ON THE CERTAIN SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper, we introduce and investigate an interesting subclass $B^{p,q}_{\Sigma}(h,\lambda)$ of bi-univalent functions in the open unit disk $U$. Furthermore, we find estimates on the $|a_2|$ and $|a_3|$ coefficients for functions in this subclass. The results presented in this paper would generalize and improve those in related works of several earlier authors.

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1. Introduction

Let $A$ denote the class of analytic functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Further, by $S$ we shall denote the class of functions in $A$ which are univalent in $U$ (for details, see [2, 3, 5]).

It is well known that every functions $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$
A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \).

Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1). Brannan and Taha [2] (see also [11]) introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses \( S^*(\alpha) \) and \( K(\alpha) \) of starlike and convex functions of order \( \alpha (0 < \alpha \leq 1) \), respectively (see [1]).

Determination of the bounds for the coefficients \( a_n \) is an important problem in geometric function theory as they give information about the geometric properties of these functions. Recently there interest to study the bi-univalent functions class \( \Sigma \) (see [3, 6, 7, 9, 10, 12]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \). The coefficient estimate problem i.e. bound of \( |a_n| \) \((n \in \mathbb{N} - \{1, 2\})\) for each \( f \in \Sigma \) is still an open problem.

Srivastava et al. [10] introduced the following two subclasses of the bi-univalent function class \( \Sigma \) and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) of functions in each of these subclasses.

**Definition 1.** [10] A function \( f(z) \) given by (1) is said to be in the \( H_{\alpha} \Sigma \) \((0 < \alpha \leq 1)\), if the following conditions are satisfied:

\[
|\arg(f'(z))| < \frac{\alpha \pi}{2} \quad (z \in U), \quad |\arg(g'(w))| < \frac{\alpha \pi}{2} \quad (w \in U),
\]

where \( g \) is the extension of \( f^{-1} \) to \( U \).

**Theorem 1.** [10] Let the function \( f(z) \) given by (1) be in the \( H_{\alpha} \Sigma \) \((0 < \alpha \leq 1)\). Then

\[
|a_2| \leq \alpha \sqrt{\frac{2}{\alpha + 2}}, \quad |a_3| \leq \frac{\alpha(3\alpha + 2)}{3}.
\]

**Definition 2** ([10]). A function \( f(z) \) given by (1) is said to be in the \( H_{\Sigma}(\beta) \) \((0 \leq \beta < 1)\), if the following conditions are satisfied:

\[
f \in \Sigma, \quad \text{Re}(f'(z)) > \beta \quad (z \in U), \quad \text{Re}(g'(w)) > \beta \quad (w \in U),
\]

where \( g \) is the extension of \( f^{-1} \) to \( U \).

**Theorem 2.** [10] Let the function \( f(z) \) given by (1) be in the \( H_{\Sigma}(\beta) \) \((0 \leq \beta < 1)\). Then

\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{3}}, \quad |a_3| \leq \frac{(1 - \beta)(5 - 3\beta)}{3}.
\]
As a generalization of two subclasses $H_\alpha \Sigma$ and $H_\Sigma (\beta)$, Frasin [7] introduced the following two subclasses of the bi-univalent function class $\Sigma$ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

**Definition 3.** [7] A function $f(z) \in \Sigma$ given by (1) is said to be in the $B_\Sigma (\alpha, \lambda)$ $(0 < \alpha \leq 1, \lambda \geq 1)$, if the following conditions are satisfied:

$$|\arg((1 - \lambda) \frac{f(z)}{z} + \lambda f'(z))| < \frac{\alpha \pi}{2} \ (z \in \mathbb{U}), \quad |\arg((1 - \lambda) \frac{g(w)}{w} + \lambda g'(w))| < \frac{\alpha \pi}{2} \ (w \in \mathbb{U}),$$

where $g$ is the extension of $f^{-1}$ to $\mathbb{U}$.

**Theorem 3.** [7] Let the function $f(z)$ given by (1) be in the $B_\Sigma (\alpha, \lambda)$ $(0 < \alpha \leq 1, \lambda \geq 1)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}, \quad |a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{2\lambda + 1}.$$

**Definition 4.** [7] A function $f(z) \in \Sigma$ given by (1) is said to be in the $B_\Sigma (\beta, \lambda)$ $(0 \leq \beta < 1, \lambda \geq 1)$, if the following conditions are satisfied:

$$\text{Re}((1 - \lambda) \frac{f(z)}{z} + \lambda f'(z)) > \beta \ (z \in \mathbb{U}), \quad \text{Re}((1 - \lambda) \frac{g(w)}{w} + \lambda g'(w)) > \beta \ (w \in \mathbb{U}),$$

where $g$ is the extension of $f^{-1}$ to $\mathbb{U}$.

**Theorem 4.** [7] Let the function $f(z)$ given by (1) be in the $B_\Sigma (\beta, \lambda)$ $(0 \leq \beta < 1, \lambda \geq 1)$. Then

$$|a_2| \leq \sqrt{\frac{2(1 - \beta)}{2\lambda + 1}}, \quad |a_3| \leq \frac{4(1 - \beta)^2}{(\lambda + 1)^2} + \frac{2(1 - \beta)}{2\lambda + 1}.$$

The object of the present paper is to introduce a new subclass of the function class $\Sigma$ and obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass which generalize and improve those in related works of several earlier authors.

2. Coefficient bounds for the function class $B_\Sigma^{p,q}(h, \lambda)$

In this section, we introduce the subclass $B_\Sigma^{p,q}(h, \lambda)$ and find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this subclass.
Let
\[ h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \text{ where } h_n > 0 \text{ for all } n \geq 2. \]  
(2)

The Hadamard product \( f(z), h(z) \) is defined as \( (f \star h)(z) = z + \sum_{n=2}^{\infty} a_n h_n z^n \), where \( f(z) \in A \) given by (1).

**Definition 5.** Let the functions \( p, q : \mathbb{U} \to \mathbb{C} \) be so constrained that
\[ \min\{\text{Re}(p(z)), \text{Re}(q(z))\} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad p(0) = q(0) = 1. \]

A function \( f(z) \in A \) given by (1) is said to be in the class \( B_{\Sigma}^{p,q}(h,\lambda) \), if the following conditions are satisfied:
\[ f \in \Sigma, \quad [(1 - \lambda) \frac{(f \star h)(z)}{z} + \lambda(f \star h)'(z)] \in p(\mathbb{U}) \quad (z \in \mathbb{U}; \ \lambda \geq 1) \]  
(3)
and
\[ [(1 - \lambda) \frac{(f \star h)^{-1}(w)}{w} + \lambda((f \star h)^{-1})'(w)] \in q(\mathbb{U}) \quad (w \in \mathbb{U}; \ \lambda \geq 1), \]  
(4)
where the function \( h(z) \) is given by (2).

**Remark 1.** There are many choices of the functions \( p(z) \) and \( q(z) \) which would provide interesting subclasses of the analytic function class \( A \). For example, if we let
\[ p(z) = q(z) = \left(\frac{1 + z}{1 - z}\right)^{\alpha} \quad (0 < \alpha \leq 1; \ z \in \mathbb{U}), \]
it is easy to verify that the functions \( p(z) \) and \( q(z) \) satisfy the hypotheses of Definition 5. If \( f(z) \in B_{\Sigma}^{p,q}(h,\lambda) \), then
\[ |\arg\left((1 - \lambda) \frac{(f \star h)(z)}{z} + \lambda(f \star h)'(z)\right)| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}; \ \lambda \geq 1) \]
and
\[ |\arg\left((1 - \lambda) \frac{(f \star h)^{-1}(z)}{w} + \lambda((f \star h)^{-1})'(w)\right)| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}; \ \lambda \geq 1). \]

Therefore for \( p(z) = q(z) = \left(\frac{1 + z}{1 - z}\right)^{\alpha} \) and \( h(z) = \frac{z}{1 - z} \), the class \( B_{\Sigma}^{p,q}(h,\lambda) \) reduce to Definition 3 and in special case \( \lambda = 1 \) it reduce to Definition 1.
If we take
\[ p(z) = q(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \leq \beta < 1; \ z \in \mathbb{U}), \]
then the functions \( p(z) \) and \( q(z) \) satisfy the hypotheses of Definition 5. If \( f(z) \in \mathcal{B}_{\Sigma}^{p,q}(h,\lambda) \), then
\[
\Re \left( (1-\lambda) \frac{(f \ast h)(z)}{z} + \lambda (f \ast h)'(z) \right) > \beta \quad (z \in \mathbb{U}; \ \lambda \geq 1)
\]
and
\[
\Re \left( (1-\lambda) \frac{(f \ast h)^{-1}(z)}{w} + \lambda ((f \ast h)^{-1})'(w) \right) > \beta \quad (w \in \mathbb{U}; \ \lambda \geq 1).
\]

Therefore for \( p(z) = q(z) = \frac{1+(1-2\beta)z}{1-z} \) and \( h(z) = \frac{z}{1-z} \), the class \( \mathcal{B}_{\Sigma}^{p,q}(h,\lambda) \) reduce to Definition 4 and in special case \( \lambda = 1 \) it reduce to Definition 2.

### 2.1. Coefficients estimates

Now, we derive the estimates of the coefficients \( |a_2| \) and \( |a_3| \) for class \( \mathcal{B}_{\Sigma}^{p,q}(h,\lambda) \).

**Theorem 5.** Let a function \( f(z) \) given by (1) be in the class \( \mathcal{B}_{\Sigma}^{p,q}(h,\lambda) \) \((\lambda \geq 1)\). Then
\[
|a_2| \leq \min \left\{ \frac{1}{h_2(\lambda+1)} \sqrt{\frac{|p'(0)|^2 + |q'(0)|^2}{2}}, \frac{1}{2h_2} \sqrt{\frac{|p''(0)| + |q''(0)|}{2\lambda + 1}} \right\}
\]
and
\[
|a_3| \leq \min \left\{ \frac{|p'(0)|^2 + |q'(0)|^2}{2h_3(\lambda+1)^2}, \frac{|p''(0)| + |q''(0)|}{4h_3(2\lambda + 1)}, \frac{|p''(0)|}{2h_3(2\lambda + 1)} \right\}.
\]

**Proof.** First of all, we write the argument inequalities in (3) and (4) in their equivalent forms as follows:
\[
(1-\lambda) \frac{(f \ast h)(z)}{z} + \lambda (f \ast h)'(z) = p(z) \quad (z \in \mathbb{U}),
\]
\[
(1-\lambda) \frac{(f \ast h)^{-1}(w)}{w} + \lambda ((f \ast h)^{-1})'(w) = q(w) \quad (w \in \mathbb{U}),
\]
respectively, where functions \( p(z) \) and \( q(w) \) satisfy the conditions of Definition 5. Furthermore, the functions \( p(z) \) and \( q(w) \) have the following Taylor-Maclaurin series expansions:

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 ... \quad (7)
\]

and

\[
q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 ... , \quad (8)
\]

respectively. Now, upon substituting from (7) and (8) into (5) and (6), respectively, and equating the coefficients, we get

\[
(\lambda + 1)a_2 h_2 = p_1, \quad (9)
\]

\[
(2\lambda + 1)a_3 h_3 = p_2, \quad (10)
\]

\[
-(\lambda + 1)a_2 h_2 = q_1 \quad (11)
\]

and

\[
2(2\lambda + 1)a_2^2 h_2^2 - (2\lambda + 1)a_3 h_3 = q_2. \quad (12)
\]

From (9) and (11), we obtain

\[
p_1 = -q_1, \quad (13)
\]

\[
a_2^2 = \frac{p_1^2 + q_1^2}{2(\lambda + 1)^2 h_2^2}. \quad (14)
\]

By adding (10) and (12), we get

\[
a_2^2 = \frac{p_2 + q_2}{2(2\lambda + 1)h_2^2}. \quad (15)
\]

Therefore, we find from the equations (14) and (15) that

\[
|a_2| \leq \frac{1}{h_2(\lambda + 1)} \sqrt{\frac{|p'(0)|^2 + |q'(0)|^2}{2}}
\]

and

\[
|a_2| \leq \frac{1}{2h_2} \sqrt{\frac{|p''(0)| + |q''(0)|}{2\lambda + 1}},
\]
respectively. So we get the desired estimate on the coefficient \(|a_2|\) asserted. Next, in order to find the bound on the coefficient \(|a_3|\), we subtract (12) from (10). We thus get

\[
2(2\lambda + 1)a_3h_3 - 2(2\lambda + 1)a_2^2h_2^2 = p_2 - q_2.
\]

Upon substituting the value of \(a_2^2\) from (14) into (16), it follows that

\[
a_3 = \frac{p_1^2 + q_1^2}{2h_3(\lambda + 1)^2} + \frac{p_2 - q_2}{2h_3(2\lambda + 1)}. \tag{17}
\]

We thus find that

\[
|a_3| \leq \frac{|p'(0)|^2 + |q'(0)|^2}{2h_3(\lambda + 1)^2} + \frac{|p''(0)| + |q''(0)|}{4h_3(2\lambda + 1)}.
\]

On the other hand, upon substituting the value of \(a_2^2\) from (15) into (16), it follows that

\[
a_3 = \frac{p_2 + q_2}{2h_3(2\lambda + 1)} + \frac{p_2 - q_2}{2h_3(2\lambda + 1)}. \tag{18}
\]

Consequently, we have

\[
|a_3| \leq \frac{|p''(0)|}{2h_3(2\lambda + 1)}.
\]

3. Corollaries and Consequences

By setting

\[
h(z) = p(z) = (\frac{1 + z}{1 - z})^\alpha \quad (0 < \alpha \leq 1, \; z \in \mathbb{U}),
\]

in Theorem 5, we obtain the following result.

**Corollary 6.** Let the function \(f(z)\) given by (1) be in the bi-univalent function class \(B_{2\Sigma}(h, \alpha, \lambda)\) \((0 < \alpha \leq 1; \; \lambda \geq 1)\). Then

\[
|a_2| \leq \min \left\{ \frac{2\alpha}{h_2(\lambda + 1)}, \sqrt{\frac{2}{2\lambda + 1}} \right\}
\]

and

\[
|a_3| \leq \frac{2\alpha^2}{h_3(2\lambda + 1)}.
\]
Remark 2. The bounds on $|a_2|$, $|a_3|$ given in Corollary 6 are better than those given by El-Ashwah\cite[Theorem 1]{El-Ashwah}.  

By setting $h(z) = \frac{z}{1-z}$ and $\lambda = 1$ in Corollary 6, we conclude the following corollary.

Corollary 7. Let the function $f(z)$ given by (1) be in the bi-univalent function class $\mathcal{H}_\alpha^\Sigma (0 < \alpha \leq 1)$. Then

$$|a_2| \leq \min\{\alpha, \sqrt{\frac{2}{3}} \alpha\} = \sqrt{\frac{2}{3}} \alpha$$

and

$$|a_3| \leq \frac{2}{3} \alpha^2.$$

Remark 3. The bounds on $|a_2|$, $|a_3|$ given in Corollary 7 are better than those given in Theorem 1. Because

$$\sqrt{\frac{2}{3}} \alpha \leq \alpha \sqrt{\frac{2}{\alpha + 2}}$$

and

$$\frac{2}{3} \alpha^2 \leq \alpha^2 + \frac{2}{3} \alpha.$$

By setting $h(z) = \frac{z}{1-z}$ in Corollary 6, we conclude the following corollary.

Corollary 8. Let the function $f(z)$ given by (1) be in the bi-univalent function class $\mathcal{B}_\Sigma(\alpha, \lambda)$ $(0 < \alpha \leq 1, \lambda \geq 1)$. Then

$$|a_2| \leq \min\{\frac{2\alpha}{\lambda + 1}, \alpha \sqrt{\frac{2}{2\lambda + 1}}\}$$

and

$$|a_3| \leq \frac{2\alpha^2}{2\lambda + 1}.$$

Remark 4. The bounds on $|a_2|$, $|a_3|$ given in Corollary 8 are better than those given in Theorem 3. Because

$$\frac{2\alpha}{\lambda + 1} \leq \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}} \quad (\lambda \geq 1 + \sqrt{2})$$

and

$$\frac{2\alpha^2}{2\lambda + 1} \leq \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{2\lambda + 1}.$$
By setting
\[ h(z) = p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1, \ z \in \mathbb{U}), \]
in Theorem 5, we obtain the following result.

**Corollary 9.** Let the function \( f(z) \) given by (1) be in the bi-univalent function class \( B_{\Sigma}(h, \beta, \lambda) \) \((0 \leq \beta < 1, \lambda \geq 1)\). Then
\[
|a_2| \leq \min\left\{ \frac{2(1 - \beta)}{h_2(\lambda + 1)}, \frac{1}{h_2} \sqrt{\frac{2(1 - \beta)}{2\lambda + 1}} \right\}
\]
and
\[
|a_3| \leq \frac{2(1 - \beta)}{h_3(2\lambda + 1)}.
\]

**Remark 5.** The bounds on \(|a_2|, |a_3|\) given in Corollary 9 are better than those given by El-Ashwah[6, Theorem 2].

By setting \( h(z) = \frac{1}{1 - z} \) and \( \lambda = 1 \) in Corollary 9, we conclude the following corollary.

**Corollary 10.** Let the function \( f(z) \) given by (1) be in the bi-univalent function class \( H_{\Sigma}(\beta) \) \((0 \leq \beta < 1)\). Then
\[
|a_2| \leq \begin{cases} \sqrt{\frac{2}{3}(1 - \beta)} & ; \ 0 \leq \beta \leq \frac{1}{3} \\ (1 - \beta) & ; \ \frac{1}{3} \leq \beta < 1 \end{cases}
\]
and
\[
|a_3| \leq \frac{2}{3}(1 - \beta).
\]

**Remark 6.** The bound on \(|a_2|, |a_3|\) given in Corollary 10 are better than those given in Theorem 2.

By setting \( h(z) = \frac{1}{1 - z} \) in Corollary 9, we conclude the following corollary.

**Corollary 11.** Let the function \( f(z) \) given by (1) be in the bi-univalent function class \( B_{\Sigma}(\beta, \lambda) \) \((0 \leq \beta < 1, \lambda \geq 1)\). Then
\[
|a_2| \leq \min\left\{ \frac{2(1 - \beta)}{\lambda + 1}, \sqrt{\frac{2(1 - \beta)}{2\lambda + 1}} \right\}
\]
and

\[ |a_3| \leq \frac{2(1 - \beta)}{2\lambda + 1}. \]

Remark 7. The bounds on \(|a_2|, |a_3|\) given in Corollary 11 are better than those given in Theorem 4. Because

\[
\frac{2(1 - \beta)}{(\lambda + 1)} \leq \sqrt{\frac{2(1 - \beta)}{2\lambda + 1}} \quad (\lambda \geq 1 - 2\beta + \sqrt{4\beta^2 - 6\beta + 2}; \ 0 \leq \beta \leq \frac{1}{3})
\]

and

\[
\frac{2(1 - \beta)}{(2\lambda + 1)} \leq \frac{4(1 - \beta)^2}{(\lambda + 1)^2} + \frac{2(1 - \beta)}{2\lambda + 1}. \]

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