COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF SPIRALLIKE FUNCTIONS

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ABSTRACT. In this article, we derive a sharp estimates for the Taylor-Maclaurin coefficients of functions in a certain subclass of spirallike functions. Also, we give several corollaries and consequences of the main results.

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1. Introduction

Let \( D \) be the unit disk \( \{ z : |z| < 1 \} \), \( A \) be the class of functions analytic in \( D \), satisfying the conditions

\[
    f(0) = 0 \quad \text{and} \quad f'(0) = 1. \tag{1}
\]

Then each function \( f \) in \( A \) has the Taylor expansion

\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}
\]

because of the conditions (1). Let \( S \) denote class of analytic and univalent functions in \( D \) with the normalization conditions (1).

Definition 1. For \( 0 \leq \alpha < 1 \) let \( S^*(\alpha) \) and \( S^c(\alpha) \) denote the class of starlike and convex univalent functions of order \( \alpha \), which are defined as the following, respectively

\[
    S^*(\alpha) = \left\{ f(z) \in S : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in D \right\}
\]

and

\[
    S^c(\alpha) = \left\{ f(z) \in S : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in D \right\}.
\]
Observe that $S^\ast(0) = S^\ast$ represent standard starlike functions. A notation of $\alpha$-starlikeness and $\alpha$-convexity were generalized onto a complex order $\alpha$ by Nasr and Aouf [7]. Špaček [10] extend the class of starlike functions by introducing the class of spirallike functions of type $\beta$ in $\mathbb{D}$ and gave the following analytical characterization of spirallikeness functions of type $\beta$ in $\mathbb{D}$.

**Theorem 1.** (Špaček [10]) Let the function $f(z)$ be in the normalized analytic function class $A$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(z)$ is a spirallike function of type $\beta$ in $\mathbb{D}$ if and only if

$$\text{Re}\left(\frac{e^{i\beta}zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}. \quad (3)$$

We denote the the class of spirallike functions of type $\beta$ in $\mathbb{D}$ by $\tilde{S}_\beta$. Libera [6] unified and extended the classes $S^\ast(\alpha)$ and $\tilde{S}_\beta$ by introducing the analytic function class $\tilde{S}_\beta^\alpha$ in $\mathbb{D}$ as follows.

**Definition 2.** (Libera [6]) Let the function $f(z)$ be in the normalized analytic function class $A$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\alpha \in [0, 1)$. We say that $f \in \tilde{S}_\beta^\alpha$ if and only if

$$\text{Re}\left(\frac{e^{i\beta}zf'(z)}{f(z)}\right) > \alpha \cos \beta \quad (z \in \mathbb{D}; \; 0 \leq \alpha < 1). \quad (4)$$

From Definition 1 and 2, we have the following inclusions:

$$\tilde{S}_\beta^0 = S^\ast(\alpha) \quad \text{and} \quad \tilde{S}_\beta^0 = \tilde{S}_\beta.$$

Libera [6] also proved the following coefficients bounds for the functions in the class $\tilde{S}_\beta^\alpha$.

**Theorem 2.** (Libera [6]) If the function $f \in \tilde{S}_\beta^\alpha$ is given by (2), then

$$|a_n| \leq \prod_{j=0}^{n-2} \left[\frac{2(1-\alpha)e^{-i\beta}\cos \beta + j}{j+1}\right] \quad (n \in \mathbb{N}\setminus\{1\}; \; \mathbb{N} := \{1, 2, 3, \ldots\}). \quad (5)$$

The coefficient estimates in (5) are sharp.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic functions in $\mathbb{D}$. The Hadamard product (convolution) of $f$ and $g$, denoted by $f * g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \; z \in \mathbb{D}.$$
Let $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The Ruscheweyh derivative \cite{8} of the $n^{th}$ order of $f$, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k z^k. \quad (6)$$

The Ruscheweyh derivative gave an impulse for various generalization of well known classes of functions. Exemplary, for $\alpha (0 \leq \alpha < 1)$ and $n \in \mathbb{N}_0$, Ahuja \cite{1, 2} defined the class of functions, denoted $R_n(\alpha)$, which consist of univalent functions of the form \eqref{2} that satisfying the condition

$$\text{Re} \left( e^{i\beta} z \left( \frac{D^n f(z)}{D^n f(z)} \right)' \right) > \alpha, \ z \in \mathbb{D}. \quad (7)$$

We denote that $R_0(\alpha) = S^*(\alpha)$. The class $R_n(0) = R_n$ was studied by Singh and Singh \cite{9}. With the aid of Ruscheweyh derivative we can generalize the spirallike functions as follows.

**Definition 3.** Let the function $f(z)$ be in the normalized analytic function class $A$. Also let $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $f(z)$ is in the class $\tilde{R}_n^\beta$ if and only if

$$\text{Re} \left( e^{i\beta} z \left( \frac{D^n f(z)}{D^n f(z)} \right)' \right) > 0, \ z \in \mathbb{D}. \quad (8)$$

Note that $\tilde{R}_0^\beta = \tilde{S}_\beta$.

**Definition 4.** Let the function $f(z)$ be in the normalized analytic function class $A$. Also, let $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\alpha \in [0, 1)$. Then $f(z)$ is in the class $\tilde{R}_n^\beta(\alpha)$ if and only if

$$\text{Re} \left( e^{i\beta} z \left( \frac{D^n f(z)}{D^n f(z)} \right)' \right) > \alpha \cos \beta, \ z \in \mathbb{D}. \quad (9)$$

Also, note that $\tilde{R}_0^\beta(\alpha) = \tilde{S}_\alpha^\beta$, $\tilde{R}_0^\beta(0) = \tilde{S}_\beta^\beta$, and $\tilde{R}_0^0(\alpha) = S^*(\alpha)$.

**Definition 5.** Let $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and let $f$ be an univalent function of the form \eqref{2} such that $D^n f(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$. We say that $f$ belongs to $\tilde{R}_n^\beta(\alpha, \lambda)$ if and only if

$$\text{Re} \left( e^{i\beta} \frac{z (D^n f(z))'}{(1-\lambda) D^n f(z) + \lambda z (D^n f(z))'} \right) > \alpha \cos \beta, \ z \in \mathbb{D}. \quad (10)$$
Definition 6. Let \( f(z) \) and \( g(z) \) are analytic functions in \( \mathbb{D} \). We say that \( f(z) \) is subordinate to \( g(z) \) in \( \mathbb{D} \) and we denote

\[
f(z) \prec g(z) \quad (z \in \mathbb{D}),
\]

if there exists a Schwarz function \( w(z) \) analytic in \( \mathbb{D} \), with

\[
w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \mathbb{D}),
\]

such that

\[
f(z) = g(w(z)) \quad (z \in \mathbb{D}).
\]

In particular, if the function \( g \) is univalent in \( \mathbb{D} \), the above subordination is equivalent to

\[
f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).
\]

After the proof of the Bieberbach Conjecture [3] (which is also known as de Branges Theorem [4]), many authors were interested in other interesting subclasses of normalized analytic function class \( \mathcal{A} \). In this paper, we obtain sharp coefficient bounds for functions in the class \( \tilde{R}^\beta_n(\alpha, \lambda) \) and we give a necessary and sufficient condition such that \( f \in \mathcal{A} \) belongs to \( \tilde{R}^\beta_n(\alpha, \lambda) \).

2. Main Results

In this section, we obtain coefficient conditions for functions in the class given by Definition 5. Also, we get sharp estimates for functions belong to \( \tilde{R}^\beta_n(\alpha, \lambda) \).

Theorem 3. Let \( \alpha \in [0, 1) \) and \( \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and let \( f(z) \) is in the form (2) such that \( D^n f(z) \neq 0 \) for \( z \in \mathbb{D}\setminus \{0\} \). Then, \( f(z) \) belongs to the class \( \tilde{R}^\beta_n(\alpha, \lambda) \) if and only if

\[
\sum_{k=1}^{\infty} \left\{ (k-1)(1-\lambda(\alpha + i \tan \beta)) + 2e^{2i\beta} - \lambda(1-\alpha) \left( 1-e^{2i\beta} \right)(k-1) \right\} A_k z^k \neq 0
\]

\[
(z \in z \in \mathbb{D}\setminus \{0\}), \tag{11}
\]

where

\[
A_k = (1+(k-1)\lambda) \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k, \quad k \in \mathbb{N}\setminus \{1\}.
\]
Proof. Let the function \( f \in S \) be defined by \( (2) \). Define a function

\[
h(z) = D^n f(z) = z + \sum_{k=2}^{\infty} A_k z^k, \quad z \in \mathbb{D}.
\]

(12)

Consider the function

\[
p(z) = e^{i\beta} \sec \beta \left( \frac{h(z)}{(1-\lambda)h(z) + \lambda z h'(z)} \right) - i \tan \beta - \alpha
\]

is an analytic function which satisfies \( p(0) = 1 \) and \( \text{Re}(p(z)) > 0 \), then \( f \in \tilde{R}_n^\beta (\alpha, \lambda) \) if and only if

\[
p(z) \neq \frac{1 - e^{2i\beta}}{1 + e^{2i\beta}}
\]

or,

\[
\frac{e^{i\beta} \sec \beta z h'(z) - (\alpha + i \tan \beta) ((1-\lambda) h(z) + \lambda z h'(z))}{(1-\alpha) ((1-\lambda) h(z) + \lambda z h'(z))} \neq \frac{1 - e^{2i\beta}}{1 + e^{2i\beta}}
\]

By using the series expansion of \( h(z) \) which is given by \( (12) \), we get the following

\[
\left(1 + e^{2i\beta}\right) \sum_{k=1}^{\infty} ((k-1)(1-\alpha \lambda - i \lambda \tan \beta) + (1-\alpha)) A_k z^k
\]

\[
\neq (1-\alpha) \left(1 - e^{2i\beta}\right) \sum_{k=1}^{\infty} (1+(k-1)\lambda) A_k z^k
\]

for \( z \neq 0 \). It is equivalent to

\[
\sum_{k=1}^{\infty} \left\{(k-1) (1-\lambda (\alpha + i \tan \beta)) + 2 e^{2i\beta} - (1-\alpha) (1 - e^{2i\beta}) (k-1) \lambda \right\} A_k z^k \neq 0,
\]

which completes the proof of Theorem 3. \( \blacksquare \)

Now, we prove our coefficient estimates for functions which belong to the class \( \tilde{R}_n^\beta (\alpha, \lambda) \).

Theorem 4. Let \( \alpha \in [0,1) \) and \( \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and let \( f(z) \) is in the form \( (2) \) such that \( D^n f(z) \neq 0 \) for \( z \in \mathbb{D}\setminus\{0\} \). If \( f(z) \) belongs to the class \( \tilde{R}_n^\beta (\alpha, \lambda) \) then

\[
|a_k| \leq \frac{\Gamma(n+1)}{\Gamma(n+k)} \frac{1}{(1-\lambda)^{k-1}} \prod_{j=0}^{k-2} \left| j (1-\lambda) + 2 (1-\alpha) e^{i\beta} \cos \beta (1+\lambda j) \right| \quad (n \in \mathbb{N}\setminus\{1\}; \ \mathbb{N} := \{1,2,3,\ldots\}).
\]

(13)

This result is sharp.
Proof. Since \( f \in \tilde{R}^\beta_n (\alpha, \lambda) \) there exists a Schwarz function \( w(z) \), which is already introduced in Definition 6, such that
\[
e^{i\beta \sec \beta} \left( \frac{z \left( D^n f (z) \right)'}{(1 - \lambda) D^n f (z) + \lambda z \left( D^n f (z) \right)'} \right) - i \tan \beta = \frac{1 + (1 - 2\alpha) w(z)}{1 - w(z)}.
\]
Consider the function \( h(z) \) defined by (12). Then, we get
\[
\sum_{k=2}^{\infty} \left[ k e^{i\beta \sec \beta} - (1 + i \tan \beta) (1 - (k - 1) \lambda) \right] A_k z^k
\]
\[
= \left( \sum_{k=1}^{\infty} \left[ k e^{i\beta \sec \beta} + (1 - 2\alpha - i \tan \beta) (1 + (k - 1) \lambda) \right] A_k z^k \right) w(z). \tag{14}
\]
The last equation (14) may be written (for \( n \in \mathbb{N} \)) as follows:
\[
\sum_{k=2}^{m} \left[ k e^{i\beta \sec \beta} - (1 + i \tan \beta) (1 - (k - 1) \lambda) \right] A_k z^k + \sum_{k=m+1}^{\infty} b_k z^k
\]
\[
= \left( \sum_{k=1}^{m-1} \left[ k e^{i\beta \sec \beta} + (1 - 2\alpha - i \tan \beta) (1 + (k - 1) \lambda) \right] A_k z^k \right) w(z). \tag{15}
\]
The last sum on the left-hand side of (15) is convergent in \( \mathbb{D} \) for \( m = 2, 3, \ldots \).

Since, by hypothesis, \(|w(z)| < 1 \ (z \in \mathbb{D})\), it is not difficult to find by appealing to Parseval’s Theorem that
\[
\sum_{k=1}^{m-1} \left| k e^{i\beta \sec \beta} (1 - 2\alpha - i \tan \beta) (1 + (k - 1) \lambda) \right|^2 |A_k|^2
\]
\[
\geq \sum_{k=2}^{m} \left| k e^{i\beta \sec \beta} - (1 + i \tan \beta) (1 - (k - 1) \lambda) \right|^2 |A_k|^2
\]
or
\[
\sum_{k=1}^{m-1} 4 (1 - \alpha) (k - \alpha (1 + (k - 1) \lambda)) |A_k|^2 \geq \frac{(m - 1)^2 (1 - \lambda)^2}{\cos^2 \beta} |A_m|^2 \tag{16}
\]
where \( A_1 = 1 \).

We claim that
\[
|A_m| \leq \frac{1}{(m - 1)! (1 - \lambda)^{m-1}} \prod_{j=0}^{m-2} \left| j (1 - \lambda) + 2 (1 - \alpha) \cos \beta e^{i\beta} (1 + j \lambda) \right|. \tag{17}
\]
For \(m = 2\), we get from (16)

\[|A_2| \leq \frac{2(1 - \alpha) \cos \beta}{1 - \lambda},\]

which is equivalent to (17). (17) is obtained for larger \(m\) from inequality (16) by the principle of the mathematical induction.

Fix \(m, m \geq 3\), and suppose that (13) holds for \(k = 2, 3, \cdots, m - 1\). Then from (16) we get the following inequality

\[|A_m|^2 \leq \frac{4(1 - \alpha) \cos^2 \beta}{(m - 1)^2 (1 - \lambda)^2} \left\{ 1 - \alpha + \sum_{k=2}^{m-1} B(k, j, \alpha) \right\} \tag{18}\]

where

\[B(k, j, \alpha) = \frac{(1 + (k - 1) \lambda)(k - \alpha(k - 1) \lambda)}{(k - 1)! (1 - \lambda)^{k-1}} \prod_{j=0}^{k-2} \left| j(1 - \lambda) + 2(1 - \alpha) \cos \beta e^{i\beta} (1 + j\lambda) \right|^2.\]

We must show that the square of the right side of (17) is equal to the right side of (18); that is

\[\prod_{j=0}^{m-2} \left| j(1 - \lambda) + 2(1 - \alpha) \cos \beta e^{i\beta} (1 + j\lambda) \right|^2 \left[ (m - 1)! (1 - \lambda)^{m-1} \right]^2 = \frac{4(1 - \alpha) \cos^2 \beta}{(m - 1)^2 (1 - \lambda)^2} \left\{ 1 - \alpha + \sum_{k=2}^{m-1} B(k, j, \alpha) \right\} \tag{19}\]

for \(m = 3, 4, \cdots\). After necessary calculations we can show that (19) is true for \(m = 3\) and proves our claim for \(m = 3\). Assume that (19) is valid for all \(k, 3 < k \leq m - 1\); then from (16) and (18) we obtain

\[|A_m|^2 \leq \frac{4(1 - \alpha) \cos^2 \beta}{(m - 1)^2 (1 - \lambda)^2} \left\{ 1 - \alpha + \sum_{k=2}^{m-2} B(k, j, \alpha) + B(m - 1, j, \alpha) \right\}\]
\[ |A_m|^2 \leq \frac{4(1-\alpha)\cos^2 \beta}{(m-1)^2(1-\lambda)^2} \{ 1-\alpha + \sum_{k=2}^{m-2} \frac{(1+(k-1)\lambda)(k-\alpha(k-1)\lambda)}{(k-1)!(1-\lambda)^{k-1}} \prod_{j=0}^{k-2} |j(1-\lambda) + 2(1-\alpha)\cos \beta e^{i\beta}(1+j\lambda)|^2 \}
+ \frac{(1+(m-2)\lambda)(m-1-\alpha(m-2)\lambda)}{(m-2)!(1-\lambda)^{m-2}} \prod_{j=0}^{m-3} |j(1-\lambda) + 2(1-\alpha)\cos \beta e^{i\beta}(1+j\lambda)|^2 \}
\]

\[
\prod_{j=0}^{m-3} |j(1-\lambda) + 2(1-\alpha)\cos \beta e^{i\beta}(1+j\lambda)|^2 \frac{\{(m-2)^2}{(m-1)^2}
+4(1-\alpha)\cos^2 \beta \frac{(1+(m-2)\lambda)(m-1-\alpha(m-2)\lambda)}{(m-1)^2(1-\lambda)^2}\}
\]

\[
\prod_{j=0}^{m-3} |j(1-\lambda) + 2(1-\alpha)\cos \beta e^{i\beta}(1+j\lambda)|^2 \frac{\{(m-2)^2}{(m-1)^2}
+4(1-\alpha)\cos^2 \beta \frac{(1+(m-2)\lambda)(m-1-\alpha(m-2)\lambda)}{(m-1)^2(1-\lambda)^2}\}
\]

\[
\prod_{j=0}^{m-3} |j(1-\lambda) + 2(1-\alpha)\cos \beta e^{i\beta}(1+j\lambda)|^2 \frac{\{(m-2)^2}{(m-1)^2}
+4(1-\alpha)\cos^2 \beta \frac{(1+(m-2)\lambda)(m-1-\alpha(m-2)\lambda)}{(m-1)^2(1-\lambda)^2}\}
\]

From equality (6) we get the desired result. □

3. Corollaries and Consequences

By choosing appropriate values of values of \(n, \lambda, \beta\) and \(\alpha\) in Theorem 4, we obtain the corresponding results for several subclasses of \(S\).

**Corollary 5.** If \(\lambda = 0\), we get the following result for function \(f \in \tilde{R}_m^{n,\lambda}(\alpha)\)

\[
|a_k| \leq \frac{\Gamma(n+1)}{\Gamma(n+k)} \prod_{j=0}^{k-2} |j(1-\lambda) + 2(1-\alpha)\cos \beta e^{i\beta}|.
\]
Corollary 6. If \( n = 0 \) and \( \lambda = 0 \), we obtain (5) which is stated in Theorem 2.

Corollary 7. If \( n = 0 \), \( \beta = 0 \) and \( \lambda = 0 \), we obtain the following result for functions belong to \( S^* (\alpha) \)

\[
|a_k| \leq \prod_{j=0}^{k-2} \frac{|j + 2 (1 - \alpha)|}{j + 1}.
\]

Corollary 8. If \( \lambda = 0 \) and \( \alpha = 0 \), we get the following result for function \( f \in \tilde{R}_n^\beta \)

\[
|a_k| \leq \frac{\Gamma (n + 1)}{\Gamma (n + k)} \prod_{j=0}^{k-2} \left| j + 2 e^{i\beta} \cos \beta \right|.
\]

Corollary 9. If \( n = 0 \), \( \lambda = 0 \) and \( \alpha = 0 \), we get the following result for spirallike functions of type \( \beta \) in \( \mathbb{D} \)

\[
|a_k| \leq \prod_{j=0}^{k-2} \frac{|j + 2 e^{i\beta} \cos \beta|}{j + 1}.
\]

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