THIRD HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. In the present paper we investigate the upper bounds of the Hankel determinant $H_3(1)$ for a class of analytic functions with respect to symmetric points, denoted $M_s(\alpha)$.

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1. Introduction

Let $A$ be the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$  \hspace{1cm} (1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Consider $S$ the subclass of $A$ consisting of univalent functions.

Recently, Selvaraj and Vasanthi [16] defined the next subclass of analytic functions with respect to symmetric points:

**Definition 1.** ([16]) Let $M_s(\alpha)$ denote the class of analytic functions $f$ of the form (1) and satisfying the condition

$$\text{Re} \left[ \frac{\alpha^2 z f''(z) + zf'(z)}{\alpha z (f(z) - f(-z))^2 + (1-\alpha)(f(z) - f(-z))} \right] > 0, \ 0 \leq \alpha \leq 1, \ z \in U. \hspace{1cm} (2)$$

In particular:

(i) for $\alpha = 0, M_s(0) \equiv S_s^*$,

$$S_s^* := \left\{ f \in A : \text{Re} \left[ \frac{zf'(z)}{f(z) - f(-z)} \right] > 0, \ z \in U \right\}.$$
These functions are called starlike functions with respect to symmetric points and were introduced by Sakaguchi [13].

(ii) for \( \alpha = 1 \), \( M_s(1) \equiv K_s \),

\[
K_s := \left\{ f \in A : \text{Re} \left[ \frac{(zf(z))'}{(f(z) - f(-z))'} \right] > 0, \ z \in U \right\}.
\]

Functions in the class \( K_s \) are called convex functions with respect to symmetric points and were introduced by Das and Singh [14].

**Definition 2.** ([10]) Let \( f \) and \( g \) be two analytic functions in \( U \). Then, the function \( f \) is said to be subordinate to \( g \), written \( f \prec g \), if there exists a function \( w \), analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in U \), so that

\[
f(z) = g(h(z)) \text{ for all } z \in U.
\]

Pommerenke [11] stated the \( q \)-th Hankel determinant as

\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \cdots & a_{n+q-1} \\
  a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+q-1} & \cdots & \cdots & a_{n+2q-2}
\end{vmatrix}, \tag{3}
\]

where \( n \leq 1 \) and \( q \leq 1 \). The Hankel determinant is useful, for example, in the study of power series with integral coefficients (see [3, 4]), meromorphic functions (see [21]) and also singularities (see [4]).

It is well known that the Fekete-Szegö functional is equivalent to \( H_2(1) \). In particular, sharp upper bounds on \( H_2(2) \) were obtained in [6, 7, 8, 20]. Recently, the third Hankel determinant \( H_3(1) \) has been considered in works [1, 12, 17].

In this paper, we determine the upper bound of \( H_3(1) \) for subclasses of analytic functions with respect to symmetric and conjugate points by using Toeplitz determinants [15] and following a method devised by Libera and Zlotkiewicz (see [18, 19]).

In our proposed investigation we shall make use of the next results.

2. Preliminary Results

Let \( P \) denote the class of analytic functions \( p \) normalized by

\[
p(z) = 1 + \sum_{k=1}^{\infty} t_k z^k \tag{4}
\]

such that \( \text{Re} \ p(z) > 0, \ z \in U \).
Lemma 1. [5]. If \( p \in P \) then the following sharp estimate holds:

\[ |t_k| \leq 2, \quad k = 1, 2, \ldots \] (5)

Lemma 2. [18, 19]. Let \( p \in P \). Then

\[ 2t_2 = t_1^2 + x(4 - t_1^2), \] (6)
\[ 4t_3 = t_1^3 + 2(4 - t_1^2)t_1x - (4 - t_1^2)t_1x^2 + 2(4 - t_1^2)(1 - |x|^2)z, \] (7)

for some complex numbers \( x, z \) with \( |x| \leq 1 \) and \( |z| \leq 1 \).

Lemma 3. [9]. If \( p \in P \), then for \( \lambda \) a complex number

\[ |t_2 - \lambda t_1^2| \leq 2 \max(1, |2\lambda - 1|). \] (8)

This result is sharp for the functions

\[ p(z) = \frac{1 + z}{1 - z} \] and \( p(z) = \frac{1 + z^2}{1 - z^2}. \] (9)

3. Main Results

Theorem 4. Let \( f \in M_s(\alpha) \). Then we have the sharp inequality

\[
|a_2a_3 - a_4| \leq \frac{1}{1 + 3\alpha} \max \left\{ \frac{1}{2}, \frac{2(4\alpha^2 + 3\alpha + 1)}{3(1 + \alpha)(1 + 2\alpha)}, \sqrt{\frac{4\alpha^2 + 3\alpha + 1}{3(1 + \alpha)(1 + 2\alpha)}} \right\}.
\]

Proof. Using the definition of subordination, \( f \in M_s(\alpha) \) if and only if

\[
\frac{2\alpha z^2 f''(z) + 2z f'(z)}{\alpha z(f(z) - f(-z))' + (1 - \alpha)(f(z) - f(-z))} = \frac{1 + \omega(z)}{1 - \omega(z)} = p(z), \quad p \in P.
\]

It follows that

\[
z + \sum_{n=2}^{\infty} n[1 + \alpha(n - 1)] a_n z^n = (1 + t_1z + t_2z^2 + \ldots)(z + (1 + 2\alpha)a_3z^3 + (1 + 4\alpha)a_5z^5 + \ldots \cdot [1 + (2n - 2)\alpha]a_{2n-1}z^{2n-1} + (1 + 2n\alpha)a_{2n+1}z^{2n+1} + \ldots).
\]

On equating the coefficients like powers of \( z \) in (10), we obtain

\[
a_2 = \frac{t_1}{2(1 + \alpha)}, \quad a_3 = \frac{t_2}{2(1 + 2\alpha)}, \quad a_4 = \frac{t_1t_2 + 2t_3}{8(1 + 3\alpha)}. \] (11)

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Assuming $t_1 = t$ and substituting for $t_2$ and $t_3$ by using Lemma 2 in (11), we have

$$a_2 = \frac{t}{2(1 + \alpha)}, \quad a_3 = \frac{t^2 + (4 - t^2)x}{4(1 + 2\alpha)}, \quad \text{(12)}$$

and

$$a_4 = \frac{1}{16(1 + 3\alpha)}(2t^3 + 3t(4 - t^2)x - t(4 - t^2)x^2 + 2(4 - t^2)(1 - |x|^2)z). \quad \text{(13)}$$

From (12) and (13) we get

$$|a_2a_3 - a_4| = A(\alpha) - 4\alpha^2t^3 - t(4 - t^2)(6\alpha^2 + 3\alpha + 1)x + t(4 - t^2)(1 + 3\alpha + 2\alpha^2)x^2 - 2(4 - t^2)(1 + 3\alpha + 2\alpha^2)(1 - |x|^2)z,$$

where

$$A(\alpha) = \frac{1}{16(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)}.$$

Applying the triangle inequality with $t \in [0, 2], |z| \leq 1$ and $\delta = |x|$, we have

$$|a_2a_3 - a_4| \leq A(\alpha)|4t^3\alpha^2 + t(4 - t^2)(6\alpha^2 + 3\alpha + 1)|
+ t(4 - t^2)(1 + 3\alpha + 2\alpha^2)\delta^2 + 2(4 - t^2)(1 + 3\alpha + 2\alpha^2)(1 - \delta^2)
\leq A(\alpha)(t - 2)(4 - t^2)(1 + 3\alpha + 2\alpha^2)\delta^2 + t(4 - t^2)(6\alpha^2 + 3\alpha + 1)\delta
+ 4t^3\alpha^2 + 2(4 - t^2)(1 + 3\alpha + 2\alpha^2) = A(\alpha)F(\delta). \quad \text{(14)}$$

Next, we maximize the function $F(\delta)$.

$F'(\delta) = 0$ implies $\delta = \frac{at}{2(2a+b)} = \delta^*$ where $a = 6\alpha^2 + 3\alpha + 1$ and $b = 2(1 + \alpha)(1 + 2\alpha)$, so we need to consider two cases.

(i) If $\delta^* > 1$, we have max$_{\delta \in [0, 1]} F(\delta) = F(1)$, therefore

$$F(\delta) \leq -2t^3(2\alpha^2 + 3\alpha + 1) + 8t(4\alpha^2 + 3\alpha + 1) = G_1(t).$$

By differentiating $G_1(t)$, we get

$$G_1'(t) = -6t^2(2\alpha^2 + 3\alpha + 1) + 8(4\alpha^2 + 3\alpha + 1).$$

Setting $G_1'(t) = 0$ we obtain $t = \pm 2\sqrt{\frac{4\alpha^2 + 3\alpha + 1}{3(2\alpha^2 + 3\alpha + 1)}}$. Since

$$G_1''(t) = -12t(2\alpha^2 + 3\alpha + 1) \leq 0,$$

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it follows that $G$ has a maximum value at $t = 2\sqrt{\frac{4\alpha^2 + 3\alpha + 1}{3(2\alpha^2 + 3\alpha + 1)}} = t'$. Hence,

$$G_1(t) \leq \frac{32(4\alpha^2 + 3\alpha + 1)}{3} \sqrt{\frac{4\alpha^2 + 3\alpha + 1}{3(2\alpha^2 + 3\alpha + 1)}}. \quad (15)$$

(ii) If $\delta^* \leq 1$, we find that $\max_{\delta \in [0,1]} F(\delta) = F(\delta^*)$. Thus,

$$F(\delta) \leq \frac{(2 + t)(a^2t^2 + 8b^2(2 - t))}{4b} + 4\alpha^2t^3 = G_2(t).$$

It follows that $G_2$ has a maximum value at $t = 0$, so

$$G_2(t) \leq 16(1 + \alpha)(1 + 2\alpha). \quad (16)$$

From the relations (14), (15) and (16) upon simplification, the theorem is proved. The result is sharp for $t_1 = t$, $t_2 = t_1^2 - 2$ and $t_3 = t_1(t_1^2 - 3)$.

**Corollary 5.** [2] If $f \in S_\ast^s$, then

$$|a_2a_3 - a_4| \leq \frac{1}{2}.$$

**Corollary 6.** [2] If $f \in K_\ast$, then

$$|a_2a_3 - a_4| \leq \frac{4}{27}.$$

**Theorem 7.** Let $f \in M_\ast(\alpha)$. Then for a complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{1 + 2\alpha} \max \left(1, \left| (1 + 2\alpha)\mu - 1 \right| \right). \quad (17)$$

**Proof.** From (11), we get

$$|a_3 - \mu a_2^2| = \frac{1}{2(1 + 2\alpha)} \left| t_2 - \frac{(1 + 2\alpha)\mu}{2(1 + \alpha)^2} t_1^2 \right|.$$

Applying Lemma 3, the theorem is proved. This result is sharp for the functions

$$\frac{\alpha z^2 f''(z) + zf'(z)}{\alpha (f(z) - f(-z))^\gamma + (1 - \alpha)(f(z) - f(-z))} = \frac{1 + z}{1 - z}$$

or

$$\frac{\alpha z^2 f''(z) + zf'(z)}{\alpha (f(z) - f(-z))^\gamma + (1 - \alpha)(f(z) - f(-z))} = \frac{1 + z^2}{1 - z^2}.$$
For $\mu = 1$, we get $H_2(1)$.

**Corollary 8.** If $f \in M_s(\alpha)$, then

$$|a_3 - a_2^2| \leq \frac{1}{1 + 2\alpha}.$$  

**Corollary 9.** [2] If $f \in S_s^*$, then

$$|a_3 - a_2^2| \leq 1.$$  

**Corollary 10.** [2] If $f \in K_s$, then

$$|a_3 - a_2^2| \leq \frac{1}{3}.$$  

**Theorem 11.** Let $f \in M_s(\alpha)$. Then we have the sharp inequality

$$|H_3(1)| \leq \frac{1}{(1 + 2\alpha)^3(1 + 3\alpha)^2(1 + 4\alpha)} \cdot 
\max \left\{ 52\alpha^4 + 124\alpha^3 + 88\alpha^2 + 25\alpha + 2.5; \frac{D_1 + D_2\sqrt{3(4\alpha^2 + 3\alpha + 1)(1 + \alpha)(1 + 2\alpha)}}{9(1 + \alpha)^2} \right\}, \quad (18)$$

where

$D_1 = 18(1 + \alpha)^2(1 + 3\alpha)^2(2\alpha^2 + 4\alpha + 1)$ and $D_2 = 2(1 + 2\alpha)(1 + 4\alpha)(4\alpha^2 + 3\alpha + 1)$.

**Proof.** Since $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and applying the triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \quad (19)$$

By comparing the coefficients on both sides of equation (10) and using Lemma 1 we have the sharp estimations

$$|a_3| \leq \frac{1}{1 + 2\alpha}, \quad |a_4| \leq \frac{1}{1 + 3\alpha} \quad \text{and} \quad |a_5| \leq \frac{1}{1 + 4\alpha}. \quad (20)$$

Using the known inequality $|a_2a_4 - a_3^2| \leq \frac{1}{(1 + 2\alpha)^2}$ (see [20]) and (20) together with Theorem 4 and Corollary 8 in (19), the theorem is proved. The inequality (18) is sharp because each of the components functionals in (19) is sharp.
Corollary 12. [2] If \( f \in S_s^* \), then
\[
|H_3(1)| \leq \frac{5}{2}.
\]

Corollary 13. [2] If \( f \in K_s \), then
\[
|H_3(1)| \leq \frac{19}{135}.
\]

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