**SOME NEW DISTORTION THEOREMS FOR STARLIKE HARMONIC FUNCTIONS OF ORDER ALPHA**

H. E. Özkan Uçar, M. Aydoğan, Y. Polatoğlu

**Abstract.** Let \( f(z) = h(z) + g(z) \) where \( h(z) \) and \( g(z) \) are analytic functions in \( U \). If \( f(z) \) satisfies the condition \(|h'(z)|^2 - |g'(z)|^2 > 0\), then \( f(z) \) is called sense-preserving harmonic univalent function and denoted by \( S_H \). We also note that \( f(z) = h(z) + g(z) \in S_H \) if and only if \( g'(z) = \omega(z)h'(z) \) where \( \omega(z) \) is second dilatation of \( f(z) \). Moreover, let \( H(U) \) be the linear space of all analytic functions defined on the simply connected domain \( U \subset \mathbb{C} \). A log-harmonic mapping \( F \) is a solution of the non-linear elliptic partial differential equation \( \frac{F_z}{F} = \omega_1(z) \frac{F_z}{F} \), where the second dilatation function \( \omega_1(z) \in H(U) \) is such that \(|\omega_1(z)| < 1\) for all \( z \in U \). It has been shown that if \( F \) is non-vanishing log-harmonic mapping, then \( F \) can be expressed on \( F = H(z)G(z) \), where \( H(z) \) and \( G(z) \) are analytic functions in \( U \) with the normalization \( H(0) = G(0) = 1 \), and the class of non-vanishing log-harmonic functions is denoted by \( S_{LH}^* \).

The aim of this paper is to give the relation between the classes \( S_H^* \) and \( S_{LH}^* \) the new distortion theorems of starlike harmony univalent functions of LH order \( \alpha \).

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1. **Introduction**

Let \( S^*(\alpha) \) denote the class of functions \( s(z) = z + a_2z^2 + \ldots \) which are analytic in the open unit disc \( U = \{ z : |z| < 1 \} \) and satisfy

\[
\Re \left( \frac{s'(z)}{s(z)} \right) > \alpha
\]

for all \( z \in U \).
Next, let $\Omega$ be the family of functions $\phi(z)$ which are analytic in $U$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in U$. Let $\mathcal{P}$ denote the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ which are regular and satisfy the conditions $\text{Re} \, p(z) > \alpha$, $p(0) = 1$ for all $z \in U$, and we note that $p(z) \in \mathcal{P}$ if and only if

$$ p(z) = \frac{1 + (1 - 2\alpha)\phi(z)}{1 - \phi(z)} \quad (2) $$

for some function $\phi(z) \in \Omega$ and every $z \in U$, see [4].

Moreover, let $f_1(z) = z + d_2 z^2 + \ldots$ and $f_2(z) = z + e_2 z^2 + \ldots$ be analytic functions in $U$. If there exists a function $\phi(z) \in \Omega$ such that $f_1(z) = f_2(\phi(z))$, we then say that $f_1(z)$ is subordinate to $f_2(z)$ and we write $f_1(z) \prec f_2(z)$.

Finally, a function $f$ is said to be a complex valued harmonic function in $U$ if both $\text{Re} \, f$ and $\text{Im} \, f$ are real harmonic in $U$. Every such $f$ can be uniquely represented by $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic with the normalization $h(0) = g(0) = 0$, $h'(0) = 1$. A complex-valued harmonic function $f$ which is not identically constant and satisfies $f = h(z) + g(z)$ is said to be sense-preserving in $U$ if it satisfies the equation

$$ g'(z) = \omega(z)h'(z) \quad (3) $$

where $\omega(z)$ is analytic in $U$ with $|\omega(z)| < 1$ for every $z \in U$ and $\omega(z)$ is called the second dilatation of $f$. The Jacobian of $f$ is defined by

$$ J_f(z) = |h'(z)|^2 - |g'(z)|^2 \quad (4) $$

Let $H(U)$ be the linear space of all analytic functions defined on the open unit disc $U$. A log-harmonic mapping $F$ is the solution of the non-linear elliptic partial differential equation

$$ \frac{F_z}{F} = \omega(z) \frac{F_z}{F} \quad (5) $$

where $\omega(z)$ is the second dilatation of $F$ and $\omega(z) \in H(U)$, $|\omega(z)| < 1$ for every $z \in U$. It has been show that if $F$ is a non-vanishing log-harmonic function, then $F$ can be expressed as

$$ F = H(z) \cdot G(z) \quad (6) $$

where $H(z)$ and $G(z)$ are analytic in $U$ with the normalization $H(0) = G(0) = 1$. The class of non-vanishing log-harmonic functions is denoted by $S^0_{LH}$. Also, the class of log-harmonic functions is denoted by $S_{LH}$. For details, see [1], [2], and [3].

In [5], Jack’s lemma states that for the (non-constant) function $\omega(z)$ which is analytic in $U$ with $\omega(0) = 0$, if $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0 \omega'(z_0) = k \omega(z_0)$, where $k$ is a real number and $k \geq 1$. 

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2. Main Results

**Theorem 1.** \( f = h(z) + \bar{g}(z) \in S_0^* \iff F = H(z)\overline{G(z)} = e^{h(z) + \bar{g}(z)} \in S_{LH}^0 \).

**Proof.** Let \( f = h(z) + \overline{g(z)} \in S_H \). Then we have
\[
\omega(z) = \frac{g'(z)}{h'(z)}. \tag{7}
\]
Now we define the function
\[
\begin{align*}
H(z) &= e^{h(z)} \\
G(z) &= e^{g(z)}
\end{align*}
\Rightarrow F = H(z) \cdot \overline{G(z)} = e^{h(z) + \bar{g}(z)}, \tag{8}
\]
then we have
\[
\begin{align*}
\log H(z) &= h(z) \implies h'(z) = \frac{H'(z)}{H(z)} \\
\log G(z) &= g(z) \implies g'(z) = \frac{G'(z)}{G(z)}, \tag{9}
\end{align*}
\]
\[
\begin{align*}
H(0) &= e^{h(0)} = e^0 = 1 \\
G(0) &= e^{g(0)} = e^0 = 1 \implies F(0) = H(0)\overline{G(0)} = 1, \tag{10}
\end{align*}
\]
\[
\omega(z) = \frac{g'(z)}{h'(z)} = \frac{G'(z)/G(z)}{H'(z)/H(z)} \iff \frac{F_z}{F} = \omega(z) \frac{F_z}{F}. \tag{11}
\]
Therefore, \( F = H(z)\overline{G(z)} \in S_{LH}^0 \).

Conversely, let \( F = H(z)\overline{G(z)} \in S_{LH}^0 \). Then we define the following functions
\[
\begin{align*}
\log H(z) &= h(z) \\
\log G(z) &= g(z)
\end{align*} \tag{12}
\]
Then,
\[
\begin{align*}
h(0) &= \log H(0) = \log 1 = 0 \\
g(0) &= \log G(0) = \log 1 = 0
\end{align*}
\]
h(z) and g(z) are analytic in U and also we have (11). Using (9) in (11) we obtain
\[
\omega(z) = \frac{g'(z)}{h'(z)} \text{ this shows that } f = h(z) + \overline{g(z)} \in S_H.
\]

**Lemma 2.** The starlike condition of \( F = H(z)\overline{G(z)} = e^{h(z) + \bar{g}(z)} \) is
\[
\text{Re}(zh'(z) - zg'(z)) > 0. \tag{13}
\]

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Proof.

\[ F = H(z)G(z) = e^{h(z) + g(z)} \]

\[ \Rightarrow F_z = h'(z)e^{h(z) + g(z)} \Rightarrow zF_z = zh'(z)e^{h(z) + g(z)} \]

\[ F_\bar{z} = g'(\bar{z})e^{h(z) + g(z)} \Rightarrow \overline{F_\bar{z}} = \overline{zg'(\bar{z})e^{h(z) + g(z)}} \]

\[ \Rightarrow \frac{zF_z - \overline{zF_\bar{z}}}{F} = \frac{e^{h(z) + g(z)} \cdot [zh'(z) - \overline{zg'(z)}]}{e^{h(z) + g(z)}} = zh'(z) - \overline{zg'(z)} \]

\[ \Rightarrow \text{Re} \left( \frac{zF_z - \overline{zF_\bar{z}}}{F} \right) = \text{Re}(zh'(z) - \overline{zg'(z)}) = \text{Re}(zh'(z) - zg'(z)) > 0. \]

Lemma 3. Let \( f = h(z) + g(z) \) be an element of \( S^*_H \). Then,

\[ \text{Re}(zh'(z) - zg'(z)) = r \frac{\partial}{\partial r} \log |e^{h(z) - g(z)}| \quad (14) \]

Proof.

\[ e^{h(re^{i\theta}) - g(re^{i\theta})} = \left| e^{h(re^{i\theta}) - g(re^{i\theta})} \right| e^{i\theta} \]

\[ \Rightarrow \log(e^{h(re^{i\theta}) - g(re^{i\theta})}) = \log |e^{h(re^{i\theta}) - g(re^{i\theta})}|e^{i\theta} \]

\[ \Rightarrow h(re^{i\theta}) - g(re^{i\theta}) = \log |e^{h(re^{i\theta}) - g(re^{i\theta})}| + i\theta \log e = \log |e^{h(re^{i\theta}) - g(re^{i\theta})}| + i\theta \]

\[ \Rightarrow e^{i\theta} \cdot h'(re^{i\theta}) - e^{i\theta} \cdot g(re^{i\theta}) = \frac{\partial}{\partial r} \log |e^{h(re^{i\theta}) - g(re^{i\theta})}| \]

\[ \Rightarrow re^{i\theta} \cdot h'(re^{i\theta}) - re^{i\theta} \cdot g'(re^{i\theta}) = r \frac{\partial}{\partial r} \log |e^{h(re^{i\theta}) - g(re^{i\theta})}| \]

\[ \Rightarrow zh'(z) - zg'(z) = r \frac{\partial}{\partial r} \log |e^{h(z) - g(z)}| \]

\[ \Rightarrow \text{Re}(zh'(z) - zg'(z)) = r \frac{\partial}{\partial r} \log |e^{h(z) - g(z)}| \]

\[ \Rightarrow \text{Re}(zh'(z) - zg'(z)) = r \frac{\partial}{\partial r} \log |e^{h(z) - g(z)}|. \]

Theorem 4. Let \( f = h(z) + g(z) \) be an element of \( S^*_H \). The function \( f \) satisfies the condition

\[ zh'(z) - zg'(z) < \frac{2(1 - \alpha)z}{1 - z} \quad (15) \]

if and only if \( F = ze^{h(z) + g(z)} \in S^*_LH(\alpha) \).
Proof. Let \( f \) satisfies (15). We define the function \( \phi(z) \in \Omega \) by
\[
e^{h(z) - g(z)} = (1 - \phi(z))^{-2(1 - \alpha)},
\]
where \((1 - \phi(z))^{-2(1 - \alpha)}\) has the value \(1\) at \(z = 0\) (we consider the corresponding Riemann branch). Then \( \phi(z) \) is analytic and \( \phi(0) = 0 \). If we take the logarithmic derivative of (16) and after the brief calculations we get
\[
h'(z) - g'(z) = \frac{-2(1 - \alpha)(-\phi'(z))}{1 - \phi(z)}
\]
and then
\[
zh'(z) - zg'(z) = \frac{2(1 - \alpha)z\phi'(z)}{1 - \phi(z)}.
\]
On the other hand, the function \( w := \frac{2(1 - \alpha)z}{1 - z} \) maps \(|z| = r\) onto the circle with the radius \( \rho = \rho(r) = \frac{2(1 - \alpha)r}{1 - r^2} \) and the center \( c = c(r) = \left( \frac{2(1 - \alpha)r^2}{1 - r^2}, 0 \right) \). Now it is easy to realize that the subordination (15) is equivalent to \(|\phi(z)| < 1\) for all \(z \in \mathbb{U}\). Indeed, let us assume to the contrary. Then there is a \(z_1 \in \mathbb{U}\) such that \(|\phi(z_1)| = 1\). By Jack’s Lemma, \(z_1\phi'(z_1) = k\phi(z_1)\) for some \(k \geq 1\), so for such \(z_1\) we have
\[
z_1h'(z_1) - z_1g'(z_1) = \frac{2(1 - \alpha)k\phi(z_1)}{1 - \phi(z_1)} = kw(\phi(z_1)) \notin S(\mathbb{U})
\]
but this contradicts to (15); so our assumption is wrong, i.e., \(|\phi(z)| < 1\) for every \(z \in \mathbb{U}\). By using the condition (15) we get
\[
1 + zh'(z) - zg'(z) = \frac{1 + (1 - 2\alpha)\phi(z)}{1 - \phi(z)}.
\]
On the other hand, using Theorem 1, Lemma 2, and Lemma 3 and after simple calculations we get
\[
F = zh(z)G(z) = ze^{h(z) + g(z)} \in S^*_{LH}
\]
\[\Rightarrow \log F = \log z + \log H(z) + \log G(z) = \log z + \log h(z) + \log g(z)\]
\[
\Rightarrow \begin{cases} 
\frac{F_z}{F} = \frac{1 + H'(z)}{zH(z)} = \frac{1 + h'(z)}{z} \Rightarrow zF_z = 1 + zH'(z) = 1 + zh'(z) \\
\frac{F_z}{G(z)} = \frac{G'(z)}{G(z)} = g'(z) \Rightarrow zF_z = zG'(z) = zg'(z)
\end{cases}
\]
\[ \Re \left( \frac{zF_z - \overline{zF_z}}{F} \right) = \Re \left( 1 + \frac{H'(z) - \overline{G'(z)}}{H(z)} \right) = \Re(1 + z h'(z) - \overline{z g'(z)}). \] (19)

Considering (18) and (19) together we obtain the desired result.

For the converse, let \( F = z e^{h(z) + g(z)} \) be an element of \( S_{LH}^*(\alpha) \). It follows that

\[ \Re \left( \frac{zF_z - \overline{zF_z}}{F} \right) > \alpha \text{ and } \]

\[ \frac{zF_z - \overline{zF_z}}{F} = \frac{1 + (1 - 2\alpha)\phi(z)}{1 - \phi(z)}. \]

On the other hand,

\[ \Re \left( \frac{zF_z - \overline{zF_z}}{F} \right) = \Re(1 + z h'(z) - \overline{z g'(z)}) > \alpha \]

\[ \Rightarrow 1 + z h'(z) - z g'(z) = \frac{1 + (1 - 2\alpha)\phi(z)}{1 - \phi(z)} \]

\[ \Rightarrow z h'(z) - z g'(z) = \frac{2(1 - \alpha)\phi(z)}{1 - \phi(z)}. \]

This shows that \( z h'(z) - z g'(z) < \frac{2(1 - \alpha)z}{1 - z} \).

**Theorem 5.** Let \( f(z) = h(z) + \overline{g(z)} \) be an element of \( S_{H}^*(\alpha) \). Then,

\[ \frac{(1 + r)^{2\alpha - 3}}{r(1 - r)} \leq |e^{h(z) - \overline{g(z)}|} \leq \frac{(1 - r)^{2\alpha - 3}}{r(1 + r)}. \]

This inequality is sharp because if we consider the following simple calculations:

\[ h(z) - g(z) = \log(1 - z)^{-2(1 - \alpha)} \]

\[ \Rightarrow h(z) - g(z) = -2(1 - \alpha) \log(1 - z) \]

\[ \Rightarrow h'(z) - g'(z) = \frac{2(1 - \alpha)}{1 - z} \]

\[ \Rightarrow z h'(z) - z g'(z) = \frac{2(1 - \alpha)z}{1 - z} \]

\[ \Rightarrow 1 + z h'(z) - z g'(z) = 1 + \frac{2(1 - \alpha)z}{1 - z} = \frac{1 + (1 - 2\alpha)z}{1 - z} \]

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then the extremal function is the solution of the following differential equation
\[ h(z) - g(z) = \log(1 - z)^{-2(1 - \alpha)} \]
\[ g_\pi = \mathcal{F}_z - \mathcal{H}_z = 0. \]

**Proof.** The set of the values of the function \( \frac{2(1 - \alpha)z}{1 - z} \) is the closed disc with the center \( c \) and the radius \( \rho \), where
\[ c = c(r) = \left( \frac{2(1 - \alpha)r^2}{1 - r^2}, 0 \right), \quad \rho = \rho(r) = \frac{2(1 - \alpha)r}{1 - r^2}. \]

Using the subordination, we can write
\[
\begin{align*}
\left| (zh'(z) - zg'(z) + 1) - \frac{2(1 - \alpha)r^2}{1 - r^2} \right| &\leq \frac{2(1 - \alpha)r}{1 - r^2} \\
\Rightarrow \left| (zh'(z) - zg'(z)) + 1 - \frac{2(1 - \alpha)r^2}{1 - r^2} \right| &\leq \frac{2(1 - \alpha)r}{1 - r^2} \\
\Rightarrow \left| (zh'(z) - zg'(z)) - \left( \frac{2(1 - \alpha)r^2}{1 - r^2} - 1 \right) \right| &\leq \frac{2(1 - \alpha)r}{1 - r^2} \\
\Rightarrow \left| (zh'(z) - zg'(z)) - \left( \frac{2(1 - \alpha)r^2 - 1 + r^2}{1 - r^2} - 1 \right) \right| &\leq \frac{2(1 - \alpha)r}{1 - r^2} \\
\Rightarrow \left| (zh'(z) - zg'(z)) - \left( \frac{3 - 2\alpha r^2 - 1}{1 - r^2} - 1 \right) \right| &\leq \frac{2(1 - \alpha)r}{1 - r^2} \\
- \frac{2(1 - \alpha)r}{1 - r^2} \leq - \left| (zh'(z) - zg'(z)) - \left( \frac{3 - 2\alpha r^2 - 1}{1 - r^2} - 1 \right) \right| \\
\leq \text{Re} \left[ (zh'(z) - zg'(z)) - \left( \frac{3 - 2\alpha r^2 - 1}{1 - r^2} - 1 \right) \right] \\
\left| (zh'(z) - zg'(z)) - \left( \frac{3 - 2\alpha r^2 - 1}{1 - r^2} - 1 \right) \right| &\leq \frac{2(1 - \alpha)r}{1 - r^2} \\
\Rightarrow - \frac{2(1 - \alpha)r}{1 - r^2} \leq \text{Re}[zh'(z) - zg'(z)] - \frac{3 - 2\alpha r^2 - 1}{1 - r^2} \leq \frac{2(1 - \alpha)r}{1 - r^2} \\
\Rightarrow \frac{(3 - 2\alpha)r^2 - 1}{1 - r^2} - \frac{2(1 - \alpha)r}{1 - r^2} \leq \text{Re}[zh'(z) - zg'(z)] \leq \frac{(3 - 2\alpha)r^2 - 1}{1 - r^2} + \frac{2(1 - \alpha)r}{1 - r^2} \\
\Rightarrow \frac{(3 - 2\alpha)r^2 - 2(1 - \alpha)r - 1}{1 - r^2} \leq \text{Re}[zh'(z) - zg'(z)] \leq \frac{(3 - 2\alpha)r^2 + 2(1 - \alpha)r - 1}{1 - r^2}
\]
(20)
On the other hand, from Lemma 3 we have
\[ \text{Re}[zh'(z) - zg'(z)] = r \frac{\partial}{\partial r} \log |e^{h(z)-g(z)}|. \] (21)

Considering (20) and (21) together, then the inequality (20) can be written in the following form
\[
\frac{(3 - 2\alpha)r^2 - 2(1 - \alpha)r - 1}{1 - r^2} \leq r \frac{\partial}{\partial r} \log |e^{h(z)-g(z)}| \leq \frac{(3 - 2\alpha)r^2 + 2(1 - \alpha)r - 1}{1 - r^2}
\] (22)

Since
\[
\frac{(3 - 2\alpha)r^2 - 2(1 - \alpha)r - 1}{r(1 - r^2)} = -\frac{1}{r} + \frac{1}{1 - r} + \frac{2\alpha - 3}{1 + r},
\]

It follows that
\[
\int \frac{(3 - 2\alpha)r^2 - 2(1 - \alpha)r - 1}{r(1 - r^2)} \, dr = \log \frac{(1 + r)^{2\alpha - 3}}{r(1 + r)}
\] (23)

Similarly, since
\[
\frac{(3 - 2\alpha)r^2 + 2(1 - \alpha)r - 1}{r(1 - r^2)} = -\frac{1}{r} - \frac{1}{1 - r} + \frac{3 - 2\alpha}{1 - r},
\]

it follows that
\[
\int \frac{(3 - 2\alpha)r^2 + 2(1 - \alpha)r - 1}{r(1 - r^2)} \, dr = \log \frac{(1 - r)^{2\alpha - 3}}{r(1 + r)}.
\] (24)

Considering (22), (23), (24) and integrating both sides of (22) we obtain
\[
\frac{(1 + r)^{2\alpha - 3}}{r(1 - r)} \leq \left| e^{h(z)-g(z)} \right| \leq \frac{(1 - r)^{2\alpha - 3}}{r(1 + r)}.
\]

**Corollary 6.** Let \( f(z) = h(z) + \overline{g(z)} \) be an element of \( S_H^*(\alpha) \). Then,
\[
\left| (e^{h(z)-g(z)})^{\frac{1}{2\alpha(1-\alpha)}} - 1 \right| < 1.
\]

This inequality is the Marx-Strohhacker inequality [4] for the starlike harmonic univalent functions of order \( \alpha \).
Proof. Using Theorem 4, we have
\[ e^{h(z)-g(z)} = (1 - \phi(z))^{-2(1-\alpha)}. \]
This equality shows that
\[ e^{h(z)-g(z)} = \frac{1}{(1 - \phi(z))^{-2(1-\alpha)}} \Rightarrow \frac{1}{(e^{h(z)-g(z)})^{2(1-\alpha)} - 1} = | - \phi(z) | < 1. \]

Corollary 7. Let \( f(z) = h(z) + \overline{g(z)} \) be an element of \( S_{H}(\alpha) \). Then,
\[ |h'(z) - g'(z)| < \frac{2(1 - \alpha)}{1 - r}. \]

Proof. Let \( s(z) := (e^{h(z)-g(z)})^{2(1-\alpha)} - 1 \). Then by Corollary 7 and (16), we have \( s(0) = 0, |s(z)| < 1 \) and \( s(z) = z\phi(z) \). Since
\[ z\phi(z) = \frac{1}{(e^{h(z)-g(z)})^{2(1-\alpha)} - 1} \]
?we have
\[ h(z) - g(z) = 2(1 - \alpha) \log(1 + z\phi(z)). \]
So,
\[ h'(z) - g'(z) = \frac{2(1 - \alpha)(\phi(z) + z\phi'(z))}{1 + z\phi(z)} \]
and hence
\[ |h'(z) - g'(z)| \leq \frac{2(1 - \alpha)}{1 - r}. \]

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References


H. Esra Özkanc Uçar  
Department of Mathematics and Computer Science  
İstanbul Kültür University,  
İstanbul, Turkey  
email: e.ozkan@iku.edu.tr

Melike Aydoğan  
Department of Mathematics,  
İşık University,  
İstanbul, Turkey  
email: melike.aydogan@isikun.edu.tr

Yaşar Polatolu  
Department of Mathematics and Computer Science  
İstanbul Kültür University,  
İstanbul, Turkey  
email: y.polatoglu@iku.edu.tr