COEFFICIENT ESTIMATES FOR SOME SUBCLASSES OF M-FOLD
SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In the present investigation, we consider two new general subclasses
$H_{\Sigma_m}(\tau, \gamma; \alpha)$ and $H_{\Sigma_m}(\tau, \gamma; \beta)$ of $\Sigma_m$ consisting of analytic and $m$-fold symmetric
bi-univalent functions in the open unit disk $U$. For functions belonging to the two
classes introduced here, we derive estimates on the initial coefficients $|a_{m+1}|$ and
$|a_{2m+1}|$. Several related classes are also considered and connections to earlier known
results are made.

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tions, $m$-Fold symmetric bi-univalent functions.

1. Introduction, Definitions and Preliminaries

Let $A$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$ 

We denote by $S$ the class of all functions $f(z) \in A$ which are univalent in $U$ [3, 11, 16].
Some of the important and well-investigated subclasses of the univalent function
class $S$ include the class $S^*(\alpha)$ of starlike functions of order $\alpha$ in $U$ and the class
$K(\alpha)$ of convex functions of order $\alpha$ in $U$.

It is well-known that every function $f(z) \in S$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$
and
\[ f^{-1}(f(w)) = w \quad \left( |w| < r_0(f); \ r_0(f) \geq \frac{1}{4} \right). \]

The inverse function \( f^{-1} \) may analytically continued to \( \mathbb{U} \) as follows:
\[ f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 \quad - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \] (2)

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \). We denote by \( \Sigma \) the class of bi-univalent functions in \( \mathbb{U} \) given by (1).

For each function \( f \in \mathcal{S} \), the function
\[ h(z) = \sqrt[2m]{f(z^m)} \quad (z \in \mathbb{U}; \ m \in \mathbb{N}) \] (3)
is univalent and maps the unit disk \( \mathbb{U} \) into a region with \( m \)-fold symmetry. A function is said to be \( m \)-fold symmetric (see [7, 10]) if it has the following normalized form:
\[ f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \quad (z \in \mathbb{U}; \ m \in \mathbb{N}). \] (4)

We denote by \( S_m \) the class of \( m \)-fold symmetric univalent functions in \( \mathbb{U} \), which are normalized by the series expansion (4). The functions in the class \( \mathcal{S} \) are said to be one-fold symmetric.

Each bi-univalent function generates an \( m \)-fold symmetric bi-univalent function for each integer \( m \in \mathbb{N} \). The normalized form of \( f \) is given as in (4) and the series expansion for \( f^{-1} \), which has been recently proven by Srivastava et al. [17], is given as follows:
\[ g(w) = w - a_{m+1}w^{m+1} + \left[(m + 1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1} \]
\[-\frac{1}{2}(m + 1)(3m + 2)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \] (5)

where \( f^{-1} = g \). We denote by \( \Sigma_m \) the class of \( m \)-fold symmetric bi-univalent functions in \( \mathbb{U} \). It is easily seen that for \( m = 1 \), the formula (5) coincides with the formula (2). Here are some examples of \( m \)-fold symmetric bi-univalent functions:
\[ \left( \frac{z^m}{1 - z^m} \right)^{\frac{1}{m}}, \quad \left[ \frac{1}{2} \log \left( \frac{1 + z^m}{1 - z^m} \right) \right]^{\frac{1}{m}} \quad \text{and} \quad \left[ -\log (1 - z^m) \right]^{\frac{1}{m}} \]
with the corresponding inverse functions
\[
\left( \frac{w^m}{1-w^m} \right)^{1/m}, \quad \left( \frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right)^{1/m} \quad \text{and} \quad \left( \frac{e^{w^m} - 1}{e^{w^m}} \right)^{1/m},
\]
respectively.

In 1967, Lewin [8] investigated the class Σ and showed that \(|a_2| < 1.51\). Subsequently, Brannan and Clunie [1] conjectured that \(|a_2| \leq \sqrt{2}\). Afterwards in 1981, Styer and Wright [18] showed that there exist functions \(f(z) \in \Sigma\) for which \(|a_2| > \frac{4}{3}\).

The best known estimate for functions in \(\Sigma\) has been obtained in 1984 by Tan [19], that is, \(|a_2| \leq 1.485\). The coefficient estimate problem involving the bound of \(|a_n|\) for each \(f \in \Sigma\) given by (4) is still an open problem.

Recently, many researchers [5, 6, 9, 13, 14, 15, 17, 20, 21], following the work of Brannan and Taha [2], introduced and investigated a lot of interesting subclasses of the bi-univalent function class Σ and they obtained non-sharp estimates of the first two Taylor-Maclaurin coefficients \(|a_2|\) and \(|a_3|\).

In this paper, we derive estimates on the initial coefficients \(|a_{m+1}|\) and \(|a_{2m+1}|\) for functions belonging to the new general subclasses \(H_\Sigma_m(\tau, \gamma; \alpha)\) and \(H_\Sigma_m(\tau, \gamma; \beta)\) of \(\Sigma_m\). Several related classes are also considered and connections to earlier known results are made. These two new subclasses \(H_\Sigma_m(\tau, \gamma; \alpha)\) and \(H_\Sigma_m(\tau, \gamma; \beta)\) are defined as follows:

**Definition 1.** A function \(f(z) \in \Sigma_m\) given by (4) is said to be in the class \(H_\Sigma_m(\tau, \gamma; \alpha)\) if the following conditions are satisfied:
\[
\left| \arg \left( 1 + \frac{1}{\tau} \left[ f'(z) + \gamma zf''(z) - 1 \right] \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \mathbb{U}) \tag{6}
\]
and
\[
\left| \arg \left( 1 + \frac{1}{\tau} \left[ g'(w) + \gamma wg''(w) - 1 \right] \right) \right| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{U}) \tag{7}
\]
where the function \(g = f^{-1}\) is given by (5).

**Definition 2.** A function \(f(z) \in \Sigma_m\) given by (4) is said to be in the class \(H_\Sigma_m(\tau, \gamma; \beta)\) if the following conditions are satisfied:
\[
\Re \left( 1 + \frac{1}{\tau} \left[ f'(z) + \gamma zf''(z) - 1 \right] \right) > \beta \quad (z \in \mathbb{U}) \tag{8}
\]
and

\[
\Re \left( 1 + \frac{1}{\tau} \left[ g'(w) + \gamma w g''(w) - 1 \right] \right) > \beta \quad (w \in \mathbb{U})
\]

\[(9)\]

\[
0 \leq \beta < 1; \quad \tau \in \mathbb{C} \setminus \{0\}; \quad 0 \leq \gamma \leq 1,
\]

and where the function \( g = f^{-1} \) is given by (5).

The following lemma [3] will be required in order to derive our main results.

**Lemma 1.** If \( h \in \mathcal{P} \), then \(|c_k| \leq 2\) for each \( k \in \mathbb{N} \), where \( \mathcal{P} \) is the family of all functions \( h \), analytic in \( \mathbb{U} \), for which

\[
\Re (h(z)) > 0, \quad (z \in \mathbb{U}),
\]

where

\[
h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}).
\]

2. **Coefficient Bounds for the Functions Class** \( \mathcal{H}_{\Sigma_m}(\tau, \gamma; \alpha) \)

We begin this section by finding the estimates on the coefficients \(|a_{m+1}|\) and \(|a_{2m+1}|\) for functions in the class \( \mathcal{H}_{\Sigma_m}(\tau, \gamma; \alpha) \).

**Theorem 2.** Let \( f(z) \in \mathcal{H}_{\Sigma_m}(\tau, \gamma; \alpha) \) \((0 < \alpha \leq 1; \quad \tau \in \mathbb{C} \setminus \{0\}; \quad 0 \leq \gamma \leq 1)\) be of the form (4). Then

\[
|a_{m+1}| \leq \frac{2\alpha |\tau|}{\sqrt{\tau\alpha(m+1)(2m+1)(2\gamma m + 1) + (1 - \alpha)(m +1)^2(\gamma m + 1)^2}}
\]

\[(10)\]

and

\[
|a_{2m+1}| \leq \frac{2\alpha^2 |\tau|^2}{(m+1)(\gamma m + 1)^2} + \frac{2\alpha |\tau|}{(2m+1)(2\gamma m + 1)^2}.
\]

\[(11)\]

**Proof.** It follows from (6) and (7) that

\[
1 + \frac{1}{\tau} \left[ f'(z) + \gamma z f''(z) - 1 \right] = [p(z)]^\alpha
\]

\[(12)\]
and

\[ 1 + \frac{1}{\tau} \left[ g'(w) + \gamma wg''(w) - 1 \right] = [q(w)]^\alpha, \]  

(13)

where the functions \( p(z) \) and \( q(w) \) are in \( P \) and have the following series representations:

\[ p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \]  

(14)

and

\[ q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \]  

(15)

Now, equating the coefficients in (12) and (13), we obtain

\[ \frac{(m + 1)(\gamma m + 1)}{\tau} a_{m+1} = \alpha p_m, \]  

(16)

\[ \frac{(2m + 1)(2\gamma m + 1)}{\tau} a_{2m+1} = \alpha p_{2m} + \frac{1}{2} \alpha(\alpha - 1) p_m^2, \]  

(17)

\[ -\frac{(m + 1)(\gamma m + 1)}{\tau} a_{m+1} = \alpha q_m, \]  

(18)

and

\[ \frac{(2m + 1)(2\gamma m + 1)}{\tau} \left[ (m + 1)a_m^2 - a_{2m+1} \right] = \alpha q_{2m} + \frac{1}{2} \alpha(\alpha - 1) q_m^2. \]  

(19)

From (16) and (18), we find

\[ p_m = -q_m \]  

(20)

and

\[ 2 \frac{(m + 1)^2(\gamma m + 1)^2}{\tau^2} a_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \]  

(21)

From (17), (19) and (21), we get

\[ \frac{(2m + 1)(2\gamma m + 1)}{\tau} (m + 1) a_{m+1}^2 \]

\[ = \alpha(p_{2m} + q_{2m}) + \frac{\alpha(\alpha - 1)}{2} (p_m^2 + q_m^2) \]
\[
= \alpha(p_{2m} + q_{2m}) + \frac{(\alpha - 1)(m + 1)^2(\gamma m + 1)^2}{\tau^2} a_{m+1}.
\]  

(22)

Therefore, we have

\[
a_{m+1}^2 = \frac{\tau^2 \alpha^2(p_{2m} + q_{2m})}{\tau \alpha(m + 1)(2m + 1)(2\gamma m + 1) + (1 - \alpha)(m + 1)^2(\gamma m + 1)^2}.
\]  

(23)

Applying Lemma 1 for the coefficients \(p_{2m}\) and \(q_{2m}\), we have

\[
|a_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{\tau \alpha(m + 1)(2m + 1)(2\gamma m + 1) + (1 - \alpha)(m + 1)^2(\gamma m + 1)^2}}.
\]  

(24)

This gives the desired bound for \(|a_{m+1}|\) as asserted in (10).

In order to find the bound on \(|a_{2m+1}|\), by subtracting (19) from (17), we get

\[
2 \frac{(2m + 1)(2\gamma m + 1)}{\tau} a_{2m+1} - \frac{(2m + 1)(2\gamma m + 1)}{\tau} (m + 1) a_{m+1}^2
\]

\[
= \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha - 1)}{2} (p_m^2 - q_m^2).
\]  

(25)

It follows from (20) and (25) that

\[
a_{2m+1} = \frac{\alpha^2 \tau^2(p_m^2 + q_m^2)}{4(m + 1)(\gamma m + 1)^2} + \frac{\alpha \tau(p_{2m} - q_{2m})}{2(2m + 1)(2\gamma m + 1)}.
\]  

(26)

Applying Lemma 1 once again for the coefficients \(p_m, p_{2m}, q_m\) and \(q_{2m}\), we readily obtain

\[
|a_{2m+1}| \leq \frac{2\alpha^2|\tau|^2}{(m + 1)(\gamma m + 1)^2} + \frac{2\alpha|\tau|}{(2m + 1)(2\gamma m + 1)}.
\]  

(27)

3. Coefficient Bounds for the Functions Class \(H_{\Sigma_m}(\tau, \gamma; \beta)\)

This section is devoted to find the estimates on the coefficients \(|a_{m+1}|\) and \(|a_{2m+1}|\) for functions in the class \(H_{\Sigma_m}(\tau, \gamma; \beta)\).
Theorem 3. Let \( f(z) \in \mathcal{H}_{\Sigma_{m}}(\tau; \gamma; \beta) \) \((0 \leq \beta \leq 1; \tau \in \mathbb{C} \setminus \{0\}; 0 \leq \gamma \leq 1)\) be of the form (4). Then

\[
|a_{m+1}| \leq \sqrt{\frac{4|\tau|(1-\beta)}{(m+1)(2m+1)(2\gamma m+1)}}
\]

and

\[
|a_{2m+1}| \leq \frac{2|\tau|^2(1-\beta)^2}{(m+1)(\gamma m + 1)^2} + \frac{2|\tau|(1-\beta)}{(2m+1)(2\gamma m + 1)}.
\]

Proof. It follows from (8) and (9) that there exist \( p, q \in \mathcal{P} \) such that

\[
1 + \frac{1}{\tau} \left[ f'(z) + \gamma zf''(z) - 1 \right] = \beta + (1-\beta)p(z)
\]

and

\[
1 + \frac{1}{\tau} \left[ g'(w) + \gamma wg''(w) - 1 \right] = \beta + (1-\beta)q(w),
\]

where \( p(z) \) and \( q(w) \) have the forms (14) and (15), respectively. By suitably comparing coefficients in (30) and (31), we get

\[
\frac{(m+1)(\gamma m + 1)}{\tau} a_{m+1} = (1-\beta)p_m,
\]

\[
\frac{(2m+1)(2\gamma m + 1)}{\tau} a_{2m+1} = (1-\beta)p_{2m},
\]

\[
-\frac{(m+1)(\gamma m + 1)}{\tau} a_{m+1} = (1-\beta)q_m
\]

and

\[
\frac{(2m+1)(2\gamma m + 1)}{\tau} [(m+1)a_{m+1}^2 - a_{2m+1}] = (1-\beta)q_{2m}.
\]

From (32) and (34), we find

\[
p_m = -q_m
\]

and

\[
2 \frac{(m+1)^2(\gamma m + 1)^2}{\tau^2} a_{m+1}^2 = (1-\beta)^2(p_m^2 + q_m^2).
\]
Adding (33) and (35), we have
\[
\frac{(2m+1)(2\gamma m+1)}{\tau} \tau \frac{2}{m+1} = (1 - \beta)(p_{2m} + q_{2m}). \quad (38)
\]

Applying Lemma 1, we obtain
\[
|a_{m+1}| \leq \sqrt{\frac{4|\tau|(1 - \beta)}{(m+1)(2m+1)(2\gamma m + 1)}}. \quad (39)
\]

This is the bound on $|a_{m+1}|$ asserted in (28).

In order to find the bound on $|a_{2m+1}|$, by subtracting (35) from (33), we get
\[
2 \left( \frac{2m+1}{\tau} \right) a_{2m+1} = \frac{(2m+1)(2\gamma m + 1)}{\tau} (m+1) a_{m+1}^2
\]
\[
= (1 - \beta)(p_{2m} - q_{2m})
\]

or, equivalently,
\[
a_{2m+1} = \frac{(m+1)}{2} a_{m+1}^2 + \frac{\tau(1 - \beta)(p_{2m} - q_{2m})}{2(2m+1)(2\gamma m + 1)}. \quad (40)
\]

It follows from (36) and (37) that
\[
a_{2m+1} = \frac{\tau(1 - \beta)^2(p_m^2 + q_m^2)}{4(m+1)(\gamma m + 1)^2} + \frac{\tau(1 - \beta)(p_{2m} - q_{2m})}{2(2m+1)(2\gamma m + 1)}. \quad (41)
\]

Applying Lemma 1 once again for the coefficients $p_m$, $p_{2m}$, $q_m$ and $q_{2m}$, we easily obtain
\[
|a_{2m+1}| \leq \frac{2|\tau|(1 - \beta)}{(m+1)(\gamma m + 1)^2} + \frac{2|\tau|(1 - \beta)}{(2m+1)(2\gamma m + 1)}. \quad (42)
\]

4. Applications of the main results

For one-fold symmetric bi-univalent functions and for $\tau = 1$, Theorem 1 and Theorem 2 reduce to Corollary 1 and Corollary 2, respectively, which were proven very recently by Frasin [4] (see also [12]).

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Corollary 4. Let \( f(z) \in \mathcal{H}_\Sigma(\alpha, \gamma) \) \((0 < \alpha \leq 1; 0 \leq \gamma \leq 1)\) be of the form (1). Then
\[
|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha + 2) + 4\gamma(\alpha + \gamma + 2 - \alpha\gamma)}}
\]
and
\[
|a_3| \leq \frac{\alpha^2}{(\gamma + 1)^2} + \frac{2\alpha}{3(2\gamma + 1)}.
\]

Corollary 5. Let \( f(z) \in \mathcal{H}_\Sigma(\beta, \gamma) \) \((0 < \alpha \leq 1; 0 \leq \gamma \leq 1)\) be of the form (1). Then
\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{3(2\gamma + 1)}}
\]
and
\[
|a_3| \leq \frac{(1 - \beta)^2}{(\gamma + 1)^2} + \frac{2(1 - \beta)}{3(2\gamma + 1)}.
\]

The classes \( \mathcal{H}_\Sigma(\alpha, \gamma) \) and \( \mathcal{H}_\Sigma(\beta, \gamma) \) are defined in the following way:

Definition 3. A function \( f(z) \in \Sigma \) given by (1) is said to be in the class \( \mathcal{H}_\Sigma(\alpha, \gamma) \) if the following conditions are satisfied:
\[
|\arg (f'(z) + \gamma zf''(z))| < \frac{\alpha \pi}{2} \quad (z \in \mathbb{U})
\]
and
\[
|\arg (g'(w) + \gamma wg''(w))| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{U})
\]
\[
\left(0 < \alpha \leq 1; 0 \leq \gamma \leq 1\right),
\]
and where the function \( g = f^{-1} \) is given by (2).

Definition 4. A function \( f(z) \in \Sigma \) given by (1) is said to be in the class \( \mathcal{H}_\Sigma(\beta, \gamma) \) if the following conditions are satisfied:
\[
\Re (f'(z) + \gamma zf''(z)) > \beta \quad (z \in \mathbb{U})
\]
and
\[
\Re (g'(w) + \gamma wg''(w)) > \beta \quad (w \in \mathbb{U})
\]
\[
\left(0 \leq \beta < 1; 0 \leq \gamma \leq 1\right),
\]
and where the function \( g = f^{-1} \) is given by (2).
If we set $\gamma = 0$ and $\tau = 1$ in Theorem 1 and Theorem 2, then the classes $\mathcal{H}_{\Sigma_m}(\tau, \gamma; \alpha)$ and $\mathcal{H}_{\Sigma_m}(\tau, \gamma; \beta)$ reduce to the classes $\mathcal{H}_{\Sigma_m}^\alpha$ and $\mathcal{H}_{\Sigma_m}^\beta$ investigated recently by Srivastava et al. [17] and thus, we obtain the following corollaries:

**Corollary 6.** Let $f(z) \in \mathcal{H}_{\Sigma_m}^\alpha$ ($0 < \alpha \leq 1$) be of the form (4). Then

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(m+1)(am+m+1)}}$$

(51)

and

$$|a_{2m+1}| \leq \frac{2\alpha(2am + \alpha + m + 1)}{(m+1)(2m+1)}.$$

(52)

**Corollary 7.** Let $f(z) \in \mathcal{H}_{\Sigma_m}^\beta$ ($0 \leq \beta \leq 1$) be of the form (4). Then

$$|a_{m+1}| \leq 2\sqrt{\frac{(1-\beta)}{(m+1)(2m+1)}}$$

(53)

and

$$|a_{2m+1}| \leq 2(1-\beta) \left( \frac{(1-\beta)(2m+1) + m + 1}{(m+1)(2m+1)} \right).$$

(54)

The classes $\mathcal{H}_{\Sigma_m}^\alpha$ and $\mathcal{H}_{\Sigma_m}^\beta$ are respectively defined as follows:

**Definition 5.** A function $f(z) \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{H}_{\Sigma_m}^\alpha$ if the following conditions are satisfied:

$$\left| \arg \left\{ g'(z) \right\} \right| < \frac{\alpha \pi}{2} \quad (z \in \mathbb{U})$$

(55)

and

$$\left| \arg \left\{ g'(w) \right\} \right| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{U})$$

(56)

(0 < $\alpha \leq 1$),

and where the function $g$ is given by (5).

**Definition 6.** A function $f(z) \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{H}_{\Sigma_m}^\beta$ if the following conditions are satisfied:

$$\Re \left( f'(z) \right) > \beta \quad (z \in \mathbb{U})$$

(57)

and

$$\Re \left( g'(w) \right) > \beta \quad (w \in \mathbb{U})$$

(58)

(0 $\leq \beta < 1$),

and where the function $g$ is given by (5).
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