THE NOOR INTEGRAL OPERATOR AND $\beta$ UNIFORMLY $\alpha$-SPIRALLIKE FUNCTIONS

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Abstract. In [10, 12] Noor introduced an integral operator by using convolution. In this paper, we apply this operator on a class of analytic functions. We also apply the proposed operator on $\beta$ uniformly $\alpha$-spirallik functions to find some inclusion relations, coefficient bounds and test example.

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1. Introduction

A.W.Goodman investigated about some univalent functions geometrically [3]. For starlik functions, Let $\Gamma_w$ be the image of an arc $\Gamma_z : z = z(t) ; a < t < b$, where $w = f(z)$ and let $w_0$ be a point not on, $\Gamma_w$ is starlike with respect to $w_0$ if $\arg(w-w_0)$ is nondecreasing function of $t$. This condition is equivalent to:

$$\text{Im}\left\{\frac{z'(t)f'(z)}{f(z) - w_0}\right\} \geq 0.$$  

Similarly $\Gamma_w$ is $\alpha$-spiral($|\alpha| < \pi/2$) with respect to $w_0$ if

$$\alpha < \arg\left\{\frac{z'(t)f'(z)}{f(z) - w_0}\right\} < \alpha + \pi.$$  

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1)

which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$.

The class of starlike functions $f \in A$ with respect to origin denote by $S^*$. If $f \in A$ and $f$ be starlike with respect for every $z \in \Delta$, then $f$ is convex in $\Delta$. The
set of all convex functions \( f \in A \) denote by \( CV. \) Similarly the class of \( \alpha \)-spirallike functions \( f \in A \) with respect to origin denote by \( SP(\alpha) \). If \( f \in A \) and \( zf'(z) \in SP(\alpha) \) then \( f \) is convex \( \alpha \)-spirallike in \( \Delta \). The set of all convex \( \alpha \)-spirallike functions \( f \in A \) denote by \( CVSP(\alpha) \).

For \( |\alpha| < \pi/2 \), the function \( f(z) \) is uniformly \( \alpha \)-spirallike if the image of every circular \( \Gamma \) with center at \( \xi \) lying \( \Delta \) is \( \alpha \)-spirallike with respect to \( f(\xi) \). (see [13])

The function \( f(z) \in A \) is uniformly \( \alpha \)-spirallike in \( \Delta \) if and only if for every \( |\alpha| < \pi/2 \), we have

\[
\Re\left\{e^{-i\alpha}(z-\xi)f'(z)\right\} > \beta\left|zf'(z) - f(\xi)\right|, \quad z \in \Delta.
\] (2)

A function \( f(z) \in A \) for all \( z \in \Delta \), is said to be \( \beta \) uniformly \( \alpha \)-spiral in \( \Delta \) if for every circular arc \( \Gamma \) contained in \( \Delta \) with center at \( \xi \) \((|\xi| < \beta\)) the image of arc \( f(\Gamma) \) is \( \alpha \)-spirallike. (see [17])

The class of all \( \beta \) uniformly \( \alpha \)-spirallike function in \( \Delta \) is denote by \( USP(\alpha, \beta) \). (see [17])

**Theorem 1.** [17] Let \( f \in A \), then \( f(z) \) is in \( USP(\alpha, \beta) \) if and only if

\[
\Re\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} > \beta\left|zf'(z) - 1\right|, \quad z \in \Delta.
\]

(2)

**Theorem 2.** [17] Let \( f \in A \). \( f \in UCSP(\alpha, \beta) \) if and only if,

\[
\Re\left\{e^{-i\alpha}(1 + \frac{zf''(z)}{f'(z)})\right\} > \beta\left|zf''(z)\right|, \quad z \in \Delta.
\] (3)

Let \( f(z) \) and \( g(z) \) be analytic in \( \Delta \). Then \( f(z) \) is said to be subordinate to \( g(z) \), written \( f(z) \prec g(z) \), if there exists an analytic function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1(z \in \Delta) \) such that \( f(z) = g(w(z)) \) for \( z \in \Delta \).

\( g(z) \) is univalent in \( \Delta \), \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \) and \( f(\Delta) \subset g(\Delta) \). (see [7])

**Theorem 3.** [17] Let \( f \in A \), \( 0 < \beta < 1 \), then the function \( f(z) \) is in \( USP(\alpha, \beta) \) if and only if,

\[
e^{-i\alpha}\frac{zf'(z)}{f(z)} \prec h_{\beta}(z)\cos\alpha - i\sin\alpha,
\]
where
\[ h_\beta(z) = 1 + \frac{1}{2\sin^2(\sigma)} \left\{ \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^{2\pi} + \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right)^{2\pi} - 2 \right\} \quad (4) \]
and \( \sigma = \arccos \beta \).

Note that \( h_\beta(0) = 1 \) and \( h_\beta \) maps \( \Delta \) conformally onto the hyperbolic regain
\[ D_\beta = \{ w = u + iv : u > \beta \sqrt{(u - 1)^2 + v^2} \}. \]
Since \( D_\beta \) is a convex region, \( h_\beta \) is convex (and univalent) in \( \Delta \). (see [4, 17])

K.I. Noor and M.A. Noor defined an integral operator \( I_n : A \rightarrow A \) as follows.
\[ I_n f(z) = f^\dagger_n(z) * f(z). \quad (5) \]
where \( f^\dagger_n \) is defined by the relation
\[ \frac{z}{(1 - z)^{n+1}} * f^\dagger_n(z) = \frac{z}{(1 - z)^2}. \quad (see[10, 12]) \quad (6) \]
It is obvious that \( I_0(z) = z f'(z) \) and \( I_1(z) = f(z) \). The operator \( I_n f \) defined by (5) is called the Noor integral operator of \( n \)th order of \( f \).

J.L. Liu prove that the Noor integral operator satisfying the equation
\[ z(I_{n+1} f(z))' = (n + 1)I_n f(z) - nI_{n+1} f(z). \quad (see[5]) \quad (7) \]

Liu and Noor [6] investigated some interesting properties of the Noor integral operator and applications of the Noor integral operator. (for more details see [9, 11])

It is well known that for \( \alpha > 0 \)
\[ \frac{z}{(1 - z)^\alpha} = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} z^{m+1}, \quad (z \in \Delta). \]
where \((\alpha)_m\) is the Pochhammer symbol
\[ (\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 1, & n = 0, \alpha \neq 0 \\ \alpha(\alpha + 1)\ldots(\alpha + m - 1), & n \in \mathbb{N}. \end{cases} \]

By (6) we obtain,
\[ \sum_{m=0}^{\infty} \frac{(n + 1)_m}{m!} z^{m+1} * f^\dagger_n(z) = \sum_{m=0}^{\infty} \frac{(2)_m}{m!} z^{m+1}. \quad (8) \]

133
Then (8) implies that
\[ f_n^\dagger(z) = \sum_{m=0}^{\infty} \frac{(2)_m}{(n+1)_m} z^{m+1}, \quad (z \in \Delta). \]

Therefor, if \( f \) is of the form (1), then
\[ I_n f(z) = z + \sum_{m=2}^{\infty} \frac{(2)_{m-1}}{(n+1)_{m-1}} a_m z^m = z + \sum_{m=2}^{\infty} \frac{m!}{(n+1)_{m-1}} a_m z^m, \quad (z \in \Delta). \]

In the present paper we give some argument properties of \( \beta \) uniformly \( \alpha \)-spirallike functions and investigate some properties of the Noor integral operator.

2. Preliminary lemmas

We need the following Lemmas for our investigation.

**Lemma 4.** [15] Let \( 0 < \alpha < \beta \). If \( \beta \geq 2 \) or \( \alpha + \beta \geq 3 \), then the function
\[ h(z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\beta)_m} z^{m+1}, \quad (z \in \Delta). \]

belongs to the class of convex functions.

**Lemma 5.** [15, 16] If \( f \in CV \) and \( g \in SP(\alpha) \), then for each analytic function \( h \) in \( \Delta \) with \( h(0) = 1 \),
\[ \frac{\tilde{f} * hg(\Delta)}{\tilde{f} * g(\Delta)} \subseteq \overline{coh}(\Delta), \]
where \( \tilde{f}(z) = f(\frac{z}{2}) \) and \( \overline{coh}(\Delta) \) denotes the closed convex hull of \( h(\Delta) \).

**Lemma 6.** [2, 8] Let \( f \) be convex univalent in \( \Delta \) with \( f(0) = 1 \) and \( \text{Re}(\lambda f(z) + \mu) > 0 \) \( (\lambda, \mu \in \mathbb{C}) \). If \( p \) is analytic in \( \Delta \) with \( p(0) = 1 \), then
\[ p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} \prec f(z). \]
implies,
\[ p(z) \prec f(z), \quad (z \in \Delta). \]
**Remark 1.** Let $|\alpha| < \pi/2$ and $f$ be a convex univalent function in $\Delta$ with $g(0) = e^{-i\alpha}$ and $\text{Re}(\lambda g(z) + \mu) > 0$ ($\lambda, \mu \in \mathbb{C}$). If $p$ is analytic in $\Delta$ with $p(0) = e^{-i\alpha}$ then

$$p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} < g(z).$$

implies,

$$p(z) < g(z), \quad (z \in \Delta).$$

**Lemma 7.** [14] Let

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n < 1 + \sum_{n=1}^{\infty} C_n z^n = H(z), \quad (z \in \Delta).$$

If the function $H$ be univalent in $\Delta$ and $H(\Delta)$ be a convex set, then

$$|c_n| \leq |C_1|.$$ 

**Lemma 8.** [17] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in SP(\alpha, \beta)$ and $0 < \beta < 1$, then

$$|a_2| \leq 8 \cos \alpha \left( \frac{\sigma}{\pi \sin \sigma} \right)^2, \quad \sigma = \arccos \beta. \quad (9)$$

This result is sharp. Also it is clear that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + ..., \quad (z \in \Delta).$$

### 3. Main results

**Theorem 9.** Let $f \in A$. If $f \in SP(\alpha, \beta)$ satisfying the condition

$$\frac{e^{-i\alpha} z (I_{n-1} f(z))'}{I_{n-1} f(z)} < h_{\beta}(z) \cos \alpha - isin \alpha, \quad (z \in \Delta),$$

then,

$$\frac{e^{-i\alpha} z (I_{n+1} f(z))'}{I_{n+1} f(z)} < h_{\beta}(z) \cos \alpha - isin \alpha, \quad (z \in \Delta). \quad (11)$$

**Proof.** Let

$$p(z) = \frac{e^{-i\alpha} z (I_{n+1} f(z))'}{I_{n+1} f(z)},$$

135
where $p$ is an analytic function with $p(0) = e^{-ia}$. By using the equation (7), we have

$$p(z) + e^{-ia} n = e^{-ia} (n + 1) \frac{I_n f(z)}{I_{n+1} f(z)}. \tag{12}$$

Taking logarithmic derivative in both side of (12) and multiplying by $e^{-ia} z$, we have

$$p(z) + z p'(z) e^{ia} c = e^{-ia} (I_n f(z))'. \tag{13}$$

By applying relation (10) and Remark 1 it follows that $p(z) \prec h_\beta(z) \cos \alpha - isin \alpha$, that is the relation (11).

**Theorem 10.** If a function $f \in A$ satisfies the condition

$$\frac{e^{-ia} z \left( I_n f(z) \right)'}{I_n f(z)} \prec h_\beta(z) \cos \alpha - isin \alpha, \quad (z \in \Delta), \tag{13}$$

then,

$$\frac{e^{-ia} z \left( I_n F_c(f)(z) \right)'}{I_n F_c(f)(z)} \prec h_\beta(z) \cos \alpha - isin \alpha, \quad (z \in \Delta), \tag{14}$$

where $F_c$ be the integral operator defined by

$$F_c(f)(z) = \frac{c + 1}{c} \int_0^z t^{-1} f(t) dt, \quad (c \geq 0). \tag{15}$$

**Proof.** Let

$$p(z) = \frac{e^{-ia} z \left( I_n F_c(f)(z) \right)'}{I_n F_c(f)(z)},$$

where $p$ is analytic function with $p(0) = e^{-ia}$. From (15) we have

$$z \left( I_n F_c(f) \right)'(z) = (c + 1) I_n f(z) - c I_n F_c(f)(z). \tag{16}$$

Then by using (16), we get

$$c + p(z) = (c + 1) \frac{e^{-ia} I_n f(z)}{I_n F_c(f)(z)}. \tag{17}$$

Taking logarithmic derivatives in both side of (17) and multiplying by $e^{-ia} z$, we have

$$p(z) + \frac{z p'(z)}{e^{ia} c + e^{ia} p(z)} = \frac{e^{-ia} z (I_n f(z))'}{I_n f(z)}. \tag{18}$$

Therefore by relations (13) and (18) and Remark 1 we obtain (14) for all $z \in \Delta$ and the proof is complete.
Definition 1. A function \( f \in A \) is said to be in the class \( M_\alpha(n, \beta) \) \((|\alpha| < \pi/2, 0 \leq \beta \leq 1)\) if and only if, \( I_n(f) \in USP(\alpha, \beta) \) or equivalently
\[
\text{Re}\left\{\frac{e^{-ia}z(I_n f)'(z)}{(I_n f)(z)}\right\} > \beta \left| \frac{z(I_n f)'(z)}{(I_n f)(z)} - 1 \right|, \quad (z \in \Delta).
\]

(19)

Note that the class \( M_\alpha(n, \beta) \) unifies many subclasses of \( A \). In particular, \( M_\alpha(1, 0) = CVSP(\alpha) \), the class of convex \( \alpha \)-spirallike functions; \( M_\alpha(0, 0) = SP(\alpha) \), the class of \( \alpha \)-spirallike functions; \( M_\alpha(1, 1) = USP(\alpha, \beta) \), the class of uniformly \( \alpha \)-spirallike functions; \( M_\alpha(0, 1) = UCSP(\alpha) \), the class of uniformly convex \( \alpha \)-spirallike functions; \( M_\alpha(0, \beta) = UCSP(\alpha, \beta) \) and \( M_\alpha(1, \beta) = USP(\alpha, \beta) \).

Also, by a simple computation, if \( 0 < \beta_1 < \beta_2 < 1 \) then \( M_\alpha(n, \beta_2) \subset M_\alpha(n, \beta_1) \).

Theorem 11. The function \( k(z) = \frac{z}{(1 - A z)^{1-\alpha}} \) is in \( M_\alpha(1, \beta) \) if and only if
\[
|A| \leq \frac{\cos \alpha}{\beta \sqrt{2} + \sin \alpha}.
\]

(20)

Proof. By using (2) \( k(z) \in USP(\alpha, \beta) \), if and only if
\[
\text{Re}\left\{\frac{e^{-ia} 1 + A i z}{1 - A z}\right\} \geq \beta \left| \frac{A z(i + 1)}{1 - A z} \right|.
\]

(21)

It is suffices to consider \( |z| = 1 \) in the above relation, by setting \( |A| = r \) and \( A z = re^{i\theta} \) we have \( A i z = re^{i(\theta + \frac{\pi}{2})} \). It follows from (21),
\[
\text{Re}\left\{\frac{e^{-ia} 1 + re^{i(\theta + \frac{\pi}{2})}}{1 - re^{i\theta}}\right\} \geq \frac{\beta r \sqrt{2}}{|1 - re^{i\theta}|}.
\]

(22)

After simplification, we see that
\[
\text{Re}\left\{\frac{e^{-ia} 1 + re^{i(\theta + \frac{\pi}{2})}}{1 - re^{i\theta}}\right\} = \frac{\cos(1 - r \cos \theta - r \sin \theta) + r \sin(a)(\sin \theta + \cos \theta - r)}{|1 - re^{i\theta}|^2}.
\]

(23)

By using (22), (23), it is equivalent to
\[
\frac{\cos(1 - r \cos \theta - r \sin \theta) + r \sin(a)(\sin \theta + \cos \theta - r)}{(1 - 2r \cos \theta + r^2)^\frac{3}{2}} \geq \beta \sqrt{2}.
\]

(24)

The minimum value of the expression in the left hand side of the equation (24) occur at \( \theta = \pi \) and this minimum value is \( \cos \alpha - r \sin \alpha \), so we have
\[
r \leq \frac{\cos \alpha}{\beta \sqrt{2} + \sin \alpha}.
\]

(25)

Hence, a necessary and sufficient condition for (20) is (25).
Example 1. The function \( \varphi(z) = z + a_m z^m \in UCSP(\alpha, \beta) \) if and only if it satisfies (3). It is sufficient to consider \(|z| = 1\) in the above relation, by setting \(|ma_m| = r\) and \(ma_m z^{m-1} = re^{i\theta}\), we have

\[
Re \left\{ e^{-i\alpha} \frac{1 + mre^{i\theta}}{1 + re^{i\theta}} \right\} \geq \frac{\beta(m-1)r}{|1 + re^{i\theta}|}. \tag{26}
\]

After simplifying and separating the real part of the expression of (26), we get

\[
\frac{\cos \alpha (1 + mr^2 + mrcos \theta + rcos \theta) - rsin \alpha sin \theta (m-1)}{(1 + r^2 + 2rcos \theta)^{\frac{1}{2}}} \geq \beta(m-1)r.
\]

The minimum of the expression in the left hand side of the above equation occurs at \( \theta = \pi \) and this minimum value is \( \cos \alpha (1 - mr) \), hence

\[
r \leq \frac{\cos \alpha}{(m-1)\beta + m \cos \alpha}.
\]

After by solving this equation for \( r = |ma_m|\), we have

\[
|a_m| \leq \frac{\cos \alpha}{m(m-1)\beta + m^2 \cos \alpha}.
\]

Since the function \( f(z) \in A \) is \( \beta \) uniformly convex \( \alpha \)-spirallike in \( \Delta \) if and only if \( zf'(z) \) is \( \beta \) uniformly \( \alpha \)-spirallike in \( \Delta \), yields; if \( f \in USP(\alpha, \beta) \) then,

\[
|a_m| \leq \frac{\cos \alpha}{(m-1)\beta + m \cos \alpha}.
\]

If \( \varphi \in USP(\alpha, \beta) \), then

\[
I_n \varphi(z) = z + \frac{m!}{(n+1)m-1} a_m z^m,
\]

is in \( UCSP(\alpha, \beta) \) for \( n \in \{3, 4, ...\} \). Moreover \( I_n \varphi \notin UCSP(\alpha, \beta) \) for \( n \in \{1, 2\} \). It would be interesting to check this property of the Noor integral operator for other functions in \( USP(\alpha, \beta) \).

Theorem 12. The function \( f(z) = z + a_m z^m \) is in \( M_\alpha(n, \beta) \) if and only if

\[
|a_m| \leq \frac{(n+1)m-1 \cos \alpha}{m!((m-1)\beta + m \cos \alpha)}, \quad (m \geq 2).
\]

138
Proof. Let $I_n f(z) = z + b_m z^m = z + \frac{m!}{(n+1)m-1}a_m z^m$. It is suffices to consider $|z| = 1$ in the above relation, by setting $|b_m| = r$ and $b_m z^{m-1} = re^{i\theta}$, then (19) for this $f$ will be

$$\Re\left\{e^{-i\alpha}\frac{1+mr e^{i\theta}}{1+re^{i\theta}}\right\} \geq \frac{\beta r(m-1)}{|1+re^{i\theta}|}.$$ 

By the same steps of theorem 11, we get the desired result.

**Remark 2.** For particular value of $m,n,\beta$, Theorem 12 provides functions belonging to the class $M_\alpha(n,\beta)$. For example for $m=2,n=0,\beta=1$, we have

$$|a_2| \leq \frac{\cos\alpha}{2+4\cos\alpha},$$

so the function $f(z) = z + \frac{\cos\alpha}{2+4\cos\alpha} z^2$ is in UCSP($\alpha$).

**Theorem 13.** Let $f$ with the form (1), be in the class $M_\alpha(n,\beta)$, then

$$|a_2| \leq \frac{n+1}{2}\cos\alpha\left(\frac{\sigma}{\pi\sin^2\sigma}\right)^2, \quad (\sigma = \arccos\beta),$$

and

$$|a_m| \leq \frac{(n+1)m-1}{(m-1)(2m-1)}\cos\alpha\left(\frac{\sigma}{\pi\sin^2\sigma}\right)^2 \prod_{t=3}^{m-1} \left(1 + \cos\alpha\left(\frac{\sigma}{t-2}(\pi\sin^2\sigma)\right)^2\right), \quad (\sigma = \arccos\beta).$$

Proof. Let $f$ is given by (1), belongs to $M_\alpha(n,\beta)$, also $I_n f(z) = z + \sum_{m=2}^{\infty} b_m z^m = F(z)$, where

$$b_m = \frac{(2)m-1}{(n+1)m-1}a_m. \quad (29)$$

We define

$$\varphi(z) = e^{-i\alpha}\frac{zF'(z)}{F(z)} = e^{-i\alpha} + \sum_{m=1}^{\infty} c_m z^m.$$ 

Then by using theorem 3, we have $e^{i\alpha}\varphi(z) \prec e^{i\alpha}(\cos\alpha z - is\alpha)$, where $h_\beta$ is given by (4) depending on $\beta$ and the function $h_\beta$ is univalent in $\Delta$ and $h_\beta(\Delta) = D_3$.

Using Rogosinski lemma 7 and relation (9) of lemma 8 for function $e^{-i\alpha}\varphi(z)$, we have $|e^{-i\alpha}c_m| \leq |a_2|$. Now, writing $e^{-i\alpha}\varphi(z)F(z) = zF'(z)$ and comparing the coefficients of $z^n$ on both sides, we get

$$(m - 1)b_m = \sum_{k=1}^{m-1} e^{i\alpha}c_{m-k}b_k.$$
Form the above equality, we get $|b_2| = |c_1| \leq |a_2|$. By using the equation (29) we obtain (28).

Further

$$|b_3| = \frac{1}{2}|e^{ia_2}c_2 + e^{ia_1}b_2| \leq \frac{1}{2}(|c_1| + |c_2|)b \leq \frac{1}{2}a_2(1 + a_2).$$

By using the induction, we have

$$|b_k| \leq \frac{a_2}{k-1}(1 + a_2)(1 + \frac{a_2}{2})...(1 + \frac{a_2}{k-2}), \quad k = 3, 4, ..., m - 1.$$  

Then,

$$(m - 1)|b_m| \leq \sum_{k=1}^{m-1} |c_{m-k}||b_k| \leq a_2 \sum_{k=1}^{m-1} |b_k|$$

$$\leq a_2(1 + a_2 + \frac{a_2}{2}(1 + a_2) + \frac{a_2}{3}(1 + a_2)(1 + \frac{a_2}{2}) + ...$$

$$+ \frac{a_2}{m-2}(1 + a_2)(1 + \frac{a_2}{2})...(1 + \frac{a_2}{m-3}))$$

$$= a_2(1 + a_2)(1 + \frac{a_2}{2})...(1 + \frac{a_2}{m-2}),$$

and hence

$$|b_m| \leq \frac{a_2}{m-1} \prod_{t=3}^{m} (1 + \frac{a_2}{t-2}), \quad (m \geq 3).$$

By using (29) and (9) we obtain (27).

**Theorem 14.** Assume that $n_1 \leq n_2, n_1, n_2 \in \mathbb{N} \cup \{0\}$, then

$$M_\alpha(n_1, k) \subseteq M_\alpha(n_2, k),$$

for all $k \in (0, \infty)$ and $|z| < 1/2$.

**Proof.** Let $f \in M_\alpha(n, k)$. By definition 1 and theorem 3 we have

$$\frac{z(I_{n_1}f(z))'}{I_{n_1}f(z)} = h_\beta(w(z)) \cos \alpha - i \sin \alpha,$$  

(30)

where $h_\beta(\Delta) = D_\beta$ and $|w(z)| < 1$ in $\Delta$ with $w(0) = 0$.

Let us denote

$$f_{n_1, n_2} = \sum_{m=0}^{\infty} \frac{(n_1 + 1)}{(n_2 + 1)} z^{m+1}, \quad (z \in \Delta).$$  

(31)
Then by (31), we have
\[ f_{n_2}^\dagger(z) = f_{n_1}^\dagger(z) \ast f_{n_1,n_2}^\dagger(z). \]

Applying (5), (30), (31) and the properties of convolution, we get
\[
e^{-i\alpha}z\left(\frac{I_{n_2}f(\hat{z})'}{I_{n_2}f(\hat{z})}\right) = e^{-i\alpha}z\left(\frac{f_{n_2}^\dagger \ast f(\hat{z})'}{f_{n_2}^\dagger \ast f(\hat{z})}\right) = e^{-i\alpha} \frac{f_{n_1,n_2}(\frac{\hat{z}}{2}) \ast z(I_{n_1}f(z))'}{f_{n_1,n_2}(\frac{\hat{z}}{2}) \ast I_{n_1}f(z)} = f_{n_1,n_2}(\frac{\hat{z}}{2}) \ast (h_\beta(w(z))\cos\alpha - is\alpha)I_{n_1}f(z).
\] (32)

Moreover, it follows from (30) that \( I_{n_1}f \in USP(\alpha) \subseteq SP(\alpha) \subseteq S^* \) and obtain from lemma 4, \( f_{n_1,n_2} \in CV \). Then by using lemma 5 to (32), we obtain
\[
\frac{f_{n_1,n_2}(\frac{\hat{z}}{2}) \ast (h_\beta(w(z))\cos\alpha - is\alpha)I_{n_1}f(z)}{f_{n_1,n_2}(\frac{\hat{z}}{2}) \ast I_{n_1}f(z)} \subseteq \hat{co}(h_\beta(w(z))\cos\alpha - is\alpha), \quad (z \in \Delta).
\]

Hence the function (32) is subordinated to \( h_\beta(z)\cos\alpha - is\alpha \), so \( f \in M_{\alpha}(n_2, \beta) \) for \( |z| < 1/2 \).

**Corollary 15.** The Theorem 14 are satisfied

\[ USP(\alpha, \beta) = M_{\alpha}(1, \beta) \subset M_{\alpha}(n, \beta), \]

for all \( |z| < 1/2, \beta \in (0, 1) \) and all \( n \in \mathbb{N} \).

**References**


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