq t \leq \delta \). Then we can write

\[
\frac{1}{n} \sum_{k=1}^{n} f \left( |d(x; A_k, B_k) - L| \right) = \frac{1}{n} \sum_{|d(x; A_k, B_k) - L| \leq \delta} f \left( |d(x; A_k, B_k) - L| \right)
+ \frac{1}{n} \sum_{|d(x; A_k, B_k) - L| > \delta} f \left( |d(x; A_k, B_k) - L| \right). \]

Moreover, using the definiton of the modulus function \( f \), we have

\[
\frac{1}{n} \sum_{k=1}^{n} f \left( |d(x; A_k, B_k) - L| \right) < \varepsilon + \left( \frac{2f(1)}{\delta} \right) \frac{1}{n} \sum_{k=1}^{n} |d(x; A_k, B_k) - L|. \]

Thus, for any \( \gamma > 0 \),

\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f \left( |d(x; A_k, B_k) - L| \right) \geq \gamma \right\}
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left( |d(x; A_k, B_k) - L| \right) \geq \frac{(\gamma - \varepsilon) \delta}{2f(1)} \right\}. \]

Since \( A_k \overset{I^W}{\sim} B_k \), it follows the later set, and hence, the first set in above expression to \( I \). This proves that \( A_k \overset{I^W(f)}{\sim} B_k. \)
(ii) If \( \lim_{t \to \infty} \frac{f(t)}{t} = \alpha > 0 \), then we have \( f(t) \geq \alpha t \) for all \( t \geq 0 \). Suppose that \( A_k I \sim B_k \). Since
\[
\frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \frac{1}{n} \sum_{k=1}^{n} \alpha (|d(x; A_k, B_k) - L|) = \alpha \left( \frac{1}{n} \sum_{k=1}^{n} |d(x; A_k, B_k) - L| \right).
\]
It follows that for each \( \varepsilon > 0 \), we have
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} |d(x; A_k, B_k) - L| \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \alpha \varepsilon \right\}.
\]
Since \( A_k I \sim B_k \), it follows that the later set belongs to \( I \), and therefore, the theorem is proved.

**Theorem 3.** Let \( I \subset 2^\mathbb{N} \) be a non-trivial in \( \mathbb{N} \) and \( f \) be a modulus function. Then,

(i) If \( A_k I \sim B_k \) then \( A_k I \sim B_k \) and

(ii) If \( f \) is bounded, then \( A_k I \sim B_k \) \( \Leftrightarrow \) \( A_k I \sim B_k \).

**Proof.** (i)–Suppose \( A_k I \sim B_k \) and \( \varepsilon > 0 \) be given. Then we can write
\[
\frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \frac{f(\varepsilon)}{n} \cdot \{|k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon|\}.
\]
Therefore, for any \( \gamma > 0 \), we have
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \gamma \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \gamma \right\}.
\]
Since $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$, it follows that the later set belongs to \( \mathcal{I} \), and therefore $A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$.

(ii) Suppose that $f$ is bounded and $A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$. Since $f$ is bounded there exists a real number $M$ such that $\sup f(t) \leq M$. And for $\varepsilon > 0$, we can write

\[
\frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) = \frac{1}{n} \left[ \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \right] \\
+ \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \quad \text{if } |d(x; A_k, B_k) - L| < \varepsilon \\
\leq \frac{M}{n} \sum_{k \in I} |d(x; A_k, B_k) - L| + f(\varepsilon).
\]

Now if $\varepsilon \to 0$, the theorem is proved. Since $A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$, it follows that the later set belongs to \( \mathcal{I} \), and therefore $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$.

**Theorem 4.** Let $\mathcal{I} \subset 2^\mathbb{N}$ be a non-trivial in $\mathbb{N}$, $\theta = \{k_r\}$ be a lacunary sequence and $f$ be a modulus function. If $\liminf_r q_r > 1$, then $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k \implies A_k \overset{\mathcal{I}_W(N_f^\theta)}{\sim} B_k$.

**Proof.** Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r = \frac{k_r}{k_r - 1} \geq 1 + \delta$ for sufficiently large $r$, which implies that

\[
\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta}.
\]

Let $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$. For a sufficiently large $r$, we obtain the following:

\[
\frac{1}{k_r} \sum_{t=1}^{k_r} f(|d(x; A_k, B_k) - L|) \geq \frac{1}{k_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) \\
= \left( \frac{h_r}{k_r} \right) \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) \\
\geq \left( \frac{\delta}{1 + \delta} \right) \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|).
\]
which gives for any \( \varepsilon > 0 \),

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f \left( |d(x; A_k, B_k) - L| \right) \geq \varepsilon \right\}
\]

\[
\subseteq \left\{ r \in \mathbb{N} : \frac{1}{k_r} \sum_{k=1}^{k_r} f \left( |d(x; A_k, B_k) - L| \right) \geq \frac{\varepsilon \delta}{1 + \delta} \right\}.
\]

Since \( A_k \sim_{I^w(\omega_f)} B_k \), it follows that the later set belongs to \( I \), and therefore \( A_k \sim_{I^w(N'_\theta)} B_k \).

**Theorem 5.** Let \((X, \rho)\) be a metric space. Let \( I \subset P(\mathbb{N}) \) be a non-trivial in \( \mathbb{N} \), \( \theta = \{k_r\} \) be a lacunary sequence, \( A_k, B_k \) be non-empty closed subsets of \( X \); and \( f \) be a modulus function. Then,

(i) If \( A_k \sim_{I^w(N_\theta)} B_k \), then \( A_k \sim_{I^w(N'_\theta)} B_k \); and

(ii) \( \lim_{t \to \infty} \frac{f(t)}{t} = \alpha > 0 \), then \( A_k \sim_{I^w(N_\theta)} B_k \iff A_k \sim_{I^w(N'_\theta)} B_k \).

**Proof.** The proof is similar to the proof of theorem 3.1.

**Theorem 6.** Let \((X, \rho)\) be a metric space, \( I \subset 2^{\mathbb{N}} \) be a non-trivial in \( \mathbb{N} \), \( \theta = \{k_r\} \) be a lacunary sequence, \( A_k, B_k \) be non-empty closed subsets of \( X \); and \( f \) be a modulus function. Then,

(i) If \( A_k \sim_{I^w(N'_\theta)} B_k \), then \( A_k \sim_{I^w(S_\theta)} B_k \); and

(ii) If \( f \) is bounded, then \( A_k \sim_{I^w(N'_\theta)} B_k \iff A_k \sim_{I^w(S_\theta)} B_k \).

**Proof.** (i) Suppose that \( A_k \sim_{I^w(N'_\theta)} B_k \) and \( \varepsilon > 0 \) be given. Since

\[
\frac{1}{h_r} \sum_{k \in I_r} f \left( |d(x; A_k, B_k) - L| \right) \geq \frac{1}{h_r} \sum_{k \in I_r} f \left( |d(x; A_k, B_k) - L| \right)
\]

\[
\geq f(\varepsilon) \frac{1}{h_r} \sum_{|d(x; A_k, B_k) - L| \geq \varepsilon} f \left( |d(x; A_k, B_k) - L| \right)
\]

\[
\geq f(\varepsilon) \frac{1}{h_r} \sum_{k \in I_r} \{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}
\]

\[
\geq f(\varepsilon) \frac{1}{h_r} \sum_{k \in I_r} \{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}
\]

\[
\geq f(\varepsilon) \frac{1}{h_r} \sum_{k \in I_r} \{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}
\]

\[
\geq f(\varepsilon) \frac{1}{h_r} \sum_{k \in I_r} \{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}
\]
If follows that for any $\gamma > 0$, if we denote sets

$$A(\varepsilon, \gamma) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon \right\} \right| \geq \gamma \right\}$$

$$B(\varepsilon, \gamma) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) \geq \gamma f(\varepsilon) \right\}.$$

Then $A(\varepsilon, \gamma) \subset B(\varepsilon, \gamma)$. Since $A_k \mathcal{I}_W(N') \sim B_k$, so $B(\varepsilon, \gamma) \in \mathcal{I}$. But then, by the definition of an ideal, $A(\varepsilon, \gamma) \in \mathcal{I}$, and therefore, $A_k \mathcal{I}_W(N') \sim B_k$.

(ii) Suppose that $f$ is bounded and let $A_k \mathcal{I}_W(S_\theta) \sim B_k$. Since $f$ is bounded there exists a positive real number $M$ such that $|f(x)| \leq M$ for all $x \geq 0$. Further, using the fact

$$\frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) = \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|)$$

$$+ \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|)_{|d(x; A_k, B_k) - L| < \varepsilon}$$

$$\leq \frac{M}{h_r} \left| \{ k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon \} \right| + f(\varepsilon).$$

Now if $\varepsilon \to 0$, the theorem is proved. Since $A_k \mathcal{I}_W(S_\theta) \sim B_k$, it follows the later set, and hence, the first set in above expression to $\mathcal{I}$. This proves that $A_k \mathcal{I}_W(N') \sim B_k$.

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