ON AN INTEGRAL OPERATOR

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ABSTRACT. In this paper we define the integral operator $J_{\alpha,\beta}(z)$, considered for analytic functions $g_i$ in the open unit disk $U$, and we will prove, using Becker criterion, its univalence. Also, we will present some properties of the operator.

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1. Introduction and Preliminaries

Let the open unit disk $U = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathcal{A}$ the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $U$ and satisfy the condition $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{H}(U)$ the space of holomorphic functions in $U$ and by $\mathcal{S} \subset \mathcal{A}$ the subclass of univalent and regular functions from $\mathcal{A}$.

The following sufficient condition for univalency of an analytic function in the unit disk was given by Becker in [1]:

**Theorem 1.** Let $f \in \mathcal{A}$. If for all $z \in U$ we have:

$$\left(1 - |z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

then the function $f$ is univalent in $U$.

In [3], N.N. Pascu introduced the integral operator $L_\alpha : \mathcal{H}(U) \to \mathcal{H}(U)$ defined as:

$$L_\alpha(z) = \frac{z^{1-\alpha}}{\alpha} \int_0^z t^{\frac{1}{\alpha} - 2} g(t) dt.$$
Starting from this, V. Pescar and G.L. Aldea in [4] also present the operator $J_{\alpha,\beta}: \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U})$,

$$J_{\alpha,\beta}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z \frac{t^{\frac{1}{\alpha}-2}}{t^{\frac{1}{\alpha}}-2} (g(t))^{\beta} \mathrm{dt}.$$ \quad (4)

The goal of our paper is to go further with the generalization and for this we will introduce the integral operator $J_{\alpha,\beta}(z)$, given by:

$$J_{\alpha,\beta}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z \frac{t^{\frac{1}{\alpha}-2} (g_1(t))^{\beta_1} \ldots (g_n(t))^{\beta_n}}{t^{\frac{1}{\alpha}}-2} \mathrm{dt},$$ \quad (5)

with $\alpha$ and at least one $\beta_i$ unequal with 0. We will study the univalence for it and present some properties obtained from here.

2. Main results

**Theorem 2.** Let the function $g_i \in A$ of the form (1), $M$ be a positive real number $(M \geq 1)$ and $\alpha, \beta_i, \ i = 1, n$, be complex numbers with $\alpha$ and at least one of $\beta_i$ nonequal with 0.

If we have:

i) $\left| \frac{g_i'(z)}{g_i(z)} \right| \leq M, \ i = 1, n$;

ii) $M - 1 \leq \frac{1 - \frac{1}{\alpha} + \sum_{i=1}^n \beta_i - 2}{\left| \sum_{i=1}^n \beta_i \right|}$,

then the function $z^{\frac{1}{\alpha}-1}J_{\alpha,\beta_i}(z)$ is in the class $\mathcal{S}$.

**Proof.** We may write the operator (5) as:

$$J_{\alpha,\beta}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z \frac{t^{\frac{1}{\alpha}+\sum_{i=1}^n \beta_i-2}}{t^{\frac{1}{\alpha}}-2} \left( \frac{g_1(t)}{t} \right)^{\beta_1} \ldots \left( \frac{g_n(t)}{t} \right)^{\beta_n} \mathrm{dt}.$$

We consider now:

$$G_{\alpha,\beta}(z) = \frac{1}{\alpha} \int_0^z \frac{t^{\frac{1}{\alpha}+\sum_{i=1}^n \beta_i-2}}{t^{\frac{1}{\alpha}}-2} \left( \frac{g_1(t)}{t} \right)^{\beta_1} \ldots \left( \frac{g_n(t)}{t} \right)^{\beta_n} \mathrm{dt}.$$
We have:

\[ G'_{\alpha,\beta_i}(z) = \frac{1}{\alpha} z^\frac{1}{\alpha} + \sum_{i=1}^{n} \left( \frac{g_1(z)}{z} \right)^{\beta_1} \cdots \left( \frac{g_n(z)}{z} \right)^{\beta_n} \]

and:

\[ G''_{\alpha,\beta_i}(z) = \frac{1}{\alpha} z^\frac{1}{\alpha} + \sum_{i=1}^{n} \left( \frac{g_1(z)}{z} \right)^{\beta_1} \cdots \left( \frac{g_n(z)}{z} \right)^{\beta_n} \cdot \left( \frac{1}{\alpha} + \sum_{i=1}^{n} \beta_i - 2 \right) \frac{1}{z} \sum_{i=1}^{n} \beta_i \left( \frac{g'_i(z)}{g_i(z)} - \frac{1}{z} \right) \]

So we obtain:

\[ \frac{G''_{\alpha,\beta_i}(z)}{G'_{\alpha,\beta_i}(z)} = \frac{1}{z} \left( \frac{1}{\alpha} + \sum_{i=1}^{n} \beta_i - 2 \right) + \sum_{i=1}^{n} \beta_i \left( \frac{g'_i(z)}{g_i(z)} - \frac{1}{z} \right) , \]

hence:

\[ \left( 1 - |z|^2 \right) \left| \frac{G''_{\alpha,\beta_i}(z)}{G'_{\alpha,\beta_i}(z)} \right| = \left( 1 - |z|^2 \right) \left| \frac{1}{\alpha} + \sum_{i=1}^{n} \beta_i - 2 + z \sum_{i=1}^{n} \beta_i \left( \frac{g'_i(z)}{g_i(z)} - \frac{1}{z} \right) \right| \]

\[ \leq \left| \frac{1}{\alpha} + \sum_{i=1}^{n} \beta_i - 2 \right| + \sum_{i=1}^{n} |\beta_i| \left| \left( \frac{zg'_i(z)}{g_i(z)} - 1 \right) \right| \]

Using successively the properties i) and ii) for the function \( g_i \), we have:

\[ \left( 1 - |z|^2 \right) \left| \frac{G''_{\alpha,\beta_i}(z)}{G'_{\alpha,\beta_i}(z)} \right| \leq \left| \frac{1}{\alpha} + \sum_{i=1}^{n} \beta_i - 2 \right| + (M - 1) \sum_{i=1}^{n} |\beta_i| \leq 1. \]

Hence, by Becker univalence criterion, we prove that the operator \( G_{\alpha,\beta_i}(z) \) is in the class \( S \), so \( z^{\frac{1}{\alpha} - 1} J_{\alpha,\beta_i}(z) \) is in the class \( S \).

**Remark 1.** Because \( z^{\frac{1}{\alpha} - 1} J_{\alpha,\beta_i}(z) \in S \), there exists \( a_i, i = 1, \ldots, n \) such as we may write:

\[ z^{\frac{1}{\alpha} - 1} J_{\alpha,\beta_i}(z) = z + \sum_{i=2}^{\infty} a_i z^i, \quad z \in \mathcal{U}, \]

so it is obviously that the operator \( J_{\alpha,\beta_i}(z) \) is of the form:

\[ J_{\alpha,\beta_i}(z) = z^{2-\frac{1}{\alpha}} + \sum_{i=2}^{\infty} a_i z^{i+1-\frac{1}{\alpha}}, \quad z \in \mathcal{U}. \]
Remark 2. For \( \beta_i = 0, i = \frac{2}{n}, \) we obtain the Pescar and Aldea’s operator (4) and, of course, for \( \beta_1 = 1, \beta_i = 0, i = \frac{2}{n}, \) \( J_{\alpha,\beta}(z) \) becomes Pascu’s operator.

Corollary 3. Let the function \( g_i \in A \) of the form (1), \( M \) be a positive real number \( (M \geq 1) \) and \( \alpha \) be complex number, \( \alpha \neq 0. \)

If:

\[
\begin{align*}
\text{i) } & \left| \frac{g_i'(z)}{g_i(z)} \right| \leq M, \quad i = 1, n; \\
\text{ii) } & M - 1 \leq \frac{1 - |\frac{1}{n} + \alpha - 2|}{n},
\end{align*}
\]

then the function:

\[
J_{\alpha}(z) = \frac{1}{\alpha} \int_{0}^{z} t^{\frac{1}{n} - 2} g_1(t) \cdots g_n(t) dt
\]

is in the class \( S. \)

Proof. We consider \( \beta_i = 1, i = \frac{2}{n}, \) in theorem 2.

Corollary 4. Let the function \( g_i \in A \) of the form (1). If:

\[
\left| \frac{g_i'(z)}{g_i(z)} \right| \leq \frac{2}{n}, \quad i = 1, n,
\]

then the function:

\[
J(z) = \int_{0}^{z} t^{-1} g_1(t) \cdots g_n(t) dt
\]

is in the class \( S. \)

Proof. We consider \( \alpha = \beta_i = 1, i = \frac{2}{n}, \) in theorem 2.

References


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