A SANDWICH THEOREM ON THE $\phi$-LIKE FUNCTIONS INVOLVING $I_N \ast \mathcal{L}_C(A, B)$ OPERATOR

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Abstract. In this paper, we introduce a new convolution operator $I_n \ast \mathcal{L}_C(a, b)$. Several subordination and superordination results involving this operator are proved.

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1. Introduction

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \]  \hspace{1cm} (1)

which are analytic in the open unit disk $U := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $H(U)$ be the linear space of all analytic functions in $U$. For a positive integer number $n$ and $a \in \mathbb{C}$, we let

\[ H[a, n] := \left\{ f \in H(U) : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\}. \]

Let $f, g \in A$, where $f$ is given by (1) and $g$ is defined by

\[ g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \]

Then the Hadamard product (or convolution) $f \ast g$ of the functions $f$ and $g$ is defined by

\[ (f \ast g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z). \]
For two functions $f$ and $g$, analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$, and we denote it by $f(z) \prec g(z)$, if there exists a Schwarz function $w$, which is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for $(z \in U)$, such that [1-15]

$$f(z) = g(w(z)), \quad (z \in U).$$

Indeed, it is known that

$$f(z) \prec g(z) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the differential subordination $\varphi(p(z), zp'(z)) \prec h(z)$ then $p$ is called a solution of the differential subordination [12-18]. The univalent function $q$ is called a dominant of the solutions of the differential subordination, $p \prec q$. If $p$ and $\varphi(p(z), zp'(z))$ are univalent in $U$ and satisfy the differential superordination $h(z) \prec \varphi(p(z), zp'(z))$ then $p$ is called a solution of the differential superordination. An analytic function $q$ is called subordinant of the solution of the differential superordination if $q \prec p$ [19-26].

Denote by $D^\alpha : A \to A$ the operator defined by

$$D^\alpha f(z) := \frac{z}{(1-z)^{\alpha+1}} \ast f(z), \quad \alpha > -1,$$

where $(\ast)$ refers to the Hadamard product or convolution. Then implies that

$$D^n f(z) = \frac{z}{n!} \left(\frac{z^{-1} f^{(n)}(z)}{1-z}\right), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

We note that $D^0 f(z) = f(z)$ and $D' f(z) = zf'(z)$. The operator $D^n f$ is called Ruscheweyh derivative of $n$’th order of $f$ [27-29]. Ali et al [2, 3] defined and studied an integral operator $I_n : A \to A$ analogous to $D^n f$ as follows: Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$, $n \in \mathbb{N}_0$ and let $f_n^{(-1)}$ be defined such that

$$f_n(z) \ast f_n^{(-1)}(z) = \frac{z}{(1-z)}.$$  \hspace{1cm} \text{(2)}$$

Then

$$I_n f(z) = f_n(z) \ast f_n^{(-1)}(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} \ast f(z).$$
Note that \( I_0 f(z) = zf'(z) \) and \( I_1 f(z) = f(z) \). The operator \( I_n \) is called the Noor Integral of \( n \)'th order of \( f \). Using (1), (2) and a well-known identity for \( D^n f \), we have

\[
(n + 1) I_n f(z) - n I_{n+1}(z) = z(I_{n+1}(z))'.
\]

(3)

Using hypergeometric functions \( _2F_1 \), (2) becomes

\[
I_n f(z) = \left[ \frac{z}{1-\frac{z}{2}} \right] \cdot f(z),
\]

where \( _2F_1(a, b; c, z) \) is defined by

\[
_2F_1(a, b; c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k,
\]

For two functions \( f_j(z), (j = 1, 2) \), given by

\[
f_j(z) = z + \sum_{k=2}^{\infty} a_{kj} z^k, \quad (j = 1, 2).
\]

In terms of the Pochhammer symbol (or the shifted factorial), define \( (k)_n \) by

\[
(k)_0 = 1, \quad (k)_n = k(k + 1)(k + 2) \cdots (k + n - 1), \quad (n \in \mathbb{N}),
\]

and then define a function \( \phi_c(a, b) \) by

\[
\phi_c(a, b) = 1 + \sum_{n=1}^{\infty} (c + n) \frac{(a)_n}{(b)_n n!} (z^n + 1).
\]

(4)

Hence

\[
Z(\phi_c(a, b))' = a\phi_c(a + 1, b) - a \phi_c(a, b)
\]

(6)

It is easy to see that

\[
Z(\phi_c(a, b))' = a \phi_c(a + 1, b) - a \phi_c(a, b)
\]

(7)

we define the Hadamard product (or convolution) of \( I_n f(z) \) and \( L_c(a, b) f(z) \) by

\[
I_n f(z) * L_c(a, b) f(z) = \left[ \frac{z}{(1-z)^{n+1}} \right] \cdot f(z) * \phi_c(a, b) * \frac{f(z)}{z}
\]

\[
= \left( \frac{f(z)}{z} \right)^2 (1-z)^{n+1} \cdot \phi_c(a, b)
\]

\[
= \left[ \left( \frac{f(z)}{z} \right)^2 (1-z)^{n+1} \right] \cdot \left[ 1 + \sum_{n=1}^{\infty} \frac{(c + n)(a)_n}{(b)_n n} \right]
\]

(8)
Furthermore, we have
\[
\frac{z[I_n+1 \ast \mathcal{L}_c(a,b)f(z)]'}{\phi[I_n+1 \ast \mathcal{L}_c(a,b)f(z)]} = (n+1)(c+1)\frac{z[I_n \ast \mathcal{L}_c(a,b)f(z)]'}{[I_n \ast \mathcal{L}_c(a,b)f(z)]} - (n+1)(c+1).
\]

**Definition 1.** Let \( \phi \) be an analytic function in a domain containing \( f(U) \), \( \phi(0) = 0 \) and \( \phi'(0) > 0 \). The function \( [I_n \ast \mathcal{L}_c(a,b)f] \in A \) is called \( \phi \)-like if
\[
\operatorname{Re} \frac{z[I_n \ast \mathcal{L}_c(a,b)f(z)]'}{\phi[I_n \ast \mathcal{L}_c(a,b)f(z)]} > 0, \quad (z \in U).
\] (9)

**Definition 2.** Let \( \phi \) be analytic function in a domain containing \( f(U) \), \( \phi(0) = 0 \), \( \phi'(0) = 1 \) and \( \phi(\omega) \neq 0 \) for \( \omega \in f(U) - 0 \). Let \( q(z) \) be a fixed analytic function in \( U \), \( q(0) = 1 \). The function \( [I_n \ast \mathcal{L}_c(a,b)f] \in A \) is called \( \phi \)-like with respect to
\[
\frac{z[I_n \ast \mathcal{L}_c(a,b)f(z)]'}{\phi[I_n \ast \mathcal{L}_c(a,b)f(z)]} \prec q(z), \quad (z \in U).
\] (10)

### 2. Preliminaries

To derive our main results, we need the following definitions and lemmas.

**Definition 3.** A function \( L(z,t) \), \( (z \in U, \ t \geq 0) \) is said to be a subordination chain if \( L(0,t) \) is analytic and univalent in \( U \) for all \( t \geq 0 \), \( L(z,0) \) is continuously differentiable on \([0,1)\) for all \( z \in U \) and \( L(z,t_1) \prec L(z,t_2) \) for all \( 0 \leq t_1 \leq t_2 \).

**Remark 1.** Denote by \( Q \) the set of all functions \( f \) that are analytic and injective on \( \overline{U} - E(f) \), where
\[
E(f) = \{ \varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty \},
\] and such that \( f'(\varepsilon) \neq 0 \) for \( \varepsilon \in \partial U - E(f) \). The subclass of \( Q \) for which \( f(0) = a \), \( (a \in \mathbb{C}) \), is denoted by \( Q(a) \).

**Lemma 1.** The function \( L(z,t) : U \times [0,\infty) \to \mathbb{C} \) of the form
\[
L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots, \quad (a_1(t) \neq 0; \ t \geq 0),
\] and \( \lim_{t \to \infty} |a_1(t)| = \infty \) is a subordination chain if and only if
\[
\operatorname{Re} \left( \frac{z\partial L/\partial z}{\partial L/\partial t} \right) > 0, \quad (z \in U; \ t \geq 0).
\]
Lemma 2. Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition $\text{Re}(H(is,t)) \leq 0$ for all real $s$ and for all $t \leq -\frac{n(1 + s^2)}{2}$, $(n \in \mathbb{N})$.

If the function
\[ p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots, \]
is analytic in $U$ and $\text{Re}(H(p(z), zp'(z))) > 0$, $(z \in U)$, then $\text{Re}(p(z)) > 0$, $(z \in U)$.


Lemma 3. Let $k, \gamma \in \mathbb{C}$ with $k \neq 0$ and let $h \in H(U)$ with $h(0) = c$. If $\text{Re}(kh(z) + \gamma) > 0$, $(z \in U)$, then the solution of the following differential equation:
\[ q(z) + \frac{zq'(z)}{kq(z) + \gamma} = h(z), \quad (z \in U, \; q(0) = c), \]
is analytic in $U$ and satisfies the inequality given by $\text{Re}(kq(z) + \gamma) > 0$, $(z \in U)$.


Lemma 4. Let $p \in Q(a)$ and
\[ q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad (q \neq a, \; n \in \mathbb{N}). \]
be analytic in $U$. If $q$ is not subordinate to $p$, then there exists two points
\[ z_0 = r_0 e^{i\theta} \in U, \quad \text{and} \quad \varepsilon_0 \in \partial U/\mathcal{E}(f), \]
such that $q(U_{r_0}) \subset p(U)$, $q(z_0) = p(\varepsilon_0)$ and $z_0q'(z_0) = m_0 \varepsilon_0 p'(\varepsilon_0)$, $(m \geq n)$.


Lemma 5. Let $q \in H[a,1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$. Also set
\[ \phi(q(z), zq'(z)) \equiv h(z), \quad (z \in U). \]

Let
\[ L(z,t) := \phi(q(z), tzq'(z)), \]
be a subordination chain and $p \in H[a,1] Q(a)$. Then $h(z) \prec \phi(p(z), zp'(z))$ implies that $q(z) \prec p(z)$. Furthermore, if $\phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q(a)$, then $q$ is the best subordinate.

3. Main Results

We begin by presenting our first subordination property given by Theorem 6, below. For convenience, let

\[ A_0 := \{ f \in A : [I_n \ast L_c(a, b)]f(z) \neq 0, \quad (z \in U) \}. \]

**Theorem 6.** Let \( f, g \in A \) and \( a \in \mathbb{C}, \ \text{Re}(nc) > 0 \). Further let

\[ \Re \left( 1 + \frac{z\varphi''(z)}{\varphi'(z)} \right) > -\delta, \quad (z \in U, \ \varphi(z) := \frac{z[I_n \ast L_c(a, b)g(z)]'}{\phi[I_n \ast L_c(a, b)g(z)]}, \quad (11) \]

where

\[ \delta := \frac{1 + (nc)^2 - |1 - (nc)^2|}{4\Re(nc)}. \quad (12) \]

Then the subordination

\[ \frac{z[I_n \ast L_c(a, b)f(z)]'}{\phi[I_n \ast L_c(a, b)f(z)]} \prec \frac{z[I_n \ast L_c(a, b)g(z)]'}{\phi[I_n \ast L_c(a, b)g(z)]}, \]

implies that

\[ \frac{z[I_{n+1} \ast L_{c+1}(a, b)f(z)]'}{\phi[I_{n+1} \ast L_{c+1}(a, b)f(z)]} \prec \frac{z[I_{n+1} \ast L_{c+1}(a, b)g(z)]'}{\phi[I_{n+1} \ast L_{c+1}(a, b)g(z)]}. \]

Furthermore, the function \( \frac{z[I_{n+1} \ast L_{c+1}(a, b)g(z)]'}{\phi[I_{n+1} \ast L_{c+1}(a, b)g(z)]} \) is the best dominant.

**Proof.** Let the functions \( F, G \) and \( Q \) be defined by

\[ F := \frac{z[I_{n+1} \ast L_{c+1}(a, b)f(z)]'}{\phi[I_{n+1} \ast L_{c+1}(a, b)f(z)]}, \quad G := \frac{z[I_{n+1} \ast L_{c+1}(a, b)g(z)]'}{\phi[I_{n+1} \ast L_{c+1}(a, b)g(z)]}, \quad \]

\[ Q := 1 + \frac{z\varphi''(z)}{\varphi'(z)}. \quad (13) \]

We assume here, without loss of generality, that \( G \) is analytic and univalent on \( \bar{U} \) and \( G'(\varepsilon) \neq 0, \ (|\varepsilon| = 1) \). If not, then we replace \( F \) and \( G \) by \( F(\rho z) \) and \( G(\rho z) \), respectively, with \( 0 < \rho < 1 \). These new functions have the desired properties on \( \bar{U} \), and we can use them in the proof of our result. Therefore, the result would follow by letting \( \rho \to 1 \). We first show that \( \Re(Q(z)) > 0, \ (z \in U) \). By virtue of (1) and the definitions of \( G \), we know that

\[ \varphi(z) = G(z) + \frac{1}{nc}zG'(z). \quad (14) \]
Differentiating both sides of (14) with respect to $z$ yields

$$\varphi'(z) = \left(1 + \frac{1}{nc}\right) G(z) + \frac{1}{nc} z G''(z).$$  \hspace{1cm} (15)

Combining (13) and (15), we easily get

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} = Q(z) + \frac{zQ'(z)}{Q(z) + nc} = h(z), \quad (z \in U).$$  \hspace{1cm} (16)

It follows from (11) and (16) that

$$\text{Re} (h(z) + nc) > 0, \quad (z \in U).$$  \hspace{1cm} (17)

Moreover, by Lemma 2.5, we conclude that the differential equation (16) has a solution $Q \in H(U)$ with $h(0) = Q(0) = 1$. Let $H(u, v) := u + \frac{v}{u+nc} + \delta$, where $\delta$ is given by (12). From (16) and (17), we obtain

$$\text{Re}(H(Q(z), zQ'(z))) > 0, \quad (z \in U).$$

To verify the condition that

$$\text{Re}(H(is, t)) \leq 0, \quad \left(s \in \mathbb{R}; \ t \leq -\frac{n(1+s^2)}{2}\right),$$  \hspace{1cm} (18)

we proceed as follows:

$$\text{Re}(H(is, t)) = \text{Re}\left(is + \frac{t}{is + nc} + \sigma\right) = \frac{tn}{|is + nc|^2} + \sigma \leq -\frac{\psi(n, s)}{2|is + nc|^2},$$

where

$$\psi(n, s) := (n - 2\delta)s^2 - 4\sigma ns - 2\sigma n^2 + n.$$  \hspace{1cm} (19)

For $\delta$ given by (12), we note that the coefficient of $s^2$ in the quadratic expression $\psi(n, s)$ given by (19) is positive or equal to zero. Furthermore, we observe that the quadratic expression $\psi(n, s)$ by $s$ in (19) is a perfect square, which implies that (18) holds. Thus, by Lemma 2.4, we conclude that $\text{Re}(Q(z)) > 0, \ (z \in U)$. Let $f \in H(U)$, then $f$ is convex if and only if $f'(0) \neq 0$ and $\text{Re}\left\{1 + (f''(z))/(f'(z))\right\} > 0, \ z \in U$. Now by the definition of $Q$, we know that $G$ is convex. To prove $F \prec G$, let the function $L$ be defined by

$$L(z, t) := G(z) + \frac{t}{n} zG'(z), \quad (z \in U; \ 0 \leq t < \infty).$$  \hspace{1cm} (20)
Since \( G \) is convex and \( n > 0 \), then

\[
\frac{\partial L(z,t)}{\partial z} \bigg|_{z=0} = G'(0) \left( 1 + \frac{t}{n} \right) \neq 0, \quad (z \in U; \ 0 \leq t < \infty),
\]

and

\[
\Re \left( \frac{z\partial L/\partial z}{\partial L/\partial t} \right) = \Re(n + tQ(z)) > 0, \quad (z \in U).
\]

Therefore, by Lemma 2.3, we deduce that \( P \) is a subordination chain. It follows from the definition of subordination chain that \( \varphi(z) = G(z) + \frac{1}{n} z G'(z) = L(z,0) \) and \( L(z,0) \prec L(z,t), (0 \leq t < \infty) \), which implies that

\[
L(\varepsilon, t) \notin L(U,0) = \varphi(U), \quad (\varepsilon \in U; \ 0 \leq t < \infty). \tag{21}
\]

If \( F \) is not subordinate to \( G \), by Lemma 2.6, we know that there exist two points \( z_0 \in U \) and \( \varepsilon_0 \in \partial U \) such that

\[
F(z_0) = G(\varepsilon_0) \quad \text{and} \quad z_0 F(z_0) = t\varepsilon_0 G'(\varepsilon_0), \quad (0 \leq t < \infty). \tag{22}
\]

Hence, by virtue of (1) and (22), we have

\[
L(\varepsilon_0, t) = G(\varepsilon_0) + \frac{t}{n} z_0 G'(\varepsilon_0) = F(z_0) + \frac{1}{n} z_0 F'(z_0) = \frac{I_{n+1} f(z_0)}{z_0} \in \varphi(U).
\]

This contradicts to (21). Thus, we deduce that \( F \prec G \). Considering \( F = G \), we see that the function \( G \) is the best dominant.

By similarly applying the method of proof of Theorem 3.1, as well as (1), we easily get the following result.

**Corollary 7.** Let \( f, g \in A \) and \( n > -1 \). Further let

\[
\Re \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\bar{\omega}, \quad (z \in U; \ \chi(z) := \frac{I_n g(z)}{z}),
\]

where

\[
\bar{\omega} := \frac{1 + (n + 1)^2 - |1 - (n + 1)^2|}{4(n + 1)}. \tag{23}
\]

Then the subordination \( \frac{I_{n+1} f(z)}{z} \prec \frac{I_n g(z)}{z} \), implies that \( \frac{I_{n+1} f(z)}{z} \prec \frac{I_{n+1} g(z)}{z} \). Furthermore, the function \( \frac{I_{n+1} g(z)}{z} \) is the best dominant.

If \( f \) is subordinate to \( F \), then \( F \) is superordinate to \( f \). We now derive the following superordination result.

72
Theorem 8. Let $f, g \in A_p$ and $n > 0$. Further let
\[
Re \left( 1 + \frac{z \varphi''(z)}{\varphi'(z)} \right) > -\delta, \quad \left( z \in U; \varphi(z) := \frac{I_{n+1}g(z)}{z} \right),
\] (24)
where $\delta$ is given by (12). If the function $\frac{I_{n+1}f(z)}{z}$ is univalent in $U$ and $\frac{I_nf(z)}{z} \in Q$, then the subordination
\[
\frac{I_{n+1}g(z)}{z} \prec \frac{I_nf(z)}{z},
\]
implies that
\[
\frac{I_ng(z)}{z} \prec \frac{I_nf(z)}{z}.
\]
Furthermore, the function $\frac{I_ng(z)}{z}$ is the best subordinate.

Proof. Suppose that the functions $F$ and $G$ and $Q$ are defined by (13). By applying the similar method as in the proof of Theorem 3.1, we get $Re(Q(z)) > 0$, $(z \in U)$. Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L$ be defined by (20). Since $n > 0$ and $G$ is convex, by applying a similar method as in Theorem 3.1, we deduce that $L$ is subordination chain. Therefore, by Lemma 2.7, we conclude that $G \prec F$. Moreover, since the differential equation
\[
\varphi(z) = G(z) + \frac{1}{n} z G''(z) = \phi(G(z), z G'(z)),
\]
has a univalent solution $G$, it is the best subordinate.

Applying a similar proof as in Theorem 3.2, and using (1), the following results are easily obtained.

Corollary 9. Let $A_p = \{ f \in H(U) : f(z) = a + \sum_{k=p}^{\infty} a_k z^k \}$, $f, g \in A_p$ and $n > 0$. Further let
\[
Re \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\bar{\omega}, \quad \left( z \in U; \chi(z) := \frac{I_ng(z)}{z} \right),
\]
where $\bar{\omega}$ is given by (23). If the function $\frac{I_nf(z)}{z}$ is univalent in $U$ and $\frac{I_{n+1}f(z)}{z} \in Q$, then the subordination
\[
\frac{I_ng(z)}{z} \prec \frac{I_nf(z)}{z},
\]
implies that
\[
\frac{I_{n+1}g(z)}{z} \prec \frac{I_{n+1}f(z)}{z}.
\]
Furthermore, the function $\frac{I_{n+1}g(z)}{z}$ is the best subordinate.
Combining the above mentioned subordination and super ordination results involving the operator $I_n$, the following "sandwich-type results" are derived.

**Corollary 10.** Let $f, g_k \in A, (k = 1, 2)$ and $n \geq 0$. Further let
\[
\Re \left( 1 + z \frac{\varphi''(z)}{\varphi'(z)} \right) > -\delta, \quad \left( z \in U; \varphi(z) := \frac{I_{n+1}g_k(z)}{z}, \ k = 1, 2 \right),
\]
where $\delta$ is given by (12). If the function $\frac{I_{n+1}f(z)}{z}$ is univalent in $U$ and $\frac{I_nf(z)}{z} \in Q$, then the subordination chain
\[
\frac{I_{n+1}g_1(z)}{z} \prec \frac{I_{n+1}f(z)}{z} \prec \frac{I_{n+1}g_2(z)}{z},
\]
implies that
\[
\frac{I_ng_1(z)}{z} \prec \frac{I_nf(z)}{z} \prec \frac{I_ng_2(z)}{z}.
\]
Furthermore, the functions $\frac{I_ng_1(z)}{z}$ and $\frac{I_ng_2(z)}{z}$ are, respectively, the best subordinate.

**Corollary 11.** Let $f, g_k \in A, (k = 1, 2)$ and $n \geq 0$. Further let
\[
\Re \left( 1 + z \frac{\chi''(z)}{\chi'(z)} \right) > -\bar{\omega}, \quad \left( z \in U; \chi_k(z) := \frac{I_ng_k(z)}{z}, \ k = 1, 2 \right),
\]
where $\bar{\omega}$ is given by (12). If the function $\frac{I_nf(z)}{z}$ is univalent in $U$ and $\frac{I_{n+1}f(z)}{z} \in Q$, then the subordination chain
\[
\frac{I_ng_1(z)}{z} \prec \frac{I_nf(z)}{z} \prec \frac{I_ng_2(z)}{z},
\]
implies that
\[
\frac{I_{n+1}g_1(z)}{z} \prec \frac{I_{n+1}f(z)}{z} \prec \frac{I_{n+1}g_2(z)}{z}.
\]
Furthermore, the functions $\frac{I_{n+1}g_1(z)}{z}$ and $\frac{I_{n+1}g_2(z)}{z}$ are, respectively, the best subordinate.

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75


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