NON-ARCHIMEDEAN STABILITY OF GENERALIZED JENSEN'S AND QUADRATIC EQUATIONS

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Abstract. We use the operatorial approach to provide a proof of the Hyers-Ulam stability for the equations

\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a_{\lambda}) = Nf(x), \quad x, y \in E, \]
\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a_{\lambda}) = Nf(x) + Nf(y), \quad x, y \in E, \]

where \( E \) is a normed space, \( F \) is a non-Archimedean Banach space, \( \Phi \) is a finite group of automorphisms of \( E \), \( N = |\Phi| \) designates the number of its elements, and \( \{a_{\lambda}, \lambda \in \Phi\} \) are arbitrary elements of \( E \). These equations provides a common generalization of many functional equations such as Cauchy's, \( \Phi \)-Jensens's, \( \Phi \)-quadratic, Lukasik's equation. Some applications of our results will be illustrated.

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1. Introduction

In [50], Ulam posed the question of the stability of Cauchy's equation: If a function \( f \) approximately satisfies Cauchy's functional equation \( f(x + y) = f(x) + f(y) \) when does it has an exact solution which \( f \) approximates. The problem has been considered for various equations, also for mappings with many different types of domains and ranges by a number of authors including Hyers [22, 23], Aoki [2], T. M. Rassias [41], J.M. Rassias [39, 40], Gajda [19] Gavrutà [20] and others. For definitions,
approaches, and results on Hyers-Ulam-Rassias stability we refer the reader to, e.g., ([18],[24],[29],[31],[43],[44],[51]-[53]).

The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y), \]

is called a quadratic functional equation. The first stability theorem for the Eq. (1) was proved by Skof [46] for mappings \( f \) from a normed space \( X \) into a Banach space \( Y \). Cholewa [12] extended Skof’s theorem by replacing \( X \) by an abelian group \( G \). Skof’s result was later generalized by Czerwik [14] in the spirit of Hyers-Ulam-Rassias. Since then, a number of stability results have been obtained for quadratic functional equations and Jensen’s functional equation ([1],[4],[6]-[10],[26]-[28],[33],[38]). Informations and applications about the Eq. (1) and its further generalizations can be found e.g. in ([13],[14],[17],[32],[42],[45],[47]-[49]).

The stability problem for the functional equation
\[ \frac{1}{|\Phi|} \sum_{\lambda \in \Lambda} f(x + \lambda y) = f(x) + g(y), \quad x, y \in X, \]

where \( X \) is an abelian group, \( \Phi \) is is a finite subgroup of the automorphism group of \( X \) and \( f, g : X \to \mathbb{C} \) was posed and solved by Badora in [4]. Equation (2) is a joint generalization of Cauchy’s functional equation (\( \Phi = \{id\} \), \( g = f \)), Jensen’s equation (\( \Phi = \{id, -id\} \), \( g = 0 \)) and the quadratic equation (\( \Phi = \{id, -id\} \), \( g = f \)). This result was published (with a different proof and \( h = f \) ) by Ait Sibaha et al. in [1] and generalized by Charifi et al. in ([6],[7]).

In [10], the authors gave an explicit description of the solutions \( f : S \to H \) each of the following generalized equations
\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a\lambda) = Nf(x), \quad x, y \in S, \]
\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a\lambda) = Nf(x) + Nf(y), \quad x, y \in S, \]

where \( S \) is an abelian monoid, \( H \) is an abelian group and \( \Phi \) is a finite subgroup of automorphisms of \( S \), and \( f, g : X \to H \), which covers the functional equations
\[ f(x + y + a) = f(x) + f(y), \quad x, y \in S, \]
\[ \Phi = \{id\} \]
\[ f(x + y + a) + f(x + \sigma(y) + b) = 2f(x), \quad x, y \in S, \]
\[ \Phi = \{id, \sigma\} \]
\[ f(x + y + a) + f(x + \sigma(y) + b) = 2f(x) + 2f(y), \quad x, y \in S, \]
\[ \Phi = \{id, \sigma\} \]
where \(a, b\) are fixed elements of \(S\) and \(\sigma\) is an involution of \(S\) i.e. \(\sigma(x + y) = \sigma(y) + \sigma(x)\) and \(\sigma(\sigma(x)) = x\) for all \(x \in S\).

In 1897, Hensel [21] has introduced a normed space which does not have the Archimedean property. Let \(p\) be a fixed prime number and \(x\) be a non-zero rational number, there exists a unique integer \(v_p(x)\in \mathbb{Z}\) such that \(x = p^{v_p(x)} \frac{a}{b}\) where \(a\) and \(b\) are integers co-prime to \(p\). The function defined in \(\mathbb{Q}\) by \(|x|_p = p^{-v_p(x)}, x \in \mathbb{Q}\) is called a \(p\)-adic, a ultrametric or simply a non-Archimedean absolute value on \(\mathbb{Q}\).

By a non-Archimedean field we mean a field \(K\) equipped with a function (valuation) \(|.| : K \rightarrow [0, +\infty)\), called a non-Archimedean absolute value on \(K\) and satisfying the following conditions:

(i) \(|x| = 0 \iff x = 0\), \(x \in K\),

(ii) \(|xy| = |x||y|, x, y \in K\),

(iii) \(|x + y| \leq \max(|x|, |y|), x, y \in \mathbb{K}\).

We assume, throughout this paper that this value absolute is non-trivial i.e., there exists an element \(k\) of \(K\) such that, \(|k| \neq 0, 1\).

**Definition 1.** By a non-Archimedean vector space, we mean a vector space \(E\) over a non-Archimedean field \(K\) equipped with a function \(|.| : E \rightarrow [0, +\infty)\) called a non-Archimedean norm on \(E\) and satisfying the following properties:

(i) \(\|x\| = 0 \iff x = 0\), \(x \in E\),

(ii) \(\|kx\| = |k|\|x\|, (k, x) \in K \times E\),

(iii) \(\|x + y\| \leq \max(\|x\|, \|y\|), x, y \in E\).

Due to the fact that

\[
\|x_m - x_n\| \leq \max \{\|x_j - x_{j-1}\|, n + 1 \leq j \leq m\}, m > n,
\]

a sequence \((x_n)\) is Cauchy if and only if \((x_{n+1} - x_n)\) converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are \(p\)-adic numbers. A key property of \(p\)-adic numbers is that they do not satisfy the Archimedean axiom: for all \(x\) and \(y > 0\), there exists an integer \(n\) such that \(x < ny\).

In [3], Arriola and Beyer initiated the stability of Cauchy’s functional equation over \(p\)-adic fields. Moslehian and T.M. Rassias [37] proved the Hyers Ulam Rassias stability of Cauchy’s functional and the quadratic functional equations in non-Archimedean normed space. For various aspects of the theory of stability in non-Archimedean normed space we can refer to ([8],[9],[16],[36],[37]).

Let \(\mathbb{K}\) be an ultrametric field of characteristic zero, \(E\) be a \(\mathbb{K}\)-vector space and \(F\) be a complete ultrametric \(\mathbb{K}\)-vector space (in particular in the field of \(p\)-adic numbers).
As continuation of some previous works, the purpose of the present paper is to prove the Hyers–Ulam stability of the functional equations (3) and (4) for mappings $f$ from a normed space $E$ into a non-Archimedean Banach space $F$.

2. Preliminaries

To formulate our results we introduce the following notation and assumptions that will be used throughout the paper:

Let $\mathbb{K}$ be an ultrametric field of characteristic zero (in particular in the field of $p$-adic numbers), $E$ be a $\mathbb{K}$-vector space, $F$ be a complete ultrametric $\mathbb{K}$-vector space and let $F^E$ denotes the vector space consisting of all maps from $E$ into $F$. We let $\Phi$ denotes a finite group of automorphisms of $E$, $N$ designates the number of its elements and $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of $E$.

We now recall the definition and some necessary notions of multi-additive mappings, using the sequel.

A function $A : E \rightarrow F$ is additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in E$.

Let $k \in \mathbb{N}$, be a function $A_k : E^k \rightarrow F$ is $k$-additive if it is additive in each variable, in addition we say that $A_k$ is symmetric if it satisfies $A_k(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(k)}) = A_k(x_1, x_2, ..., x_k)$ for all $(x_1, x_2, ..., x_k) \in E^k$ and all permutations $\pi$ of $k$ elements.

Some informations concerning on such mappings can be found for instance in [31].

Let $A_k : E^k \rightarrow F$ be a $k$-additive and symmetric function and let $A_k^* : E \rightarrow F$ defined by $A_k^*(x) = A_k(x, x, ..., x)$ for all $x \in E$. Such a function $A_k^*$ will be called a monomial function of degree $k$ (if $A_k^* \neq 0$). We note that it is easily seen that $A_k^*(rx) = r^k A_k^*(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$.

A function $P : E \rightarrow F$ is called a GP function (generalized polynomial function) of degree $m \in \mathbb{N}$ iff there exist $A_0 \in E$ and symmetric $k$-additive functions $A_k : E^k \rightarrow F$ (for $1 \leq k \leq m$) such that

$$A_k^* \neq 0 \text{ and } P(x) = A_0 + \sum_{k=1}^{m} A_k^*(x) \text{ for all } x \in E.$$

For $h \in E$ we define the linear difference operator $\Delta_h$ on $F^E$ by

$$\Delta_h(f)(x) = f(x + h) - f(x),$$

for all $f \in F^E$ and $x \in E$. Notice that these difference operators commute ($\Delta_h \Delta_{h'} = \Delta_{h'} \Delta_h$ for all $h, h' \in E$) and if $h \in E$, $n \in \mathbb{N}$ then $\Delta_h^n$ the n-th iterate of $\Delta_h$ satisfies

$$\Delta_h^n(f)(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh), \text{ for all } x, h \in E \text{ and } f \in F^E.$$
Now we note some results for later use.

**Theorem 1.** [5] Let $n \in \mathbb{N}$, $f \in F^E$ and $\delta \in \mathbb{R}^+$. Then the following statements are equivalent.

i) $\|\Delta_n^h f(x)\| \leq \delta$ for all $x, h \in E$.

ii) There is, up to a constant, a unique GP function $P$ of degree at most $n - 1$ such that $\|f(x) - f(0) - P(x)\| \leq \delta$ for all $x \in E$.

**Theorem 2.** [9] Let $(S, +)$ be an abelian monoid, $\Phi$ be a finite subgroup of the group of automorphisms of $S$, $N = \text{card}(\Phi)$, $(H, +)$ be an abelian group uniquely divisible by $(N + 1)!$ and $a_\lambda \in S$ ($\lambda \in \Phi$). Then the function $f : S \to G$ is a solution of equation

$$\sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \ x, y \in S, \quad (8)$$

if and only if $f$ has the following form

$$f(x) = A_0 + \sum_{i=1}^{N} A_i^*(x), \ x \in S, \quad (9)$$

where $A_0 \in G$ and $A_k : S^k \to G$, $k \in \{1, 2, ..., N\}$ are symmetric and $k$-additive functions satisfying the two conditions:

i) $\sum_{i=\max \{j\} \choose k}^{N \choose j} \sum_{\lambda \in \Phi} A_i(x, x, ..., x, a_\lambda, ..., a_\lambda, \lambda y, \lambda y, ..., \lambda y) = 0, \ x, y \in S, \quad 0 \leq k \leq N - 1, \ 0 \leq j \leq N - k, \ 2 \leq \max\{j + 1, k + 1, k + j\}$

and

ii) $\sum_{\lambda \in \Phi} \sum_{i=1}^{N} A_i^*(a_\lambda) = N A_0$.

**Theorem 3.** [8] Let $\Phi$ be a finite subgroup of the group of automorphisms of $E$, $N = \text{card}(\Phi)$, $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of $E$ and $f : E \to F$ satisfying the inequality

$$\left\| \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) - N f(x) - \sum_{\lambda \in \Phi} f(\lambda y) \right\| \leq \delta,$$

for all $x, y \in E$. Then there exists a unique GP function $P : E \to F$ of degree at most $N$ solution of the equation

$$\sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = N f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \ x, y \in E, \quad (10)$$

such that

$$\|f(x) - P(x)\| \leq \frac{\delta}{|N|} \text{ for all } x \in E.$$
Lemma 4. [8] Let $\Phi$ be a finite automorphism group of $E$, $N = \text{card}\Phi$, $\delta, \delta' \in \mathbb{R}^+$, $a_\lambda \in E$ ($\lambda \in \Phi$), and $f \in F^E$ such that

$$\left\| \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) - N f(x) - \sum_{\lambda \in \Phi} f(\lambda y) \right\| \leq \delta, \ x, y \in E. \quad (11)$$

Then, there exists a mapping $h \in F^E$ which satisfies

$$\| \Delta^N_y f(x) - h(y) \| \leq \frac{\delta}{|N|}, \ x, y \in E,$$

and

$$\| \Delta^{N+1}_y f(x) \| \leq \frac{\delta'}{|N|}, \ x, y \in E. \quad (12)$$

Furthermore, if $\left\| \sum_{\lambda \in \Phi} (\lambda y) \right\| \leq \delta'$, $y \in E$, then $\| \Delta^N_y f(x) \| \leq \max\left(\frac{\delta}{|N|}, \frac{\delta'}{|N|}\right)$, $x, y \in E$.

In the next two theorems the solutions of the functional equations (3) and (4), respectively, will be expressed in terms of $GP$ functions.

Theorem 5. [10] Let $(S, +)$ be an abelian monoid, $\Phi$ be a finite subgroup of the group of automorphisms of $S$, $N = \text{card}(\Phi)$, $(H, +)$ be an abelian group uniquely divisible by $N!$ and $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of $S$. Then the function $f : S \to H$ is a solution of the equation (3) if and only if $f$ has the following form

$$f(x) = A_0 + \sum_{i=1}^{N-1} A_i^k(x), \ x \in S, \quad (13)$$

where $A_0 \in H$ and $A_k : S^k \to H$, $k \in \{1, 2, ..., N - 1\}$ are $k$-additive and symmetric functions which satisfy the following conditions

$$\sum_{i=\max(k+j,k+1)}^{N-1} \binom{i-k}{j} \sum_{\lambda \in \Phi} A_i(x, ..., a_\lambda, ..., a_\lambda, \lambda y, ..., \lambda y) = 0 \text{ for } x, y \in S,$$

$$0 \leq k \leq N - 2, \ 0 \leq j \leq N - k - 1.$$

Theorem 6. [10] Let $(S, +)$ be an abelian semigroup, $\Phi$ be a finite subgroup of the group of automorphisms of $S$, $N = \text{card}(\Phi)$, $(H, +)$ be an abelian group uniquely divisible by $(N + 1)!$ and $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of $S$. Then the function $f : S \to H$ is a solution of the equation (4) if and only if $f$ has the following form

$$f(x) = A_0 + \sum_{i=1}^{N} A_i^k(x), \ x \in S, \quad (14)$$
where $A_0 \in H$ and $A_k : S^k \to H$, $k \in \{1, 2, ..., N\}$ are symmetric and $k$-additive functions satisfying the three conditions:

1) $\sum_{\lambda \in \Phi} N \sum_{k=1}^{N} A_k^*(a_{\lambda}) = N A_0$,

2) $\sum_{2 \leq i = \max(k+j,k+1)}^{N} (i \choose j) (i-k)^{\choose j} \sum_{\lambda \in \Phi} A_i(x, ..., a_{\lambda}, ..., a_{\lambda}, \lambda y, ..., \lambda y) = 0$, $x, y \in S$, $1 \leq k \leq N - 1$, $0 \leq j \leq N - k$ and

3) $\sum_{k=1}^{N} (i \choose k) \sum_{\lambda \in \Phi} A_k(\lambda x, ..., \lambda x, a_{\lambda}, ..., a_{\lambda}) = N A_i^*(x)$, $x \in S$, $1 \leq i \leq N$.

3. Main results

The following lemma will be used in the proof of our main results namely Theorems 8 and 11.

Lemma 7. Let $K$ be an ultrametric field of characteristic zero and $\overline{K}$ its completion, $F$ be a complete ultrametric $K$-vector space, $\delta \in \mathbb{R}^+$ and $P$ be a polynomial function of degree $n$, $n \geq 1$, with rational variable and with coefficients in $F$. Suppose that

$$\|P(z)\| \leq \delta \text{ for all } z \in \mathbb{Q}.$$  \hspace{1cm} (15)

Then, there exists a prime number $p$ such that $\mathbb{Q}_p \subset \overline{K}$ and

$$P(z) = P(0) \text{ for all } z \in \mathbb{Q}_p,$$

i.e. all non-constant coefficients of $P$ are zero.

Proof. There exist $a_0, a_1, ..., a_n \in F$ such that

$$P(z) = \sum_{i=0}^{n} a_i z^i, \ z \in \mathbb{Q}.$$

The theorem of Ostrowski shows that there exists a prime number $p$ for which $\mathbb{Q}_p \subset \mathbb{K}$. An extension by continuity of the external law of $F$ from $K$ to $\mathbb{K}$ allows us to write,

$$\|P(z)\| \leq \delta \text{ for } z \in \mathbb{Q}_p.$$

Let $\varphi : F \to \mathbb{Q}_p$ be a continuous $\mathbb{Q}_p$-linear functional. Taking into account the previous inequality we have for all $z \in \mathbb{Q}_p$:

$$\|\varphi(P(z))\| \leq \delta \|\varphi\| \text{ for } z \in \mathbb{Q}_p,$$

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which means that
\[ \left\| \sum_{i=0}^{n} \varphi(a_i)z^i \right\| \leq \delta \| \varphi \| \text{ for } z \in \mathbb{Q}_p. \]

It results, since a polynomial function is bounded if and only if it is constant, that
\[ \varphi(a_i) = 0 \text{ for } 1 \leq i \leq n \text{ i.e. } P(z) = P(0) \text{ for all } z \in \mathbb{Q}_p. \]

In the following theorem, using the operatorial approach we obtain the non-Archimedean stability in the sense of Hyers-Ulam of the generalised $\Phi$-Jensen functional equation.

**Theorem 8.** Assume that $\Phi$ is a finite subgroup of the group of automorphisms of $E$, $N = \text{card}(\Phi)$, $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of $E$ and $f : E \to F$ satisfying the following inequality:

\[ \left\| \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) - Nf(x) \right\| \leq \delta, \quad (16) \]

for all $x, y \in E$. Then there exists, up to a constant, a unique GP function $P : E \to F$ solution of the equation $(3)$, of degree at most $N - 1$, such that

\[ \|f(x) - f(0) - P(x)\| \leq \frac{\delta}{|N|^2} \text{ for all } x \in E. \]

**Proof.** Suppose that $f$ satisfies the inequality $(16)$. Letting $y = 0$ and $x = 0$ in $(16)$, respectively, we get

\[ \left\| \sum_{\lambda \in \Phi} f(x + a_\lambda) - Nf(x) \right\| \leq \delta, \quad x \in E, \]

and

\[ \left\| \sum_{\lambda \in \Phi} f(\lambda y + a_\lambda) - Nf(0) \right\| \leq \delta, \quad y \in E. \]

By replacing, in the last inequality, $y$ by $\mu y$ we obtain

\[ \left\| N^2f(0) - N \sum_{\nu \in \Phi} f(\nu y) \right\|
\leq \max \left\{ \left\| N^2f(0) - \sum_{\mu \in \Phi} \sum_{\lambda \in \Phi} f(\mu \lambda y + a_\lambda) \right\|, \left\| \sum_{\nu \in \Phi} \sum_{\lambda \in \Phi} f(\nu y + a_\lambda) - N \sum_{\nu \in \Phi} f(\nu y) \right\| \right\}
\leq \delta, \quad (17) \]
for all $y \in E$. It follows, by taking $g := f - f(0)$ and the use of (16) and (17) that

$$\left\| \sum_{\lambda \in \Phi} g(x + \lambda y + a\lambda) - Ng(x) - \sum_{\lambda \in \Phi} g(\lambda y) \right\|$$

$$= \left\| \sum_{\lambda \in \Phi} f(x + \lambda y + a\lambda) - Nf(x) + Nf(0) - \sum_{\lambda \in \Phi} f(\lambda y) \right\|$$

$$\leq \max \left\{ \left\| \sum_{\lambda \in \Phi} f(x + \lambda y + a\lambda) - Nf(x) \right\| , \left\| Nf(0) - \sum_{\lambda \in \Phi} f(\lambda y) \right\| \right\}$$

$$\leq \frac{\delta}{|N|^2},$$

for all $x, y \in E$. In virtue of Theorem 3, there exists, in the class of function $g : E \to F$ with $g(0) = 0$, a GP function $P$ of degree at most $N$ solution of the functional equation

$$\sum_{\lambda \in \Phi} g(x + \lambda y + a\lambda) = Ng(x) + \sum_{\lambda \in \Phi} g(\lambda y) \quad (18)$$

such that

$$\|g(x) - P(x)\| \leq \frac{\delta}{|N|^2} \text{ for all } x \in E. \quad (19)$$

According to Theorem 2, $P(x) = \sum_{i=1}^{N} A^*_i(x)$ with

$$\sum_{\lambda \in \Phi} \sum_{i=1}^{N} A^*_i(a\lambda) = 0 \quad (20)$$

and

$$\sum_{i=\max(k,j)}^{N} \binom{i}{k} \binom{i-k}{j} \sum_{\lambda \in \Phi} \sum_{k} A_i(x,...,x,a\lambda,...,a\lambda,\lambda y,...,\lambda y) = 0 \quad (21)$$

for all $x, y \in E, 0 \leq k \leq N-1, 0 \leq j \leq N-k$ and $2 \leq \max = \max(k+1, j+1, k+j)$. In addition by (17),

$$\left\| \sum_{\lambda \in \Phi} P(\lambda y) \right\| \leq \max \left\{ \left\| \sum_{\lambda \in \Phi} (P(\lambda y) - g(\lambda y)) \right\| , \left\| \sum_{\lambda \in \Phi} g(\lambda y) \right\| \right\}$$

$$\leq \frac{\delta}{|N|^2}$$

for all $y \in E$. In view of Lemma 4, Theorem 1 and Lemma 7, we have

$$A_N = 0 \quad (22)$$
and by Lemma 7,
\[ \sum_{\lambda \in \Phi} A_i^\ast(\lambda y) = 0, \quad y \in E, \quad 1 \leq i \leq N - 1. \] (23)

Taking into account of (20) (21), (22) and (23) we get
\[ N - 1 \sum_{i=\max(k+j,k+1)} \binom{i}{k} \binom{i-k}{j} \sum_{\lambda \in \Phi} A_i(x,...,x,a_{\lambda},...,a_{\lambda},\lambda y,...,\lambda y) = 0, \quad x,y \in E, \]

\[ 0 \leq k \leq N - 2, \quad 0 \leq j \leq N - k - 1. \] This shows, using Theorem 5, that \( P \) is a solution of the Eq. (3).

The uniqueness is giving by Lemma 7. In fact, let \( Q \) be another GP function of degree at most \( N - 1 \), solution of Eq. (3) and satisfying the inequality (19) then we get
\[ \| P(x) - Q(x) \| \leq \max(\| P(x) - g(x) \|, \| g(x) - Q(x) \|) \]
\[ \leq \frac{\delta}{|N|^2}, \quad x \in E. \]

According to Lemma 7 we get \( P - Q \) is constant. This completes the proof.

**Corollary 9.** Assume that \( a, b \) are arbitrary elements of \( E \) and \( f : E \to F \) satisfying the following inequality:
\[ \| f(x + y + a) + f(x + \sigma(y) + b) - 2f(x) \| \leq \delta, \] (24)

for all \( x, y \in E \). Then there exists, up to a constant, a unique GP function \( P : E \to F \) solution of the equation (6), of degree at most 1, such that
\[ \| f(x) - f(0) - P(x) \| \leq \frac{\delta}{|4|} \quad \text{for all} \quad x \in E. \]

**Proof.** The proof follows on putting \( \Phi = \{ I, \sigma \} \) in Theorem 8.

**Corollary 10.** Let \( p \) be a prime number, \( \mathbb{C}_p = \mathbb{Q}_p + i\mathbb{Q}_p, \) \((i^2 = -1), j \) be a primitive cube root of unity, \( a \) be a nonzero complex number and \( f : \mathbb{C}_p \to \mathbb{C}_p \), be a continuous function satisfying the following inequality
\[ \| f(x + y + j a) + f(x + jy + j^2 a) + f(x + j^2 y + a) - 3f(x) \| \leq \delta, \quad x, y \in \mathbb{C}_p, \] (25)

for all \( x, y \in \mathbb{C}_p \). Then there exists, up to a constant, a unique GP function \( P : \mathbb{C}_p \to \mathbb{C}_p \) of degree at most 2, solution of the equation
\[ f(x + y + j a) + f(x + jy + j^2 a) + f(x + j^2 y + a) = 3f(x), \quad x, y \in \mathbb{C}_p, \] (26)
such that

$$\|f(x) - P(x)\| \leq \frac{\delta}{|9|}, \ x \in E.$$ 

Now we investigate the non-Archimedean stability, in the sense of Hyers-Ulam, of the equation (4).

**Theorem 11.** Assume that $\Phi$ is a finite subgroup of the group of automorphisms of $E$, $N = \text{card}(\Phi)$, $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of $E$ and $f : E \to F$ satisfying the following inequality:

$$\left\| \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) - N f(x) - N f(y) \right\| \leq \delta, \ (27)$$

for all $x, y \in E$. Then there exists a unique GP function $P : E \to F$ solution of the equation (4), of degree at most $N$, such that

$$\|f(x) - P(x)\| \leq \frac{\delta}{|N|^2} \text{ for } x \in E.$$ 

**Proof.** Suppose that $f$ satisfies the inequality (27). Letting $x = y = 0, y = 0$ and $x = 0$, respectively, in (27) we obtain

$$\left\| \sum_{\lambda \in \Phi} f(a_\lambda) - 2N f(0) \right\| \leq \delta,$$

$$\left\| \sum_{\lambda \in \Phi} f(x + a_\lambda) - N f(x) - N f(0) \right\| \leq \delta,$$

$$\left\| \sum_{\lambda \in \Phi} f(\lambda x + a_\lambda) - N f(x) - N f(0) \right\| \leq \delta,$$

for all $x, y \in E$. Taking into account the above inequalities and (27) we get that

$$\left\| N^2 f(x) + N \sum_{\mu \in \Phi} f(\mu y) - N^2 f(0) - N \sum_{\nu \in \Phi} f(x + \nu y) \right\|$$

$$\leq \max \left\{ \left\| N^2 f(x) + N \sum_{\mu \in \Phi} f(\mu y) - \sum_{\lambda \in \Phi} \sum_{\mu \in \Phi} f(x + \lambda \mu y + a_\lambda) \right\|, \right.$$  

$$\left\| \sum_{\nu \in \Phi} \sum_{\lambda \in \Phi} f(x + \nu y + a_\lambda) - N^2 f(0) - N \sum_{\nu \in \Phi} f(x + \nu y) \right\| \right\}$$

$$\leq \delta,$$
for all $x, y \in E$. With the notation $g := f - f(0)$ we can reformulate the previous inequality to
\[
\left\| \sum_{\mu \in \Phi} g(x + \mu y) - N g(x) - \sum_{\mu \in \Phi} g(\mu y) \right\| \leq \frac{\delta}{|N|},
\]
for all $x, y \in E$. Theorem 3 shows that there exists a GP function $Q : E \to F$ of degree at most $N$ solution of the equation
\[
\sum_{\mu \in \Phi} g(x + \mu y) = Ng(x) + \sum_{\mu \in \Phi} g(\mu y), \quad x, y \in E
\]
such that
\[
\|g(x) - Q(x)\| \leq \frac{\delta}{|N|^2}, \quad x \in E. \tag{28}
\]
Then there exist $k$-additive and symmetric functions $A_k : E^k \to F$, $k \in \{1, 2, ..., N\}$ such that $Q(x) = \sum_{i=1}^{N} A_k^i(x)$, $x \in E$ and we have
\[
\sum_{\mu \in \Phi} Q(x + \mu y) = NQ(x) + \sum_{\mu \in \Phi} Q(\mu y), \quad x, y \in E.
\]
Let $P$ be the GP function defined by
\[
P(x) = Q(x) + \frac{1}{N} \sum_{\lambda \in \Phi} \sum_{i=1}^{N} A_k^i(a_\lambda), \quad x \in E,
\]
so we have the following inequality
\[
\|f(x) - P(x)\| = \left\| g(x) - Q(x) - \frac{1}{N} \left( \sum_{\lambda \in \Phi} f(a_\lambda) + 2N f(0) \right) \right\|
\leq \max\left( \frac{\delta}{|N|^2}, \frac{\delta}{|N|} \right)
\leq \frac{\delta}{|N|^2},
\]
for all $x \in E$. To prove that $P$ is a solution of the equation (4) we define the functions $I_P, J_P : E \times E \to F$ by the formulas
\[
I_P(x, y) = \sum_{\nu \in \Phi} P(x + \nu y + a_\nu) - NP(x) - NP(y), \quad x, y \in E
\]
and

\[ J_P(x, y) = I_P(x, y) - I_P(0, y), \quad x, y \in E. \]

We have therefore

\[
I_P(0, 0) = \sum_{\nu \in \Phi} P(a_{\nu}) - 2N P(0)
\]

\[
= \left\{ \sum_{\nu \in \Phi} Q(a_{\nu}) + \sum_{\nu \in \Phi} \sum_{i=1}^{N} A_i^\epsilon(a_{\nu}) \right\} - 2 \left\{ \sum_{\nu \in \Phi} \sum_{i=1}^{N} A_i^\epsilon(a_{\nu}) \right\}
\]

\[
= 0.
\]

Furthermore we have,

\[
\|I_P(x, y)\| \leq \max \left\{ \left\| \sum_{\lambda \in \Phi} P(x + \lambda y + a_{\lambda}) - f(x + \lambda y + a_{\lambda}) \right\|, \right.
\]

\[
\left. \|NP(x) - N f(x)\|, \|NP(y) - N f(y)\|, \delta \right\}
\]

\[
\leq \max\left( \frac{\delta}{|N|^2}, \delta \right)
\]

\[
\leq \frac{\delta}{|N|^2},
\]

for all \( x, y \in E \). Replacing \( P \) by its expression (as a GP function) in \( I_P(0, y) \), \( I_P(x, y) \) we get, that for all \( x, y \in E \)

\[
I_P(0, y) = \sum_{\lambda \in \Phi} P(\lambda y + a_{\lambda}) - NP(0) - NP(y)
\]

\[
= \sum_{\lambda \in \Phi} \sum_{i=1}^{N} A_i^\epsilon(\lambda y + a_{\lambda}) - N \sum_{i=1}^{N} A_i^\epsilon(y) - NP(0)
\]

\[
= \sum_{i=1}^{N} \sum_{j=0}^{i} \left( \sum_{\lambda \in \Phi} A_i(\lambda y, \ldots, \lambda y, a_{\lambda}, \ldots, a_{\lambda}) - N A_i^\epsilon(y) - NP(0) \right)
\]

\[
= \sum_{j=1}^{N} \left( \sum_{i=j}^{N} \left( \sum_{\lambda \in \Phi} A_i(\lambda y, \ldots, \lambda y, a_{\lambda}, \ldots, a_{\lambda}) - N A_i^\epsilon(y) \right) \right)
\]

and

\[
J_P(x, y) = \sum_{\lambda \in \Phi} \sum_{k=1}^{N} \sum_{i=max(k+j,k+1)\leq N} \left( \begin{array}{c} i \rule{0cm}{0.8cm} \\
\end{array} \right) \left( \begin{array}{c} i-j \rule{0cm}{0.8cm} \\
\end{array} \right) A_i(x, \ldots, x, a_{\lambda}, \ldots, a_{\lambda}, \lambda y, \ldots, \lambda y).
\]
Making the substitution \( y \) by \( Zy, Z \in \mathbb{Q} \) in \( I_P(0, y) \) we obtain a polynomial function \( R(Z) \) with rational variable and with coefficients in \( F \),

\[
R(Z) = \sum_{j=1}^{N} Z^j \left( \sum_{i=j}^{N} \binom{i}{j} \sum_{\lambda \in \Phi} A_i(\lambda y, \ldots, \lambda y, a\lambda, \ldots, a\lambda) - N A_j^*(y) \right), \quad y \in E, \ Z \in \mathbb{Q}.
\]

(29)

It satisfies

\[
\|R(Z)\| \leq \frac{\delta}{|N|^2}, \ Z \in \mathbb{Q}.
\]

In view of Lemma 7, \( R(Z) = 0 \), \( Z \in \mathbb{Q}_p \). Consequently \( J_P(x, y) = I_P(x, y), \ x, y \in E \).

In addition, a similar reasoning, making the substitution \( x \) by \( Zx, Z \in \mathbb{Q} \) in \( J_P(x, y) \), we can show that \( I_P(x, y) = 0 \), \( x, y \in E \) which means that \( (p, q) \) is a solution of the equation (4).

It is left to prove the uniqueness statement. Let \( T \) be another GP function of degree at most \( N \), solution of the Eq. (4) such that

\[
\|g(x) - T(x)\| \leq \frac{\delta}{|N|^2}, \ x \in E.
\]

(30)

From (28) and (30) we infer that we have

\[
\|P(x) - T(x)\| = \|P(x) - g(x) + g(x) - T(x)\| \\
\leq \max \{\|P(x) - g(x)\|, \|g(x) - T(x)\|\} \\
\leq \frac{\delta}{|N|^2},
\]

for all \( x \in E \). So, by Lemma 7 we conclude that \( T - P \) is a constant, and by the fact that \( T \) and \( P \) are solution of the Eq. (4) we get \( T = P \). This completes the proof of Theorem 11.

**Corollary 12.** Assume that \( a, b \) are arbitrary elements of \( E \) and \( f : E \to F \) satisfying the following inequality:

\[
\|f(x + y + a) + f(x + \sigma(y) + b) - 2f(x) - 2f(y)\| \leq \delta,
\]

(31)

for all \( x, y \in E \). Then there exists a unique GP function \( P : E \to F \) solution of the equation (7), of degree at most 2, such that

\[
\|f(x) - P(x)\| \leq \frac{\delta}{4} \quad \text{for all} \ x \in E.
\]
Proof. The proof follows on putting $\Phi = \{I, \sigma\}$ in Theorem 11.

**Corollary 13.** Let $w$ be a primitive $N^{th}$ root of unity, $N \geq 2$, let $a$ be a complex constant, $p$ be a prime number, $\mathbb{C}_p = \mathbb{Q}_p + i\mathbb{Q}_p$, $i^2 = -1$ and $f : \mathbb{C}_p \to \mathbb{C}_p$ be a continuous function satisfying the inequality

$$\left\| \sum_{n=0}^{N-1} f(x + w^n y + w^{n+1}a) - N f(x) - N f(y) \right\| \leq \delta, \quad x, y \in \mathbb{C}_p.$$ 

Then there exist a unique GP function $P : \mathbb{C}_p \to \mathbb{C}_p$, of degree at most $N$, solution of the equation,

$$\sum_{n=0}^{N-1} f(x + w^n y + w^{n+1}a) = N f(x) + N f(y), \quad x, y \in \mathbb{C}_p,$$

such that

$$\|f(z) - P(z)\| \leq \frac{\delta}{|N|^2}, \quad z \in \mathbb{C}_p.$$

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