PRESERVING PROPERTIES AND ESTIMATION OF THE COEFFICIENTS FOR FUNCTIONS THAT BELONG TO THE SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In this paper we give preserving properties and estimation of the coefficients for functions that belong to the subclass of analytic functions $T_{\gamma}(f, g; \alpha, \beta)$.

2010 Mathematics Subject Classification: 30C45, 30C50.

Keywords: Alexander integral operator, Bernardi integral operator, $I_{c+\delta}$ integral operator, $L_a$ operator.

1. Introduction

Let $S$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

that are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$.

Let $T$ denote the subclass of $S$ consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0).$$

Definition 1.1. [5] Let $I_A$ be a Alexander integral operator defined as:

$I_A : A \to A, \quad I_A(F) = f$, where

$$f(z) = \int_{0}^{z} \frac{F(t)}{t} dt.$$

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Definition 1.2. [1] Let $I_a$ be a Bernardi integral operator defined as:

$$I_a : A \rightarrow A, \quad I_a(F) = f, \quad a = 1, 2, 3, \ldots,$$

where

$$f(z) = \frac{z^a + 1}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{1.4}$$

Definition 1.3. [1] Let $L_a$ be a generalization of the previously integral operator defined as:

$$L_a : A \rightarrow A, \quad L_a(F) = f, \quad a \in \mathbb{C}, \quad \text{Re} \, a \geq 0,$$

where

$$f(z) = \frac{z^a + 1}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{1.5}$$

Definition 1.4. [5] Let $I_{c+\delta}$ be the integral operator defined as: $I_{c+\delta} : A \rightarrow A, \quad 0 < \delta \leq 1, \quad 1 \leq \delta < \infty$, $0 < c < \infty$,

$$f(z) = I_{c+\delta}(F)(z) = (c + \delta) \int_0^z u^{c+\delta-2} F(uz) du. \tag{1.6}$$

Remark 1.1. [5] For $\delta = 1$ and $c=1, 2, \ldots$, from the integral operator $I_{c+\delta}$ we obtain the Bernardi integral operator defined by (1.4).

Definition 1.5. [5] Let $F \in A$, $F(z) = z + b_2 z^2 + \cdots + b_n z^n + \ldots$, and $a \in \mathbb{R}^*$. We define the integral operator $L : A \rightarrow A$ by

$$f(z) = L(F)(z) = \frac{z^a + 1}{z^a} \int_0^z F(t) \left(t^{a-1} + t^{a+1}\right) dt. \tag{1.7}$$

2. Preliminary results

Lemma 2.1. [3] For $-1 \leq \alpha < 1$, $0 \leq \gamma \leq 1$ and $\beta \geq 0$, we let $S_{\gamma}(f, g; \alpha, \beta)$ be the subclass of $S$ consisting of functions $f(z)$ of the form (1.1) and functions $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0), \tag{2.1}$$
and satisfying the analytic criterion:

\[
\text{Re} \left\{ \frac{z (f * g)'(z) + \gamma z^2 (f * g)''(z)}{(1 - \gamma) (f * g)(z) + \gamma z (f * g)'(z)} - \alpha \right\} > \\
> \beta \left| \frac{z (f * g)'(z) + \gamma z^2 (f * g)''(z)}{(1 - \gamma) (f * g)(z) + \gamma z (f * g)'(z)} - 1 \right|.
\]

Further, we define the class \( T \mathcal{S}_\gamma(f, g; \alpha, \beta) \) by

\[
T \mathcal{S}_\gamma(f, g; \alpha, \beta) = \mathcal{S}_\gamma(f, g; \alpha, \beta) \cap T.
\]

**Lemma 2.2.** [3] A function \( f(z) \) of the form (1.1) is in the class \( T \mathcal{S}_\gamma(f, g; \alpha, \beta) \) if

\[
\sum_{k=2}^{\infty} \left[ k(1 + \beta) - (\alpha + \beta) \right] \left[ 1 + \gamma(k-1) \right] |a_k| b_k \leq 1 - \alpha,
\]

where \(-1 \leq \alpha < 1, \quad \beta \geq 0 \) and \( 0 \leq \gamma \leq 1 \).

**Lemma 2.3.** [3] A necessary and sufficient condition for \( f(z) \) of the form (1.2) to be in the class \( T \mathcal{S}_\gamma(f, g; \alpha, \beta) \) is that

\[
\sum_{k=2}^{\infty} \left[ k(1 + \beta) - (\alpha + \beta) \right] \left[ 1 + \gamma(k-1) \right] a_k b_k \leq 1 - \alpha.
\]

**Corollary 2.1.** [3] Let the function \( f(z) \) be defined by (1.2) be in the class \( T \mathcal{S}_\gamma(f, g; \alpha, \beta) \). Then

\[
a_k \leq \frac{1 - \alpha}{\left[ k(1 + \beta) - (\alpha + \beta) \right] \left[ 1 + \gamma(k-1) \right] b_k}, \quad (k \geq 2).
\]

3. **Main results**

**Theorem 3.1.** The Alexander integral operator defined by (1.3) preserves the class \( T \mathcal{S}_\gamma(f, g; \alpha, \beta) \), that is: If \( F \in T \mathcal{S}_\gamma(f, g; \alpha, \beta) \), then \( f(z) = I_A F(z) \in T \mathcal{S}_\gamma(f, g; \alpha, \beta) \), for \( F(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \).

**Proof.** Let \( F \in T, \quad F(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \). Then

\[
f(z) = I_A F(z) = \int_0^{z} \frac{F(t)}{t} dt =
\]
\[ = \int_{0}^{z} \frac{1}{t} \left( t - \sum_{k=2}^{\infty} a_k t^k \right) dt = \]

\[ = \int_{0}^{\infty} \frac{1}{\eta} \left( \eta - \sum_{k=2}^{\infty} a_k \eta^k \right) d\eta \]

\[ = z - \sum_{k=2}^{\infty} \frac{a_k}{k} z^k, \text{ with} \]

\[ c_k = \frac{a_k}{k} \geq 0, k \geq 2. \] It follows that \( f \in T \). We have now to prove that \( f \in TS_\gamma(f, g; \alpha, \beta) \). Using Lemma 2.3 we need to prove that:

\[
\sum_{k=2}^{\infty} [k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] a_k b_k \leq 1 - \alpha.
\]

for \( k \geq 2, -1 \leq \alpha < 1, \beta \geq 0 \) and \( 0 \leq \gamma \leq 1 \). This means:

\[
\sum_{k=2}^{\infty} [k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] \frac{a_k}{k} b_k \leq 1 - \alpha.
\]

But we have \( \frac{a_k}{k} \leq a_k, \) for \( k \geq 2, \) and by using (2.4) and (3.2), we observe that inequality (3.1) is fulfilled. This means that \( f \in TS_\gamma(f, g; \alpha, \beta) \).

**Theorem 3.2.** The integral operator \( I_{c+\delta} \) defined by (1.6) preserves the class \( TS_\gamma(f, g; \alpha, \beta) \), that is: If \( F \in TS_\gamma(f, g; \alpha, \beta) \), then \( f(z) = I_{c+\delta}(F)(z) \in TS_\gamma(f, g; \alpha, \beta) \), for \( F(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0 \).

**Proof.** Let \( F \in TS_\gamma(f, g; \alpha, \beta) \), \( F(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0 \).

We have, from Lemma 2.3:

\[
\sum_{k=2}^{\infty} [k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] a_k b_k \leq 1 - \alpha.
\]

From (1.6) we obtain \( f(z) = I_{c+\delta}(F)(z) = z - \sum_{k=2}^{\infty} \frac{c + \delta}{c + k + \delta - 1} a_k z^k, \) where \( 0 < c < \infty, 1 \leq \delta < \infty \).
We also remark that for $0 < c < \infty$, $k \geq 2$ and $1 \leq \delta < \infty$, we have

\[(3.4) \quad 0 < \frac{c + \delta}{c + k + \delta - 1} < 1\]

Thus $f \in T$ and by using Lemma 2.3 we have only to prove that.

\[(3.5) \quad \sum_{k=2}^{\infty} [k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] \frac{c + \delta}{c + k + \delta - 1} a_k b_k \leq 1 - \alpha.\]

where $-1 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \gamma \leq 1$, $0 < c < \infty$ and $1 \leq \delta < \infty$.

By using the relation (3.4) we have

\[\frac{c + \delta}{c + k + \delta - 1} \cdot a_k < a_k,\]

for $0 < c < \infty$, $k \geq 2$, $1 \leq \delta < \infty$, and thus from (3.3) we conclude that the condition (3.5) take place and thus the proof it is complete.

The following theorem is proved similarly (see Remark 1.1):

**Corollary 3.1.** The Bernardi integral operator defined by (1.4) preserves the class $TS_\gamma(f, g; \alpha, \beta)$, that is: If $F \in TS_\gamma(f, g; \alpha, \beta)$, then $f(z) = I_a F(z) \in TS_\gamma(f, g; \alpha, \beta)$, for $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$.

**Theorem 3.3.** Let $F \in TS_\gamma(f, g; \alpha, \beta)$ with $-1 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \gamma \leq 1$, $F(z) = z - \sum_{k=2}^{\infty} b_k z^k$, $b_k \geq 0$. For $f(z) = L_a F(z)$, $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$, $z \in U$, where the integral operator $L_a$ it is defined by (1.5), we have:

\[a_k \leq \frac{1 - \alpha}{[k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] b_k \cdot \frac{a + 1}{a + k}}, \quad k \geq 2.\]

**Proof.** For $f = L_a(F)(z)$ with $F(z) = z - \sum_{k=2}^{\infty} b_k z^k$ and $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ we have

\[a_k = b_k \cdot \frac{a + 1}{a + k},\]

where $a \in \mathbb{C}$, $\text{Re} \ a \geq 0$, $k \geq 2$.

The coefficient bounds for the functions belonging to the class $TS_\gamma(f, g; \alpha, \beta)$ are

\[b_k \leq \frac{1 - \alpha}{[k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] b_k}, \quad (k \geq 2).\]
For $k \geq 2$ we obtain
\[
a_k = |b_k| \cdot \left| \frac{a+1}{a+k} \right| \leq \frac{1 - \alpha}{[k(1 + \beta) - (\alpha + \beta)][1 + \gamma(k - 1)]} \cdot \left| \frac{a+1}{a+k} \right| = \frac{1 - \alpha}{[k(1 + \beta) - (\alpha + \beta)][1 + \gamma(k - 1)]} \cdot \left| \frac{a+1}{a+k} \right|.
\]
Hence the theorem is proved.

**Theorem 3.4.** Let $F \in TS_{\gamma}(f,g;\alpha,\beta)$ with $-1 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \gamma \leq 1$, $F(z) = z - \sum_{k=2}^{\infty} b_k z^k$, $b_k \geq 0$. For $f(z) = L(F)(z)$, $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$, $z \in U$, where the integral operator $L$ it is defined by (1.7), we have:
\[
a_2 \leq \frac{1 - \alpha}{(2 - \alpha + \beta)(1 + \gamma)} \cdot \frac{a+1}{a+2},
\]
\[
a_3 \leq \left[ \frac{1 - \alpha}{(3 - \alpha + 2\beta)(1 + 2\gamma)} + 1 \right] \cdot \frac{a+1}{a+3},
\]
\[
a_k \leq \frac{(1 - \alpha)(a+1)}{a+k} \cdot (r_k + r_{k-2}),
\]
where
\[
r_k = \frac{1}{[k(1 + \beta) - (\alpha + \beta)][1 + \gamma(k - 1)]} \cdot b_k
\]
\[
r_{k-2} = \frac{1}{[(k - 2)(1 + \beta) - (\alpha + \beta)][1 + \gamma(k - 3)]} \cdot b_{k-2}.
\]

**Proof.** For $f = L(F)(z)$ with $F(z) = z - \sum_{k=2}^{\infty} b_k z^k$ and $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ we have:
\[
a_2 = b_2 \cdot \frac{a+1}{a+2},
\]
\[
a_3 = (b_3 + 1) \cdot \frac{a+1}{a+3},
\]
\[
a_k = (b_k + b_{k-2}) \cdot \frac{a+1}{a+k}.
\]
where \(a \in \mathbb{R}^*, \ k \geq 4\).
The coefficient bounds for the functions belonging to the class \(TS_\gamma(f,g;\alpha,\beta)\) are:
\[
b_k \leq \frac{1 - \alpha}{[k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] b_k}, \quad (k \geq 2).
\]
For \(k \geq 4\) we obtain:
\[
a_k = (b_k + b_{k-2}) \cdot \frac{a + 1}{a + k} \leq
\]
\[
= \frac{1 - \alpha}{[k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] b_k} \cdot \frac{a + 1}{a + k} + \frac{1 - \alpha}{[(k - 2)(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 3)] b_{k-2}} \cdot \frac{a + 1}{a + k},
\]
\[
a_k \leq \frac{(1 - \alpha)(a + 1)}{a + k} \cdot \frac{1}{[(k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] b_k} + \frac{1}{[(k - 2)(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 3)] b_{k-2}} = \frac{(1 - \alpha)(a + 1)}{a + k} \cdot (r_k + r_{k-2}),
\]
where
\[
r_k = \frac{1}{[(k(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 1)] b_k},
\]
\[
r_{k-2} = \frac{1}{[(k - 2)(1 + \beta) - (\alpha + \beta)] [1 + \gamma(k - 3)] b_{k-2}}.
\]
For \(k = 2\) we have:
\[
a_2 = b_2 \cdot \frac{a + 1}{a + 2} \leq
\]
\[
\leq \frac{1 - \alpha}{(2 - \alpha + \beta)(1 + \gamma)} b_2 \cdot \frac{a + 1}{a + 2}.
\]
Similarly for \(k = 3\) we have:
\[
a_3 \leq \left[ \frac{1 - \alpha}{(3 - \alpha + 2\beta)(1 + 2\gamma)} b_3 + 1 \right] \cdot \frac{a + 1}{a + 3}.
\]
Hence the theorem is proved.

We remark that for suitable values of the functions \(f\) and \(g\) and the parameters \(\alpha\) and \(\beta\) we obtain similarly results for the following subclasses:

i) \(TS_0\left( f, \frac{z}{1 - z}; \alpha, 1 \right) = S_p T(\alpha)\) and \(TS_0\left( f, \frac{z}{(1 - z)^2}; \alpha, 1 \right) = \)
\[ TS_1 \left( f, \frac{z}{1-z}; \alpha, 1 \right) = UCT(\alpha) \ (-1 \leq \alpha < 1) \] (see Bharati et al. [4]);

ii) \[ TS_1 \left( f, \frac{z}{(1-z)^2}; 0, \beta \right) = UCT(\beta) \ (\beta \geq 0) \] (see Subramanian et al. [12]);

iii) \[ TS_0 \left( f, z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta \right) = TS_\gamma (\alpha, \beta) \ (-1 \leq \alpha < 1, \beta \geq 0, c \neq 0, -1, -2, \ldots) \] (see Murugusundaramoorthy and Magesh [6] and [7]);

iv) \[ TS_0 \left( f, z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta \right) = TS(n, \alpha, \beta) \ (-1 \leq \alpha < 1, \beta \geq 0, n \in N_0 = N \cup \{0\}, N = \{1, 2, \ldots\} \] (see Rosy and Murugusundaramoorthy [10]);

v) \[ TS_0 \left( f, z + \sum_{k=2}^{\infty} \frac{k + \lambda - 1}{\lambda} z^k; \alpha, \beta \right) = D(\alpha, \beta, \lambda) \ (-1 \leq \alpha < 1, \beta \geq 0, \lambda > -1) \] (see Shams et al. [11]);

vi) \[ TS_0 \left( f, z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)] z^k; \alpha, \beta \right) = TS_\lambda (n, \alpha, \beta) \ (-1 \leq \alpha < 1, \beta \geq 0, \lambda \geq 0), n \in N_0 \] (see Aouf and Mostafa [2]);

vii) \[ TS_\gamma \left( f, z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta \right) = TS_\gamma (n, \alpha, \beta) \ (-1 \leq \alpha < 1, \beta \geq 0, 0 \leq \gamma \leq 1, c \neq 0, -1, -2, \ldots) \] (see Murugusundaramoorthy et al. [8]);

viii) \[ TS_0 (f, g; \alpha, \beta) = H_T(g, \alpha, \beta) \ (-1 \leq \alpha < 1, \beta \geq 0) \] (see Raina and Bansal [9]);

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