APPLICATIONS OF CARLSON SHAFFER OPERATOR IN UNIVALENT FUNCTION THEORY

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Abstract. In this research paper, we introduce some new classes of k-starlike functions and k-uniformly close-to-convex functions in the unit disk \( E = \{ z : |z| < 1 \} \) by using Carlson-Shaffer operator. Some inclusion relationships, coefficient bounds and other interesting properties of these classes are investigated. Some known results are derived as special cases.

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1. Introduction

Let \( A \) be the class of functions \( f(z) \) given by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1.
\]  

analytic in \( E = \{ z : |z| < 1 \} \). Let \( S, C, S^*, K \) be the subclasses of \( A \) of univalent, convex, starlike and close-to-convex functions respectively. The convolution (Hadamard product) given by

\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad |z| < 1,
\]

where \( f(z) \) is given by (1.1) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \), see [2].

Let \( f \) and \( g \) be analytic in \( E \). The function \( f \) is subordinate to \( g \), written \( f \prec g \) or \( f(z) \prec g(z) \), if \( g \) is univalent in \( E \), \( f(0) = g(0) \) and \( f(E) \subset g(E) \), see [7].
Let incomplete beta function \( \phi(a, c; z) \), see [9] defined by

\[
\phi(a, c; z) = z {}_2F_1(1, a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}} z^n, \quad |z| < 1, \ c \neq 0, -1, -2, \ldots, \quad (1.3)
\]

where \((a)_n\) is Pochhammer symbol defined in terms of the Gamma functions, by

\[
(a)_k = \frac{\Gamma(a + n)}{\Gamma(n)} = \begin{cases} 1, & n = 0, \\ n(n+1)(n+2)\ldots(n+n-1), & n \in N. \end{cases} \quad (1.4)
\]

Further for \( f(z) \in A \), then a linear operator \( L(a, c) : A \to A \), see [1] defined as

\[
L(a, c)f(z) = \phi(a, c; z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}} z^n, \quad |z| < 1, \quad (1.5)
\]

where \( \phi(a, c; z) \) is given by (1.3). It follows from (1.3) and (1.5) that

\[
z(L(a, c)f(z))' = aL(a+1, c)f(z) - (a-1)L(a, c)f(z). \quad (1.6)
\]

\( L(a, c)f \) is a polynomial for \( a = 0, -1, -2, \ldots \). For \( a \neq 0, -1, -2, \ldots \), root test implies that

\[
\lim_{n \to \infty} \left| \frac{(a)_n}{(c)_n} \right|^\frac{1}{n} = 1.
\]

This shows that infinite series for \( L(a, c)f \) and \( f \) has same radius of convergence. There is \( 1-1 \) mapping of \( A \) onto itself with \( L(a, a) \) as identity and \( L(c, a) \) is the continuous inverse of \( L(a, c) \) \( (a \neq 0, -1, -2, \ldots) \). Furthermore, if \( h(z) = zf'(z) \), then \( f(z) = L(1, 2)h(z) \) and \( h(z) = L(2, 1)f(z) \). Carlson-Shaffer operator generalizes other linear operators.

In 1999, Kanas and Wiśniowska [3] introduced the conic domain \( \Omega_k, \ k \geq 0 \) and studied it comprehensively, defined as

\[
\Omega_k = \left\{ u + iv : u > k \sqrt{(u - 1)^2 + v^2} \right\}. \quad (1.7)
\]

Extremal functions for the conic regions \( \Omega_k \) are given as

\[
p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi} \left( \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right), & k = 1, \\ 1 + \frac{2}{\pi} \sinh^2 \left( \frac{1}{2} \arcsin k \right) \arctan h\sqrt{z}, & 0 < k < 1, \\ 1 + \frac{1}{k^2-1} \sin \left( \frac{\pi}{2bR(t)} \right) \frac{u(x)}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-(tx)^2}} dx \right) + \frac{1}{k^2-1}, & k > 1, \end{cases} \quad (1.8)
\]
where, \( u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{t}z}, \ t \in (0,1), |z| < 1 \) and \( z \) can be chosen such that \( k = \cosh \left( \frac{\pi R(t)}{4R(t)} \right) \), \( R(t) \) is Legendre’s complete elliptic integral of \( R(t) \), see [3], [4].

If \( p_k(z) = 1 + \delta_k z + .... \), then from (1.8) one can have

\[
\delta_k = \begin{cases} 
\frac{8(\arccos k)^2}{\pi^4(1-k^2)}, & 0 \leq k < 1 \\
\frac{8}{\pi^2}, & k = 1 \\
\frac{\pi^2}{4(k^2-1)\sqrt{t(1+t)k^2(t)}}, & k > 1
\end{cases}
\] (1.9)

Later, Kanas and Wisniowska [3] defined the class of functions which maps open unit disk \(|z| < 1\) into these conic regions and denoted this class by \( P(p_k) \) as,

\[ p \] satisfies the condition \( p(0) = 1 \) belongs to the class \( P(p_k) \), if \( p(z) \prec p_k(z) \), \(|z| < 1\). That is, \( p(E) \subset p_k(E) = \Omega_k \). \( p(z) \in P(p_k) \) holds following property that \( \Re (p(z)) > \frac{k}{k+1}. \)

Now we define the following classes.

**Definition 1.1** If a function \( f \) is analytic in \(|z| < 1\) and defined by (1.1), then \( f \in k - UCV(a,c) \) if and only if

\[ L(a,c)f \in k - UCV \quad (c \neq 0, -1, -2, ...). \] (1.10)

**Special Cases**
1. \( 0 - UCV(1,1) \equiv C \), see [17].
2. \( k - UCV(1,1) \equiv k - UCV \), we refer [3].

**Definition 1.2** If \( f \) is analytic in \(|z| < 1\) and defined by (1.1), then \( f \in k - UT(a,c) \) if and only if

\[ L(a,c)f \in k - ST \quad (c \neq 0, -1, -2, ...). \] (1.11)

**Special Cases**
1. \( 0 - UT(a,c) \equiv T(a,c) \) introduced and studied in [18],
2. \( 0 - UT(1,1) \equiv S^* \), see [12].
3. \( k - UT(2,1) \equiv k - UCV \), we refer [3].
4. \( k - UT(1,1) \equiv k - ST \) introduced and studied in [3].
5. \( 0 - UT(2,1) \equiv C \), see [17].

The relationship between the classes of is \( k - UCV(a,c) \) and \( k - UT(a,c) \) is given as

\[ f \in k - UCV(a,c) \ if \ and \ only \ if \ z f' \in k - UT(a,c). \] (1.11)
Definition 1.3 If $f$ is analytic in $|z| < 1$ and defined by (1.1), then $f \in k - UK(a,c)$ if and only if
\[ L(a,c)f \in k - UK \quad (c \neq 0, -1, -2,...). \] (1.12)

Special Cases

(i) $0 - UK(1,1) \equiv K$,
(ii) $k - UK(1,1) \equiv k - UK$, see [14].
(iii) $0 - UK(2,1) \equiv C^*$, we refer [15].
(iv) $k - UK(1,1) \equiv k - UC^*$, introduced in [14].
(v) We take $g(z) = f(z)$ in (1.12), we obtain the class $k - UCV(a,c)$.

Definition 1.4 If $f$ is an analytic function in $|z| < 1$ and defined by (1.1), then $f \in k - UC^*(a,c)$ if and only if
\[ L(a,c)f \in C^* \quad (c \neq 0, -1, -2,...). \] (1.13)

Special Cases

(i) $0 - UC^*(1,1) \equiv C^*$, see [15].
(ii) $k - UC^*(1,1) \equiv k - UC^*$, we refer [14].
(iii) We take $g(z) = f(z)$ in (1.13), we obtain the class $k - UCV(a,c)$.

The relationship between the classes of is $k - UC^*(a,c)$ and $k - UK(a,c)$ is given as
\[ f \in k - UC^*(a,c) \quad if \ and \ only \ if \quad zf' \in k - UK(a,c). \] (1.14)

2. PRELIMINARY CONCEPTS

To prove our results, we need the following lemmas.

Lemma 2.1 [5] Let $f(z) = z + a_2z^2 + a_3z^3 + ... \in k - ST$. Then
\[ |a_2| \leq |\delta_k|. \]

This coefficient bound is also holds for the classes of $k - UCV$, $k - UK$ and $k - UC^*$.

Lemma 2.2 [11] If $a$, $b$ and $c$ are real and satisfy
\[ -1 \leq a \leq 1, b \geq 0 \quad and \quad c > 1 + \max\{2 + |a + b - 2|, 1 - (a - 1)(b - 1)\}, \]
then
\[ zF(a, b; c; z) \in S^*, \quad (2.1) \]

where

\[
F(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}
\]
is the Guassian hypergeometric function.

**Lemma 2.3** [18] If \( a \) and \( c \) are real and satisfy

\[-1 \leq a \leq 1 \quad \text{and} \quad c > 3 + |a|,\]

then \( \phi(a, c; z) \) defined by (1.3) is convex in \( E \).

**Lemma 2.4** [16] The class \( S^* \) and \( K \) are closed under convex convolution.

**Lemma 2.5** [5] Let \( 0 \leq k < \infty \) and \( \beta, \delta \) be any complex numbers with \( \beta \neq 0 \) and \( \Re(\frac{\beta k}{k+1} + \delta) > \delta \) where \( \gamma \) is defined as:

If \( h(z) \) is analytic in \( E \), \( h(0) = 1 \) and it satisfies

\[
h(z) + \frac{zh'(z)}{\beta h(z) + \delta} < p_{k,\gamma}, \quad (2.2)
\]

and \( q_{k,\gamma} \) is an analytic solution of

\[
q_{k,\gamma}(z) + \frac{zq'_{k,\gamma}(z)}{\beta q_{k,r}(z) + \delta} = p_{k,r}(z), \quad (2.3)
\]

then \( q_{k,\gamma}(z) \) is univalent, \( h(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z) \), and \( q_{k,\gamma}(z) \) is best dominant of (2.2).

**Lemma 2.6** [17] If \( f(z) \in C \) and \( g \in S^* \), then for any analytic function in \( E \) with \( F(0) = 1 \).

\[
\frac{f \ast Fg}{f \ast g}(E) \subset C_0 F(E), \quad f \in C, \ g \in S^*, \quad (2.4)
\]

where \( C_0 F(E) \) denotes the closed convex hull of \( F(E) \) (the smallest convex set which contain \( F(E) \)).

**Lemma 2.7** [10] Let \( P \) be a complex function in \( E \), with \( \Re(P(z)) > 0 \) for \( z \in E \) and \( h \) be a convex function in \( E \). If \( p(z) \) be a analytic function in \( E \), with \( p(0) = h(0) \) then,

\[
p(z) + P(z)zp'(z) < h(z). \quad (2.5)
\]
3. Main Results

**Theorem 3.1** For \( a \geq 1 \)

\[
k - UT(a + 1, c) \subset k - UT(a, c).
\]

**Proof.** Let \( f(z) \in k - UT(a + 1, c) \).

Let

\[
\frac{z(L(a, c)f(z))'}{L(a, c)f(z)} = p(z).
\]

(3.1)

Then \( p(z) \) is analytic with \( p(0) = 1 \).

From (2.6) and (3.1), we have

\[
aL(a + 1, c)f(z) - (a - 1)L(a, c)f(z) = p(z)L(a, c)f(z),
\]

or

\[
aL(a + 1, c)f(z) = L(a, c)f(z)[(a - 1) + p(z)].
\]

Differentiating logarithmically, we get

\[
\frac{a(L(a + 1, c)f(z))'}{L(a + 1, c)f(z)} = \frac{z(L(a, c)f(z))'}{L(a, c)f(z)} + \frac{zp'(z)}{p(z) + (a - 1)}
\]

(1)

\[
= p(z) + \frac{zp'(z)}{p(z) + (a - 1)}.
\]

Since \( f \in k - UT(a + 1, c) \), it follows that

\[
\left\{ p(z) + \frac{p'(z)}{p(z) + (a - 1)} \right\} < pk(z),
\]

and by using Lemma, \( p(z) < pk(z) \). This proves that \( f(z) \in k - UT(a, c) \) in \( E \).

As special case we note that for \( k = 0 \) in Theorem 3.1, we obtain the known result given in [18].

**Theorem 3.2** Let \( f(z) \in k - UT(a, c) \) and

\[
F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma - 1} f(t)dt \quad (\gamma \geq 0).
\]

(3.3)

Then \( F(z) \in k - UT(a, c) \).

**Proof.** From (3.3), we note that \( F(z) \in A \) and

\[
r(L(a, c)F(z)) + z(L(a, c)F(z))' = (\gamma + 1)L(a, c)f(z).
\]

(3.4)
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Let
\[ h(z) = \frac{z(L(a, c)F(z))'}{L(a, c)F(z)}. \]  

(3.5)

We note that \( h(z) \) is analytic in \( E \) write \( h(0) = 1 \).

Then, from (3.4), we have
\[ r + h(z) = \frac{(r + 1) L(a, c)f(z)}{L(a, c)F(z)}. \]

Differentiating Logarithimacally, we get
\[ h(z) \prec p_k(z) \text{ in } E, \]

and this proves that \( F(z) \in k - UT(a, c) \text{ in } E. \)

**Theorem 3.3** For \( a \geq 1, \)
\[ k - UK(a + 1, c) \subset k - UK(a, c). \]

**Proof.** Let \( f(z) \in k - UK(a + 1, c). \) Then there exists \( g(z) \in k - UT(a + 1, c) \) such that
\[ z(L(a + 1, c)f(z))' = p(z). \]  

(3.6)

Using (1.6), we have
\[ aL(a + 1, c)f(z) - (a - 1)L(a, c)f(z) = p(z)(L(a, c)g(z)), \]

and differentiating we get
\[ a(L(a + 1, c)f(z))' = p'(z)(L(a, c)g(z)) + (a - 1)(L(a, c)f(z))' + p(z)(L(a, c)g(z))' \]
\[ = p'(z)(L(a, c)g(z)) + (a - 1)(L(a, c)f(z))' \]
\[ + p(z)[aL(a + 1, c)g(z) - (a - 1)L(a, c)g(z)]. \]  

(2)

Using (1.6), we can write
\[ \frac{z(L(a + 1, c)f(z))'}{L(a + 1, c)g(z)} = \frac{L(a, c)g(z)}{aL(a + 1, c)g(z)} \]
\[ + p(z) \left\{ \frac{aL(a, c)g(z)}{L(a + 1, c)g(z)} + (a - 1) \left( \frac{L(a, c)f(z)}{L(a, c)g(z)} - \frac{L(a, c)g(z)}{L(a + 1, c)g(z)} \right) \right\} \]
\[ + p(z) \left\{ 1 - (a - 1) \frac{(L(a, c)g(z))'}{L(a, c)g(z)} - \frac{L(a, c)g(z)}{L(a + 1, c)g(z)} \right\}. \]  

(3)
Since $g(z) \in k - UT(a + 1, c)$ and $k - UT(a + 1, c) \subset k - UT(a, c)$, it follows that

$$\frac{z(L(a, c)g(z))'}{L(a, c)g(z)} = p_0(z) < p_k(z).$$

From (??), (??), we get

$$\frac{z(L(a + 1, c)f(z))'}{L(a + 1, c)f(z)} = p(z) + \frac{zp'(z)}{p_0(z) + (a - 1)}.$$ (3.7)

Now $p_0(z) \in P(p_k) \subset P\left(\frac{k}{k+1}\right) \subset P$ and $a \geq 1$, so $\Re(p_0(z) + (a - 1)) > 0$. Let $h_0(z) = \frac{1}{p_0(z) + (a - 1)}$. Then $\Re h_0(z) > 0$ in $E$.

Thus, from (3.9) and $f(z) \in k - UK(a + 1, c)$, we obtain

$$[p(z) + h_0(z)(zp'(z))] \prec p_k(z).$$

Using Lemma 2.7, it gives us that

$$p(z) \prec p_k(z),$$

which proves that $f(z) \in k - UK(a, c)$ in $E$. This completes the proof.

Theorem 3.4 For $F(z)$ be defined by (3.3) and $f(z) \in k - UK(a, c)$, $z \in E$. Then

$$F(z) \in k - UK(a, c).$$

Proof. Since $f(z) \in k - UK(a, c)$, there exists $g(z) \in k - UT(a, c)$ such that

$$\frac{z(L(a, c)f(z))'}{L(a, c)g(z)} < p_k(z), z \in E.$$

Let

$$G(z) = \frac{\gamma + 1}{z\gamma} \int_0^z t^{\gamma-1} g(t)dt \quad (\gamma \geq 0).$$ (3.10)

Then, by Theorem 3.2 leads us that $G(z) \in k - UT(a, c)$ in $E$.

Let

$$H(z) = \frac{z(L(a, c)F(z))'}{L(a, c)G(z)}.$$ (3.11)

Then $H(z)$ is analytic in $E$ with $H(0) = 1$.

From (3.10) and (3.11), we have

$$H'(z)(L(a, c)G(z)) + H(z)(L(a, c)G(z))' = -\gamma(L(a, c)F(z))' + (\gamma + 1)(L(a, c)f(z))'.$$
This gives us
\[
zH'(z) + H(z)\frac{z(L(a, c)G(z))'}{L(a, c)G(z)} = -\gamma \frac{z(L(a, c)F(z))'}{L(a, c)G(z)} + (\gamma + 1)\frac{z(L(a, c)f(z))'}{L(a, c)g(z)}.
\] (3.12)

Let \( \frac{(L(a, c)G(z))'}{L(a, c)G(z)} = p_0(z), p_0(z) \in P(p_k) \subset P \) and so \( \Re(p_0(z) + \gamma) \in P \) in \( E \). It follows that
\[
\left\{ H(z) + \frac{zH'(z)}{p_0(z) + \gamma} \right\} < p_k(z).
\]
From this we have
\[
H(z) + h_1(z)(zH'(z)) < p_k(z),
\]
where \( h_1(z) = \frac{1}{p_0(z) + \gamma} \in P \).

We now apply Lemma 2.7, and this gives us \( H(z) < p_k(z) \), which proves that \( F(z) \in k - UK(a, c) \) in \( E \).

**Theorem 3.5** Let \( f \in k - UT(a, c) \) and \( \phi \in C \), then \( \phi * f \in k - UT(a, c) \).

**Proof.** Let
\[
z \frac{[L(a, c)(f * \phi)(z)']}{L(a, c)(f * \phi)(z)} = z(L(a, c)f(z))' * \phi(z)
\]
\[
= \frac{\phi(z) * z(L(a, c)f(z))'}{\phi(z) * L(a, c)f(z)}
\]
\[
= \frac{\phi(z) * h(z)(L(a, c)f(z))}{\phi(z) * L(a, c)f(z)}.
\]

Now \( \phi \in C \), \( L(a, c)f(z) \in k - UT \subset S^* \), \( h(z) \in P(p_k) \), so using Lemma 2.6 we have
\[
z(L(a, c)(f * \phi)') \in P(p_k),
\]
and therefore \( \phi * f \in k - UT(a, c) \).

**Special Cases**

(i) We take \( k = 0 \), it follows that \( S(a, c) \) is invariant under convex convolution.

(ii) For \( a = 1, \ c = 1 \) and \( k = 0 \), we get the well known result that the class \( S^* \) is closed under convolution with convex function. For this we refer [17].

Following the similar techniques, we can easily prove the following.

**Theorem 3.6** Let \( \phi \in C \) and let \( f \in k - UK(a, c) \). Then \( \phi * f \in k - UK(a, c) \).

(We include the proof for the sake of completeness).

**Proof.** Since \( f \in k - UK(a, c) \), \( \frac{z(L(a, c)f(z))'}{L(a, c)g(z)} \in P(p_k), g \in k - UT(a, c) \).
\[
\frac{z [L(a,c)(f \ast \phi)(z)]'}{L(a,c)(g \ast \phi)(z)} = \frac{\phi(z) \ast \frac{z[L(a,c)f(z)]'}{L(a,c)g(z)}L(a,c)g(z)}{\phi(z) \ast L(a,c)g(z)} = \frac{\phi(z) \ast h(z)(L(a,c)g(z))}{\phi(z) \ast L(a,c)g(z)};
\]

where \( \phi \in C, \ h \in P(p_k), \ L(a,c)g \in S^*. \) Now on using Lemma 2.6, we obtain the required result that \((f \ast \phi) \in k - UK(a,c)\) in \(E.\)

As special cases we note that when \(a = 1, \ c = 1\) and \(k = 0\) in Theorem 3.6, it follows that the class \(K\) is closed under convolution with convex function, see [17].

### Applications of Theorem 3.5 and Theorem 3.6

From Theorem 3.5 and Theorem 3.6, it follows that the classes \(k - UT(a,c)\) and \(k - UK(a,c)\) are invariant under convolution with convex function. Using this fact, it can be easily verified that these classes are closed under the integral operators given as:

(i) \(f_1(t) = \int_{0}^{t} \frac{z}{t} f(t) dt.\)

(ii) \(f_2(t) = 2 \int_{0}^{t} \frac{z}{2} f(t) dt.\)

(iii) \(f_3(t) = \frac{c+1}{z} \int_{0}^{t} t^{c-1} f(t) dt.\)

As applications of Theorem 3.5 and Theorem 3.6 we have following results.

**Theorem 3.7** Let \(a\) and \(c\) be real and satisfy

\[c \neq 0, \ -1 < c \leq 1, \text{ and } a > 3 + |c|. \tag{3.13}\]

Then

\[k - UT(a,c) \subset k - ST.\]

**Proof.** If \(f(z) \in k - UT(a,c)\). That is \(L(a,c)f(z) = \phi(a,c) \ast f(z) \in k - ST.\) Since \(a\) and \(c\) satisfy the condition (3.13), we have from that \(\phi(c,a) \in C.\) Therefore, an application of Theorem 3.5 leads to

\[f = \phi(c,z) \ast \phi(a,c)f \in k - ST.\]

As special case we take \(k = 0\), then we obtain the known result given in [18].
Using Theorem 3.6 and similar techniques we have the following.

**Theorem 3.8** Let \( a \) and \( c \) be real and satisfy (3.13). Then

\[
k - UK(a, c) \subset k - UK.
\]

**Theorem 3.9** Let \( a, c \) and \( d \) be real. If

\[
d \neq 0, -1 < d \leq 1 \text{ and } c > 3 + |d|,
\]

then

(i) \( k - UT(a, d) \subset k - UT(a, c) \),

(ii) \( k - UK(a, d) \subset k - UK(a, c) \).

**Proof.** Let

\[
f(z) \in k - UT(a, d).
\]

Then

\[
L(a, d)f(z) = \phi(a, d) * f(z) \in k - ST.
\]

Using Lemma 2.3, \( \phi(d, c) \in C \). Hence,

\[
L(a, c)f(z) = \phi(a, c) * f(z) = \phi(a, d) * \phi(d, c) * f(z) = \phi(d, c) * \phi(a, d) * f(z).
\]

Since \( \phi(a, d) * f(z) = L(a, d)f(z) \in k - ST \) and \( \phi(d, c) \in C \), it follows \( L(a, c)f(z) \in k - ST \) and consequently \( f(z) \in k - UT(a, c) \). This completes the proof.

Proof of (ii) is similar and therefore omit it.

As special case we take \( k = 0 \) in Theorem 3.9, this implies the following.

(i) \( S(a, d) \subset S(a, c) \) which has been proved in [18].

(ii) \( K(a, d) \subset K(a, c) \).

**Theorem 3.10** Let \( f \in k - UT(a, c) \) and \( f(z) \) be given by (1.1). Then

\[
|a_2| \leq \left| \frac{c}{a} \right| \delta_k,
\]

**Proof.** Since we have \( L(a, c)f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \) belongs to \( k - ST \), this implies that

\[
\left| \frac{aa_2}{c} \right| \leq \delta_k,
\]

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which gives the required result.

**Special Cases**

(i) We take $k = 0$, we have $\delta_k = 2$. This implies that $|a_2| \leq 2|\frac{c}{a}|$.

(ii) For $k = 1$, we have $\delta_k = \frac{8}{\pi^2}$, from which follows that $|a_2| \leq |\frac{c}{a}|\frac{8}{\pi^2}$.

(iii) We take $a = 2$ and $c = 1$, it follows that $L(2, 1)f = zf'$. Therefore, we have $L(2, 1)f \in k-ST$ implies that $|a_2| \leq \frac{\delta_k}{4}$.

Let $a$ and $c$ satisfy condition (3.13). Then by Theorem 3.7, $f \in k-UT(a, c)$ is starlike and hence univalent. Using this observation, we prove the following covering result.

**Theorem 3.11** Let $a$ and $c$ satisfy (3.13) and let $f \in k-UT(a, c)$. Then $f(E)$ contains the disk

$$|w| < \frac{a}{2a + |c|\delta_k}.$$  \hspace{1cm} (3.15)

**Proof.** Since $f \in k-UT(a, c)$ with $a$ and $c$ defined by (3.13) is univalent,

$$g(z) = \frac{w_0f(z)}{w_0 - f(z)} = z + \left( a_2 + \frac{1}{w_0} \right) z^2 + \left( a_3 + \frac{1}{w_{02}} \right) z^3 + \ldots$$

is also univalent, where $w_0 (w_0 \neq 0)$ is complex number such that $f(z) \neq w_0$ for $z \in E$. Hence

$$\left| \frac{1}{|w_0|} - |a_2| \right| \leq \left| a_2 + \frac{1}{w_0} \right| \leq 2.$$

Now, using Theorem 3.10, we have $|a_2| \leq \frac{|c|\delta_k}{a}$, where $\delta_k$ is given by (1.9). This gives us

$$\frac{1}{|w_0|} \leq 2 + \frac{c}{a} \delta_k = \frac{2a + |c|\delta_k}{a},$$

which implies that

$$|w_0| \geq \frac{2a + |c|\delta_k}{a}.$$

This completes the proof of theorem.

**Special Cases**

(i) We take $k = 0$, we have $\delta_k = 2$. It follows that, $f(E)$ contains the disk $|w| \leq \frac{a}{2(a + |c|)}$, which has been proved in [18].

(ii) For $k = 1$, we have $\delta_k = \frac{8}{\pi^2}$. That is $f \in 1-UT(a, c)$ implies that $f(E)$ contains the disk $|w| \leq \frac{a\pi^2}{2(\pi^2+4)}$.  

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(iii) We take \( a = 2 \) and \( c = 1 \), it follows that \( L(2, 1) f = z f' \). Therefore, we have \( L(2, 1) f \in k - ST \) implies that \( f(E) \) contains the disk \( |w| < \frac{4}{8 + \delta_k} \).

**Theorem 3.12** Let \( f \in k - UT(a, c) \) and for \( \alpha \geq 0 \), let
\[
F_{\alpha}(z) = (1 - \alpha) f(z) + \alpha z f'(z).
\]

Then \( F_{\alpha}(z) \in k - UT(a, c) \) for \( |z| < r_{\alpha} \), where
\[
r_{\alpha} = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}. \tag{3.16}
\]

**Proof.** When \( \alpha = 0 \), the proof is immediate. So we take \( \alpha > 0 \). In Theorem 3.5, we have proved that the class \( k - UT(a, c) \) is preserved under convex convolution. We define
\[
\phi_{\alpha}(z) = (1 - \alpha) \frac{z}{1 - z} + \alpha \frac{z}{(1 - z)^2} = z + \sum_{n=2}^{\infty} (1 + (n - 1)\alpha) z^n. \tag{4}
\]

It is known [10] and can easily be verified that \( \phi_{\alpha}(z) \in C \) for \( |z| < r_{\alpha} \), where \( r_{\alpha} \) is given by (3.16).

We can write
\[
F_{\alpha}(z) = (1 - \alpha) f(z) + \alpha z f'(z) = \phi_{\alpha}(z) * f(z).
\]

Since \( f \in k - UT(a, c) \), \( \phi_{\alpha} \in C \) in \( |z| < r_{\alpha} \), therefore, by Theorem 3.5, it follows that \( F_{\alpha} \in k - UT(a, c) \) in \( |z| < r_{\alpha} = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}} \).

**Special Cases**

(i) Let \( \alpha = \frac{1}{2} \) in Theorem 4.2.8. Then we have \( F_{\alpha}(z) = \frac{(zf(z))'}{2} \). This is Livingston’s operator, see [8]. In this case, \( r_{\frac{1}{2}} = \frac{1}{2} \).

(ii) For \( \alpha = 1 \) in Theorem 4.2.6. It follows that \( F_{\alpha}(z) = zf'(z) \) and \( f \in k - UT(a, c) \). In this case \( F_{\alpha}(z) \in k - UT(a, c) \) for \( |z| < r_1 = \frac{1}{2 + \sqrt{3}} \).

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