STARLIKENESS CONDITIONS FOR NORMALIZED ANALYTIC FUNCTIONS INCLUDING RUSCHEWEYH OPERATOR

S. Shams, P. Arjomandinia

Abstract. In the present paper, we introduce special subclass of analytic functions using Ruscheweyh operator. By making use of the notion of differential subordination, we find conditions on the parameters $M, \alpha, \delta$ and $\mu$ for which

$$\left| \left(1 - \alpha + \alpha(\lambda + 2) \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} \right) \left( \frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} \right)^\mu - \alpha(\lambda + 1) \left( \frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} \right)^{\mu+1} - 1 \right| < M,$$

implies that $f \in S_n^*(\delta)$, where $n \in \mathbb{N}$. The results obtained here generalize some previously results given in the literature.

2010 Mathematics Subject Classification: Primary 30C45, Secondary 30C80.

Keywords: differential subordination, normalized analytic function, Ruscheweyh operator, convex and starlike function.

1. Introduction

Let $A_n$ denote the class of all analytic functions $f(z)$ in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ which are in the form

$$f(z) = z + a_{n+1}z^{n+1} + \ldots,$$

with $A = A_1$.

Let $0 \leq \delta < 1$. The class of (normalized) starlike functions of order $\delta$, $S_n^*(\delta)$, is defined by

$$S_n^*(\delta) = \left\{ f \in A_n : \Re \left( \frac{zf'(z)}{f(z)} \right) > \delta, \ z \in D \right\},$$

with $S^*(\delta) = S_1^*(\delta)$. It is well known that $S^*(0) = S^*$, where $S^*$ is the class of (normalized) starlike functions in $D$, (see [3]). Similarly, we denote by $K_n(\delta)$ the class of (normalized) convex functions of order $\delta$ and define by

$$K_n(\delta) = \left\{ f \in A_n : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \delta, \ z \in D \right\}.$$
It is well known that $f \in K_n(\delta)$, if and only if $zf''(z) \in S^*_n(\delta)$, (see [3]).

Let $f, g$ be analytic in $\mathbb{D}$. We say that $f$ is subordinate to $g$ (or $g$ is superordinate to $f$) and written as $f \prec g$ if there exists an analytic function $w(z)$ in $\mathbb{D}$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$.

Let $f, g \in \mathcal{A}$ be given by Taylor series expansions of the forms

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in \mathbb{D}).$$

The Hadamard product (or convolution) of $f$ and $g$, denoted by $f \ast g$, is defined by

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z). \quad (2)$$

Suppose that $f \in \mathcal{A}$. The Ruscheweyh derivative operator $[4]$, $R^\lambda : \mathcal{A} \to \mathcal{A}$, is defined as follows

$$R^\lambda f(z) = \frac{z}{(1 - z)^{\lambda+1}} \ast f(z), \quad (\lambda \geq -1, z \in \mathbb{D}). \quad (3)$$

By an easy calculation we find that

$$R^0 f(z) = f(z), \quad R^1 f(z) = zf'(z) \text{ and } R^2 f(z) = \frac{z}{2} (2f'(z) + zf''(z)),$$

and so on. Using (3) and straightforward calculations we deduce that for each $\lambda \geq -1$ and $z \in \mathbb{D}$

$$z(R^\lambda f)'(z) = (\lambda + 1) R^{\lambda+1} f(z) - \lambda R^\lambda f(z). \quad (4)$$

In [6] some conditions on $M, \alpha, \delta$ and $\mu$ were determined so that

$$\left| (1 - \alpha) \left( \frac{f(z)}{z} \right)^\mu + \alpha f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} - 1 \right| < M$$

implies $f \in S^*_n(\delta)$.

Motivated by the recent work of Zhu [6], in the present paper we see that the results remain true for the functions $f \in \mathcal{A}_n$ that satisfy the following condition:

$$\left| (1 - \alpha + \alpha(\lambda + 2) \frac{R^{\lambda+2} f(z)}{R^{\lambda+1} f(z)} \left( \frac{R^{\lambda+1} f(z)}{R^\lambda f(z)} \right)^\mu - \alpha(\lambda + 1) \left( \frac{R^{\lambda+1} f(z)}{R^\lambda f(z)} \right)^{\mu+1} - 1 \right| < M. \quad (5)$$

For special choices of $\alpha$ and $\lambda$, (5) reduces to the interesting cases that will be given in the corollaries. For the similar results see [1, 2, 5].

To prove our main results we shall use the following lemmas.
Lemma 1. ([6]) Let $B(z), C(z)$ and $D(z)$ be complex functions in $\mathbb{D}$ and let $n$ be a positive integer. Suppose that $D(0) = 0, B(z) \neq 0$ and $\Re \frac{C(z)}{B(z)} \geq -n$ for all $z \in \mathbb{D}$. If $p(z) = p_nz^n + \ldots$ is analytic in $\mathbb{D}$ and satisfies

$$|B(z)zp'(z) + C(z)p(z) + D(z)| < M,$$

for all $z \in \mathbb{D}$, then $|p(z)| < N$ in $\mathbb{D}$, where

$$N = \sup \left\{ \frac{M + |D(z)|}{nB(z) + C(z)} : z \in \mathbb{D} \right\}.$$

Lemma 2. ([6]) Let $\alpha > 0, \mu > 0$ and

$$M_n(\alpha, \delta, \mu) = \begin{cases} 
\frac{(\mu + n\alpha)(1 - \delta)}{n + \mu(1 - \delta)} & ; \alpha \geq \alpha_2 \\
\frac{(\mu + n\alpha)\sqrt{2\alpha(1 - \delta) - 1}}{\sqrt{n^2\alpha^2 + 2(n\mu + (1 - \delta)\mu^2)\alpha}} & ; \alpha_1 \leq \alpha \leq \alpha_2 \\
\frac{\alpha(\mu + n\alpha)(1 - \delta)}{2\mu + (n - \mu + \mu\delta)\alpha} & ; 0 < \alpha < \alpha_1 
\end{cases}$$

where $\alpha_2 = \frac{n + \mu(1 - \delta)}{n(1 - \delta)}$ and

$$\alpha_1 = \frac{\sqrt{9\mu^2 + 2n\mu + n^2 - (18\mu^2 + 2n\mu)\delta + 9\mu^2\delta^2 - 3\mu + n + 3\mu\delta}}{2n(1 - \delta)}.$$

If $p(z)$ and $q(z)$ are analytic in $\mathbb{D}$ with $p(z) = 1 + p_nz^n + \ldots$, and $q(z) = 1 + q_nz^n + \ldots$, and satisfy $q(z) < 1 + \frac{\mu Mz}{\alpha + \mu}$ also $q(z)(1 - \alpha + \alpha p(z)) < 1 + Mz$ with $0 < M \leq M_n(\alpha, \delta, \mu)$, then $\Re(p(z)) > \delta$ for all $z \in \mathbb{D}$.

2. Main Results

Using Lemmas 1 and 2, we state and prove the following results.

Theorem 3. Suppose that $\alpha, \mu, \delta, M$ and $M_n(\alpha, \delta, \mu)$ be defined as in Lemma 2. If $f \in \mathcal{A}_n$ satisfies

$$\left(1 - \alpha + \alpha(\lambda + 2)R^{\lambda + 2}f(z)\right)\left(\frac{R^{\lambda + 1}f(z)}{R^\lambda f(z)}\right)^\mu - \alpha(\lambda + 1)\left(\frac{R^{\lambda + 1}f(z)}{R^\lambda f(z)}\right)^\mu + 1 < 1 + Mz,$$

then

$$\Re \left( (\lambda + 2)\frac{R^{\lambda + 2}f(z)}{R^{\lambda + 1}f(z)} - (\lambda + 1)\frac{R^{\lambda + 1}f(z)}{R^\lambda f(z)} \right) > \delta.$$
Proof. Let \( q(z) = \left( \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} \right)^{\mu} \). Using (4), after an easy computation, we obtain \( \frac{1}{\mu} zq'(z) = (\lambda + 2) \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda + 1) \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} - 1 \).

This gives that
\[
q(z) + \frac{\alpha}{\mu} zq'(z) = \left( 1 - \alpha + \alpha(\lambda + 2) \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} \right) \left( \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} \right)^{\mu} - \alpha(\lambda + 1) \left( \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} \right)^{\mu+1}.
\]

By the assumption of the theorem we have \( q(z) + \frac{\alpha}{\mu} zq'(z) \prec 1 + Mz \), or equivalently \( \left| \frac{\alpha}{\mu} zq'(z) + q(z) - 1 \right| < M \). From this we see that all conditions of Lemma 1 are satisfied. So, we obtain \( |q(z) - 1| < N = \frac{\mu M}{\mu + n\alpha} \), which is equivalent to \( q(z) \prec 1 + \frac{\mu M}{\mu + n\alpha} z \). Let,
\[
p(z) = (\lambda + 2) \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda + 1) \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)}.
\]

The assumption of the theorem shows that \( q(z)(1 - \alpha + \alpha p(z)) \prec 1 + Mz \).

Applying Lemma 2, we see that \( \Re p(z) > \delta \). This completes the proof.

Taking \( \lambda = -1 \) in Theorem 3 we obtain [[6], Theorem 2]:

**Corollary 4.** Let \( \alpha, \mu, \delta, M \) and \( M_n(\alpha, \delta, \mu) \) be defined as in Lemma 2. If \( f \in A_n \) satisfies
\[
(1 - \alpha) \left( \frac{f(z)}{z} \right)^{\mu} + \alpha f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \prec 1 + Mz
\]
then \( f \in S_n^*(\delta) \).

Taking \( \lambda = \delta = 0 \) and \( \mu = 1 \) in Theorem 3 we obtain the following result:

**Corollary 5.** Let \( \alpha > 0 \) and
\[
M_n(\alpha) = \begin{cases} 
\frac{(1+n\alpha)}{n+1} & ; \quad \alpha \geq \frac{n+1}{n} \\
\frac{(1+n\alpha)\sqrt{2\alpha-1}}{\sqrt{n^2\alpha^2+2(n+1)}} & ; \quad \sqrt{9+2n^2-3+n} \leq \alpha < \frac{n+1}{n} \\
\frac{\alpha(1+n\alpha)}{2(n-1)\alpha} & ; \quad 0 < \alpha < \sqrt{9+2n^2-3+n} \end{cases}.
\]

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If \( f \in A_n \) satisfies
\[
\left( 1 + \alpha + \frac{zf''(z)}{f(z)} \right) \left( \frac{zf'(z)}{f(z)} \right) - \alpha \left( \frac{zf'(z)}{f(z)} \right)^2 < 1 + Mz,
\]
then
\[
\Re \left( 1 + \frac{zf''(z)}{f(z)} \right) > \Re \left( \frac{zf'(z)}{f(z)} \right) - 1.
\]

**Theorem 6.** Let \( \mu > 0 \) and \( 0 < \beta \leq \frac{\mu + n}{\sqrt{\mu^2 + (\mu + n)^2}} \). If \( f \in A_n \) satisfies
\[
\left| \left( \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} \right)^\mu \left( \lambda + 2 \right) \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda + 1) \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} \right| - 1 < \beta,
\]
then
\[
\Re \left( \left( \lambda + 2 \right) \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda + 1) \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} \right) > \delta
\]
where,
\[
\delta = \begin{cases} 
\frac{(\mu+n)(1-\beta)}{\mu+n+\mu\beta} & ; \quad 0 < \beta < \frac{\mu+n}{2\mu+n} \\
\frac{(\mu+n)^2(1-\beta^2)-\mu^2\beta^2}{2(\mu+n)^2-2\mu^2\beta^2} & ; \quad \frac{\mu+n}{2\mu+n} \leq \beta \leq \frac{\mu+n}{\sqrt{\mu^2 + (\mu + n)^2}}.
\end{cases}
\]

**Proof.** From (6) we have
\[
\beta = \begin{cases} 
\frac{(\mu+n)\sqrt{1-\delta}}{\sqrt{n^2+2\mu n+(1-\delta)\mu^2}} & ; \quad 0 \leq \delta \leq \frac{\mu}{3\mu+n} \\
\frac{(\mu+n)(1-\delta)}{n+\mu+\mu\delta} & ; \quad \frac{\mu}{3\mu+n} < \delta < 1.
\end{cases}
\]

It is easy to show that the inequality
\[
\frac{\sqrt{9\mu^2 + 2n\mu + n^2 - (18\mu^2 + 2n\mu)\delta + 9\mu^2\delta^2 - 3\mu + n + 3\mu\delta}}{2n(1-\delta)} \leq 1
\]
is equivalent to \( \delta \leq \frac{\mu}{3\mu+n} \). Hence, it is seen that all conditions of Theorem 3 are satisfied with \( \beta = M_n(1, \delta, \mu) \) and we obtain \( \Re(p(z)) > \delta \), where \( \delta \) is given by (6) and
\[
p(z) = (\lambda + 2) \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda + 1) \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)}.
\]

Taking \( \lambda = -1 \) in Theorem 6 we obtain [[6], Theorem 3]:

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Corollary 7. Let $\mu > 0$ and $0 < \beta \leq \frac{\mu + n}{\sqrt{\mu^2 + (\mu + n)^2}}$. If $f \in A_n$ satisfies
\[
\left| f'(z) \left( \frac{f(z)}{z} \right)^{\mu - 1} - 1 \right| < \beta; \quad (z \in \mathbb{D}),
\]
then $f \in S_n^*(\delta)$, where $\delta$ is given by (6).

Finally, taking $\lambda = -1$, $\mu = 1$ and $zf'(z)$ instead of $f(z)$ in Theorem 3 we obtain [[6], Theorem 4]:

Corollary 8. Let $0 \leq \delta < 1$, $\alpha > 0$ and
\[
M_n(\alpha, \delta) = \begin{cases} 
\frac{(1+n\alpha)(1-\delta)}{n+1-\delta} & ; \quad \alpha \geq \alpha_2 \\
\frac{(1+n\alpha)\sqrt{2\alpha(1-\delta)-1}}{\sqrt{n^2\alpha^2+2(n+1-\delta)\alpha}} & ; \quad \alpha_1 \leq \alpha \leq \alpha_2 \\
\frac{\alpha(1+n\alpha)(1-\delta)}{2(n+1-\delta)\alpha} & ; \quad 0 < \alpha < \alpha_1
\end{cases}
\]
where $\alpha_2 = \frac{n+1-\delta}{n(1-\delta)}$ and
\[
\alpha_1 = \sqrt{9 + 2n + n^2 - (18 + 2n)\delta + 9\delta^2 - 3 + n + 3\delta}. \frac{2n(1-\delta)}{2n(1-\delta)}.
\]
If $f \in A_n$ satisfies
\[
|f'(z) + \alpha zf''(z) - 1| < M; \quad (z \in \mathbb{D}),
\]
with $0 < M \leq M_n(\alpha, \delta)$, then $zf' \in S_n^*(\delta)$, i.e., $f$ is convex-univalent function of order $\delta$.

References


Saied Shams
Department of Mathematics, Faculty of Science,
Urmia University,
Urmia, Iran
email: sa40shams@yahoo.com

Parviz Arjomandinia
Department of Mathematics, Faculty of Science,
Urmia University,
Urmia, Iran
email: p.arjomandinia@gmail.com