SOME PROPERTIES OF AN INTEGRAL OPERATOR

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Abstract. In this paper we will prove, using Pescar’s criterion, the univalence of an integral operator, considered for analytic functions in the open unit disk $U$.

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1. Introduction and Preliminaries

Let the unit disk $U = \{ z \in \mathbb{C} \mid |z| < 1 \}$ and $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $U$ and satisfy the condition $f(0) = f'(0) - 1 = 0$.

We denote by $S$ the subclass of $\mathcal{A}$ containing univalent and regular functions. In 1996, V. Pescar has proved univalent condition:

Theorem 1. [4] Let $\alpha \in \mathbb{C}$ with $\text{Re} \alpha > 0$ and $c \in \mathbb{C}$ with $|c| < 1$. We consider also a function $f(z)$ of the form (1) which is analytic in $U$. If:

$$|c|z^{2\alpha} + (1 - |z|)^{2\alpha} \frac{zf''(z)}{zf'(z)} \leq 1,$$

for every $z \in U$, then the function $F_\alpha(z)$ defined by:

$$F_\alpha(z) = \left( \frac{\alpha \int_0^z t^{\alpha-1} f'(t)dt}{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha}}$$

is univalent in $U$.

In [3] Ozaki and Nunokawa gave the following result:
Theorem 2. If \( f \in A \) satisfies the following inequality:

\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1, \tag{3}
\]

for every \( z \in U \), then \( f \) is univalent in \( U \).

Also, an important result that we will use in our paper is the General Schwarz Lemma. We remind it here:

Lemma 3. [1] Let the regular function \( f \) in the disk \( U_R = \{ z \in \mathbb{C} \mid |z| < R \} \), with \( |f(z)| < M \), \( M \) fixed. If \( f \) has in \( z = 0 \) one zero with multiplicity \( \geq m \), then

\[
|f(z)| \leq \frac{M}{R^m}|z|^m, \quad z \in U_R. \tag{4}
\]

It is obviously that for \( R = m = 1 \) the relation (4) becomes:

\[
\left| \frac{f(z)}{z} \right| \leq M, \quad z \in U. \tag{5}
\]

The goal of our paper is to introduce an integral operator, to prove the univalence for it and present some properties obtained from here.

2. Main results

Theorem 4. Let \( f_i \in A \), \( i = 1, n \), the functions that satisfy the inequality (3) \( \alpha_i, \gamma, c \) be complex numbers with \( \text{Re} \gamma > 0 \) and \( M_i, N_i \in \mathbb{R}_{+}^* \), \( M_i \geq 1 \).

If:

i) \( |f_i(z)| \leq M_i, \quad i = 1, n; \)
ii) \( \left| \frac{f''(z)}{f'(z)} \right| \leq N_i, \quad i = 1, n; \)
iii) \( |c| \leq 1 - \frac{1}{|\gamma|} \sum_{i=1}^{n} |\alpha_i| (2M_i + N_i + 1) , \)

then the function:

\[
G_n(z) = \left( \frac{1}{\gamma} \int_0^z u^{-\gamma} \prod_{i=1}^{n} \left( \frac{u}{f_i(u)f_i'(u)} \right)^{\alpha_i} du \right)^{\frac{1}{\gamma}}
\]

is univalent in \( U \).
Proof. Let the function \( g_n \), regular in \( \mathcal{U} \) and \( g_n(0) = g'_n(0) - 1 = 0 \), defined as:

\[
g_n(z) = \int_0^z \prod_{i=1}^n \left( \frac{u}{f_i(u)f'_i(u)} \right)^{\alpha_i} \, du.
\]

For this function we have:

\[
g'_n(z) = n \prod_{i=1}^n \left( \frac{z}{f_i(z)f'_i(z)} \right)^{\alpha_i}
\]

and:

\[
g''_n(z) = \sum_{i=1}^n \left\{ \left[ \left( \frac{z}{f_i(z)f'_i(z)} \right)^{\alpha_i} \right]' \cdot \prod_{j=1}^n \left( \frac{z}{f_j(z)f'_j(z)} \right)^{\alpha_j} \right\}
\]

\[
= \prod_{i=1}^n \left( \frac{z}{f_i(z)f'_i(z)} \right)^{\alpha_i} \cdot \sum_{i=1}^n \alpha_i \left( \frac{1}{z} - \frac{f'_i(z)}{f_i(z)} - \frac{f''_i(z)}{f'_i(z)} \right).
\]

So we have:

\[
\frac{zg''_n(z)}{g'_n(z)} = \sum_{i=1}^n \alpha_i \left( 1 - \frac{zf'_i(z)}{f_i(z)} - \frac{zf''_i(z)}{f'_i(z)} \right),
\]

hence:

\[
\left| \frac{zg''_n(z)}{g'_n(z)} \right| \leq \sum_{i=1}^n |\alpha_i| \left( 1 + \left| \frac{zf'_i(z)}{f_i(z)} \right| + \left| \frac{zf''_i(z)}{f'_i(z)} \right| \right)
\]

\[
\leq \sum_{i=1}^n |\alpha_i| \left( 1 + 2 \left| \frac{zf'_i(z)}{f^2_i(z)} \right| \cdot \frac{f_i(z)}{z} + \left| \frac{zf''_i(z)}{f'_i(z)} \right| \right).
\]

Because \( |f_i(z)| \leq M_i \) and using Schwarz Lemma, we obtain:

\[
\left| \frac{zg''_n(z)}{g'_n(z)} \right| \leq \sum_{i=1}^n |\alpha_i| \left( 1 + 2 \left| \frac{zf'_i(z)}{f^2_i(z)} \right| \cdot M_i + \left| \frac{zf''_i(z)}{f'_i(z)} \right| \right)
\]

\[
\leq \sum_{i=1}^n |\alpha_i| \left( 1 + \left| \frac{zf'_i(z)}{f^2_i(z)} \right| - 1 \right) \cdot M_i + M_i + \left| \frac{zf''_i(z)}{f'_i(z)} \right| \right).
\]

Applying inequality (3) and ii), we have:

\[
\left| \frac{zg''_n(z)}{g'_n(z)} \right| \leq \sum_{i=1}^n |\alpha_i| (2M_i + N_i + 1).
\]
From this relation, we obtain:

$$\left| c|z|^{2\gamma} + (1 - |z|)^{2\gamma} \frac{z g''_n(z)}{\gamma g'_n(z)} \right| \leq |c| + \frac{1}{|\gamma|} \sum_{i=1}^{n} |\alpha_i| (2M_i + N_i + 1).$$

So, because of iii), it results:

$$\left| c|z|^{2\gamma} + (1 - |z|)^{2\gamma} \frac{z g''_n(z)}{\gamma g'_n(z)} \right| \leq 1$$

and according with Theorem 1, we obtain that the function $G_n$ is in the class $\mathcal{S}$.

**Corollary 5.** Let $f \in \mathcal{A}$ a function that satisfy the inequality (3) $\alpha, \gamma, c$ be complex numbers with $\text{Re} \gamma > 0$ and $M, N \in \mathbb{R}_+^*$, $M \geq 1$.

If:

i) $|f(z)| \leq M$;

ii) $\left| \frac{f''(z)}{f'(z)} \right| \leq N$;

iii) $|c| \leq 1 - \frac{|\alpha|}{|\gamma|} \sum_{i=1}^{n} (2M_i + N_i + 1)$, then the function:

$$G_n(z) := \left( \gamma \int_{0}^{z} u^{\gamma-1} \left( \frac{u}{f(u)f'(u)} \right)^{\alpha} \, du \right)^{\frac{1}{\gamma}}$$

is univalent in $U$.

**Proof.** We consider $n = 1$ in Theorem 4.

**Corollary 6.** Let $f_i \in \mathcal{A}$, $i = \overline{1,n}$, the functions that satisfy the inequality (3) $\alpha, \gamma, c$ be complex numbers with $\text{Re} \gamma > 0$ and $M_i, N_i \in \mathbb{R}_+^*$, $M_i \geq 1$.

If:

i) $|f_i(z)| \leq M_i$, $i = \overline{1,n}$;

ii) $\left| \frac{f''_i(z)}{f'_i(z)} \right| \leq N_i$, $i = \overline{1,n}$;

iii) $|c| \leq 1 - \frac{|\alpha|}{|\gamma|} \sum_{i=1}^{n} (2M_i + N_i + 1)$, then the function:

$$G_n(z) := \left( \gamma \int_{0}^{z} u^{\gamma-1} \prod_{i=1}^{n} \left( \frac{u}{f_i(u)f'_i(u)} \right)^{\alpha} \, du \right)^{\frac{1}{\gamma}}$$

is univalent in $U$. 

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Proof. We consider $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$ in Theorem 4.

**Corollary 7.** Let $f_i \in A$, $i = 1, n$, the functions that satisfy the inequality (3) $\alpha_i, c$ be complex numbers and $M_i, N_i \in \mathbb{R}^*_+$, $M_i \geq 1$.

If:

i) $|f_i(z)| \leq M_i, i = 1, n$;

ii) $\left| \frac{f''_i(z)}{f'_i(z)} \right| \leq N_i, i = 1, n$;

iii) $|c| \leq 1 - \sum_{i=1}^{n} |\alpha_i| (2M_i + N_i + 1)$,

then the function:

$$G_n(z) := \int_0^z \prod_{i=1}^{n} \left( \frac{u}{f_i(u)f'_i(u)} \right)^{\alpha_i} du$$

is univalent in $U$.

**Proof.** In Theorem 4, we consider $\gamma = 1$.

**References**


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