APPLICATION OF SUPERORDINATION TO A SUBCLASS OF ANALYTIC FUNCTIONS INCLUDED DOUBLE INTEGRAL OPERATORS

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Abstract. We suppose that the normalized analytic function \( f(z) \) satisfies the differential equation

\[ f'(z) + \alpha zf''(z) + \lambda z^2 f'''(z) = g(z), \]

where \( g \) is univalent in the open unit disk \( \mathbb{D} \) and is superordinate to a convex-univalent function \( h(z) \) normalized by \( h(0) = 1 \). In addition, we assume that the function \( f(z) \) is given by a double integral operator of the form

\[ f(z) = (1 + \delta_1)(1 + \delta_2) \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} z G'(zt^{\mu}s^{\nu}) \, ds \, dt, \]

where \( G'(z) + zG''(z) = g(z) \). We shall determine the best subordinant of the solutions of differential superordination

\[ h(z) \prec f'(z) + \alpha zf''(z) + \lambda z^2 f'''(z). \]

Some special cases are given in the corollaries.

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1. Introduction

Let \( \mathcal{A} \) be the class of all analytic functions \( f(z) \) of the form

\[ f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots; \quad (z \in \mathbb{D}), \]
which satisfy the normalization condition \( f(0) = f'(0) - 1 = 0 \), and that \( S \subseteq \mathcal{A} \) be the class of normalized univalent functions. Further, suppose that \( C \) denote the class of convex-univalent functions in \( \mathbb{D} \). For two analytic functions

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k
\]

the Hadamard product (or convolution) of \( f \) and \( g \) is an analytic function in \( \mathbb{D} \) defined by \((f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k\).

For \( f, g \in \mathcal{A} \) the function \( f \) is subordinate to \( g \) (or \( g \) is superordinate to \( f \)) written as \( f(z) \prec g(z) \) if there exist an analytic function \( w(z) \) in \( \mathbb{D} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \). If \( g \) is univalent in \( \mathbb{D} \), then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(\mathbb{D}) \subseteq g(\mathbb{D}) \), (see \([3]\)).

Suppose that \( p, h \) are two analytic function in \( \mathbb{D} \) and \( \varphi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C} \). If \( p(z) \) and \( \varphi(p(z), z p'(z), z^2 p''(z); z) \) are univalent in \( \mathbb{D} \) and if \( p(z) \) satisfies the second-order superordination

\[
h(z) \prec \varphi(p(z), z p'(z), z^2 p''(z); z),
\]

then \( p \) is called a solution of the differential superordination \((1)\). An analytic function \( q(z) \) is called a subordinant of \((1)\), if \( q(z) \prec p(z) \) for all the solutions of \((1)\). The best subordinant \( \tilde{q} \) is univalent subordinant that satisfies \( q \prec \tilde{q} \) for all the subordinants \( q \) of \((1)\), (see \([4]\)).

**Definition 1.** ([3]) We denote by \( Q \) the set of all functions \( p(z) \) that are analytic and injective on \( \overline{\mathbb{D}} \setminus E(p) \), where

\[
E(p) = \{ \xi \in \partial \mathbb{D} : \lim_{z \to \xi} p(z) = \infty \},
\]

and are such that \( p'(\xi) \neq 0 \) for \( \xi \in \partial \mathbb{D} \setminus E(p) \).

We will use the following results, but we omit their proofs.

**Lemma 1.** ([5]) Let \( f, g \in \mathcal{A} \) and \( F, G \in \mathcal{C} \). If \( f \prec F \) and \( g \prec G \), then \( f * g \prec F * G \).

**Lemma 2.** ([4]) Let \( h(z) \) be convex in \( \mathbb{D} \), with \( h(0) = a, \lambda \neq 0 \) and \( \Re(\lambda) \geq 0 \). If \( p \in Q(a) = \{ p \in Q : p(0) = a \} \), \( p(z) + \frac{1}{\lambda} z p'(z) \) is univalent in \( \mathbb{D} \) and

\[
h(z) \prec p(z) + \frac{1}{\lambda} z p'(z)
\]

then \( q(z) \prec p(z) \), where

\[
q(z) = \frac{\lambda}{n \pi^{\lambda/n}} \int_0^z h(w) w^{\frac{\lambda}{n} - 1} dw.
\]

The function \( q \) is convex in \( \mathbb{D} \) and is the best subordinant.
In a recently paper [1] authors used subordination and investigated starlikeness and other properties of functions $f \in \mathcal{A}$ given by a double integral operator. In this article, using superordination, conditions on a different integral operator are investigated. Let $\delta_1 > -1$ and $\delta_2 > -1$. We consider functions $f \in \mathcal{A}$ defined by the double integral operator of the form

$$f(z) = (1 + \delta_1)(1 + \delta_2) \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} zG'(zt^{\mu} s^{\nu}) \, ds \, dt; \quad (G \in \mathcal{A}, z \in \mathbb{D}).$$  \hfill (2)

From (2) we see that

$$f'(z) = (1 + \delta_1)(1 + \delta_2) \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} g(zt^{\mu} s^{\nu}) \, ds \, dt,$$

where $g(z) = G'(z) + zG''(z)$. In addition, we will see that there are suitable parameters $\alpha, \lambda$ such that

$$f'(z) + \alpha zf''(z) + \lambda z^2 f'''(z) = g(z).$$

2. Main results

Let $h(z)$ be a convex-univalent function in $\mathbb{D}$ with $h(0) = 1$. For $\alpha \geq \lambda \geq 0$, consider $f'(z) + \alpha zf''(z) + \lambda z^2 f'''(z)$ is univalent in $\mathbb{D}$. We define the class $S(\alpha, \lambda, h)$ of functions $f \in \mathcal{A}$ as following

$$S(\alpha, \lambda, h) = \{f \in \mathcal{A} : h(z) \prec f'(z) + \alpha zf''(z) + \lambda z^2 f'''(z), \ z \in \mathbb{D}\}.$$  

Put

$$\mu = \frac{1 + \delta_2}{2}((\alpha - \lambda) - \sqrt{\Delta}), \quad \alpha - \lambda = \frac{\nu}{1 + \delta_1} + \frac{\mu}{1 + \delta_2}, \quad (1 + \delta_1)(1 + \delta_2)\lambda = \mu \nu \quad (3)$$

where $\Delta = (\alpha - \lambda)^2 - 4\lambda$. It is seen that $\Re(\mu) \geq 0$ and $\Re(\nu) \geq 0$. Now we write the solution of

$$f'(z) + \alpha zf''(z) + \lambda z^2 f'''(z) = g(z)$$

in its double integral form. The relations (3) and (4) show that

$$g(z) = f'(z) + \left(\frac{\mu \nu}{(1 + \delta_1)(1 + \delta_2)} + \frac{\nu}{1 + \delta_1} + \frac{\mu}{1 + \delta_2}\right) z f''(z) + \frac{\mu \nu}{(1 + \delta_1)(1 + \delta_2)} z^2 f'''(z)$$

$$= \frac{\nu}{1 + \delta_1} z^{1 + \frac{1 + \delta_1}{\nu}} \left(\frac{\mu}{1 + \delta_2} z^{1 + \frac{1 + \delta_1}{\nu}} f''(z) + z^{1 + \frac{1 + \delta_1}{\nu} f'(z)}\right)'$$

$$= \frac{\nu}{1 + \delta_1} z^{1 + \frac{1 + \delta_1}{\nu}} \left(\frac{\mu}{1 + \delta_2} z^{1 + \frac{1 + \delta_1}{\nu} - \frac{1 + \delta_2}{\nu}} (z^{1 + \frac{1 + \delta_2}{\nu} f'(z)})'\right).$$
Theorem 3. Let

\[
\frac{\mu}{1 + \delta_2} z^{1 + \frac{1 + \delta_1}{\nu}} f'(z) = \frac{1 + \delta_1}{\nu} \int_0^z w^{1 + \frac{1}{\nu} - 1} g(w) \, dw.
\]

We have

\[
\subordinant.
\]

Then the function \( w \) is the best subordinant.

Proof. We have

\[
(1 + \delta_1)(1 + \delta_2) = \int_0^1 s^{\delta_1} h(z t^\mu s^\nu) \, ds \, dt.
\]

Therefore

\[
(\psi_{\delta_1,\nu}(z) \ast \psi_{\delta_2,\mu}(z)) \ast h(z) = \int_0^1 t^{\nu} \, dt = \int_0^1 t^{\delta_2} h(z t^\mu) \, dt.
\]

By Theorem \([3], 2.6h\) it is seen that \( \psi_{\delta,\lambda}(z) \in C \) provided that \( \Re(\lambda) \geq 0. \)

Theorem 3. Let \( \mu \) and \( \nu \) be defined as (3) and

\[
q(z) = (1 + \delta_1)(1 + \delta_2) \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} h(z t^\mu s^\nu) \, ds \, dt.
\]

Then the function \( q(z) = (1 + \delta_1)(1 + \delta_2)(\psi_{\delta_1,\nu} \ast \psi_{\delta_2,\mu} \ast h)(z) \) is convex. If \( f \in S(\alpha, \lambda, h), f'(z) \in Q \) and \( f''(z) + \frac{\nu}{1 + \delta_1} z f''(z) \in Q \) then \( q(z) < f'(z) \) and \( q \) is the best subordinant.

Proof. We have

\[
(\psi_{\delta_1,\nu}(z) \ast \psi_{\delta_2,\mu}(z)) \ast h(z) = \int_0^1 t^{\nu} \, dt = \int_0^1 t^{\delta_2} h(z t^\mu) \, dt.
\]

The function \( q(z) \) is convex, since the functions \( \psi_{\delta_1,\nu}, \psi_{\delta_2,\mu} \) and \( h \) are convex univalent in \( \mathbb{D} \) (see \([2]\)). Put \( p(z) = f'(z) + \frac{\nu}{1 + \delta_1} z f''(z) \), then \( h(z) < p(z) + \frac{\mu}{1 + \delta_2} z p'(z) \). By Lemma 2 we obtain

\[
\frac{1 + \delta_2}{\mu z^{1 + \delta_2}} \int_0^z w^{1 + \delta_2 - 1} h(w) \, dw = (1 + \delta_2)(\psi_{\delta_2,\mu}(z) \ast h(z)) < p(z),
\]

\[
\int_0^z w^{1 + \delta_2 - 1} h(w) \, dw = (1 + \delta_2)(\psi_{\delta_2,\mu}(z) \ast h(z)) < p(z),
\]

\[
\int_0^z w^{1 + \delta_2 - 1} h(w) \, dw = (1 + \delta_2)(\psi_{\delta_2,\mu}(z) \ast h(z)) < p(z),
\]
or equivalently
\[(1 + \delta_2)(\psi_{\delta_2,\mu}(z) * h(z)) \prec f'(z) + \frac{\nu}{1 + \delta_1} z f'''(z).\]

Using again Lemma 2 we obtain
\[\frac{1 + \delta_1}{\nu z^{1+\delta_1}} \int_0^z (1 + \delta_2) w^{1+\delta_1 - 1}(\psi_{\delta_2,\mu} * h)(w) \, dw \prec f'(z)\]
or equivalently \(q(z) \prec f'(z)\). Since \(q(z) + \alpha zq'(z) + \lambda z^2 q''(z) = h(z)\), this means that \(q(z)\) is a solution of the differential superordination
\[h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) = p(z) + \alpha z p'(z) + \lambda z^2 p''(z)\] (6)
which \(f'(z)\) also satisfies (6). Therefore \(q(z)\) will be a dominant for all subordinants of \(h(z) \prec f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)\). Hence \(q(z)\) is the best subordinant of it.

**Corollary 4.** Suppose that all conditions of Theorem 3 are satisfied. Then
\[(1 + \delta_1)(1 + \delta_2) \int_0^1 \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} h(z t \nu s^{-\nu}) \, dr \, dt \, ds = \int_0^1 q(tz)dt \prec \frac{f(z)}{z}.\]

**Proof.** Consider \(p(z) = \frac{f(z)}{z}\), then \(q(z) \prec p(z) + z p'(z) = f'(z)\). Lemma 2 shows that
\[\int_0^1 q(tz)dt = \frac{1}{z} \int_0^z q(w)dw \prec p(z) = \frac{f(z)}{z}.\]

Using Theorem 3 and Corollary 4 with \(h(z) = \frac{1 + Az}{1 + Bz}\) where \(-1 \leq B < A \leq 1\), we obtain the following result.

**Corollary 5.** Suppose that all conditions of Theorem 3 are satisfied. If
\[\frac{1 + Az}{1 + Bz} \prec f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)\]
then \(q(z; A, B) \prec f'(z)\), where
\[q(z; A, B) = \frac{A}{B} - \frac{(1 + \delta_1)(1 + \delta_2)(A - B)}{B} \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} \, ds \, dt \; (B \neq 0)\]
and
\[q(z; A, 0) = 1 + \frac{A(1 + \delta_1)(1 + \delta_2)}{(1 + \delta_1 + \nu)(1 + \delta_2 + \mu)}\] \(f'(z)\),
also the functions $q(z; A, B)$ and $q(z; A, 0)$ are the best subordinants. In addition

$$\frac{A}{B} - \frac{(1 + \delta_1)(1 + \delta_2)(A - B)}{B} \int_0^1 \int_0^1 \int_0^1 s^\delta_1 t^\delta_2 \, dr \, ds \, dt < \frac{f(z)}{z}$$

if $B \neq 0$, and

$$1 + \frac{A(1 + \delta_1)(1 + \delta_2)z}{2(1 + \delta_1 + \nu)(1 + \delta_2 + \mu)} < \frac{f(z)}{z}$$

for $B = 0$.

Finally, the last theorem is about the convolution of two functions in $S(\alpha, \lambda, h)$.

**Theorem 6.** Let $\mu$ and $\nu$ are given by (3) and $f, g \in S(\alpha, \lambda, h)$. If $g'(z) \in Q$, $g'(z) + \frac{\nu}{1 + \delta_1} z g''(z) \in Q$ and $f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z), \frac{g(z)}{z} \in C$, then $f * g$ belongs to $S(\alpha, \lambda, h_1)$ where $h_1(z) = q(z) * \int_0^1 h(tz)dt$ and $q(z)$ is given by (5).

**Proof.** It is easy to see that

$$(f * g)'(z) + \alpha z (f * g)''(z) + \lambda z^2 (f * g)'''(z) = (f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)) * \frac{g(z)}{z}.$$  

Hence

$$h_1(z) = q(z) * \int_0^1 h(tz) \, dt$$

$$= (1 + \delta_1)(1 + \delta_2)(h(z) * \psi_{\delta_1, \nu}(z) * \psi_{\delta_2, \mu}(z)) * (h(z) * \psi_1(z))$$

$$= (1 + \delta_1)(1 + \delta_2)(h(z) * \int_0^1 \int_0^1 s^\delta_1 t^\delta_2 h(zrt^\mu s^\nu) \, dr \, ds \, dt)$$

(by Lemma 1) $$< (f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)) * \frac{g(z)}{z}$$

$$= (f * g)'(z) + \alpha z (f * g)''(z) + \lambda z^2 (f * g)'''(z),$$

where $\psi_1(z) = \int_0^1 \frac{dr}{1 - zr}$. This completes the proof.

**References**


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